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On normality of ASL domains

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In this note we shall give a sufficient condition for a graded ASL domain over a field to be normal. (For the detail, see §4.) As an easy corollary to our result, we can give an alternative proof of the normality of homogeneous coordinate rings of Grassmann varieties using the straightening relations. The proof of our result is based on two facts: Firstly, a graded ASL has a standard filtration whose associated graded ring is a discrete ASL ([2], Proposition 1.1); secondly, *bad* height one prime ideals of a discrete ASL on a poset H over a field correspond with special subsets of H which we call *spindles* of H (Lemma 3, §3).

It should be remarked here that, as shown in [4], every homogeneous ASL domain on a poset H over a field is not normal even if H is a wonderful poset.

§1. Preliminaries

We here recall some basic properties of a graded ASL from [2] and [3].

Let H be a finite poset, and let $\boldsymbol{N}^H$ be the set of functions from H to $\boldsymbol{N}$; a *monomial* on H is an element of $\boldsymbol{N}^H$. The *support* of a monomial M on H is the set $\operatorname{Supp} M=\{x\in H|M(x)\neq 0\}$; a monomial M is called *standard* if $\operatorname{Supp} M$ is a chain. For $x\in H$, $\dim x$ is the maximal length of a chain of H ascending from x. For the empty subset $\emptyset$ of H, we put $\min\emptyset=+\infty$ and $\max\emptyset=-\infty$; we shall agree that $-\infty<x<+\infty$ for every $x\in H$.

Let k be a ring, A a k-algebra, H a finite poset contained in A which generates A as a k-algebra. Let ψ be the map from $\boldsymbol{N}^H$ to A defined by $\psi(M)=\prod_{x\in H}x^{M(x)}$ for $M\in\boldsymbol{N}^H$. We will usually identify M and $\psi(M)$. Then we say that A is an ASL on H over k if the following axioms are satisfied: (ASL–1) The algebra A is a free k-module whose basis is the set of standard monomials; (ASL–2) if x and y in H are incomparable and if $xy=\sum r_jN_j$ $(0\neq r_j\in k)$ is the unique expression for $xy\in A$ as a linear combination of distinct standard monomials, then $\min\operatorname{Supp} N_j<x, y$ for every j.

An ASL on H over k is called discrete if $xy=0$ for all incomparable pairs x, y in H. In this note $k[H]$ denotes a discrete ASL on H over k. Note that if H is a chain, then $k[H]$ is a polynomial ring of $\#H$ variables with coefficients in k. An ASL on H over k is called graded if A is a graded ring such that every element of H is a homogeneous element of degree>0 and k is the set of homogeneous elements of degree 0; a graded ASL on H is called homogeneous if $\deg x=1$ for every $x \in H$.

Let A be an ASL on H over k. Let L_A be the maximal number of factors of standard monomials appearing in the right-hand side of the relations in (ASL–2) for all incomparable pairs of elements in H. For a monomial M on H, we define the *weight* of M to be the number $w(M)=\sum_{x\in H}(2L_A+1)^{\dim x}M(x)$, and we denote by $[M]$ the set of standard monomials which appear in the expression for $M \in A$ as a linear combination of distinct standard monomials (cf. [4]).

Let A be a graded ASL on H over k. Then the proof of Proposition 1.1 in [2] is also valid in our case; consequently, if M is a non-standard monomial, then $w(N)>w(M)$ for all $N\in[M]$. Therefore if R is an A-subalgebra of $A[t, t^{-1}]$, where t is an indeterminate, generated by $\{xt^{-w(x)}|x\in H\}$ and t, then it follows from Theorem 2.1 in [2] that R is an ASL on H over $k[t]$ and R/tR is a discrete ASL on H over k, where the embedding $q:H\to R$ is given by $q(x)=xt^{-w(x)}$.

§ 2. Weights of monomials

Let A be an ASL on a poset H over a ring k.

LEMMA 1. *Let N and N' be monomials, and choose $v\in\operatorname{Supp} N$ and $u\in\operatorname{Supp} N'$ so that $\dim v=\max\{\dim x|x\in\operatorname{Supp} N\}$ and $\dim u=\max\{\dim x|x\in\operatorname{Supp} N'\}$. Assume that $\dim v>\dim u$. Then we have the following assertions*:

i) *If $\sum_{x\in H}N'(x)\le L_A$, then $w(N)\ge 2w(N')+w(u)$.*

ii) *If $\sum_{x\in H}N'(x)\le 2L_A$, then $w(N)>w(N')$.*

PROOF. i) $w(N)\ge w(v)=(2L_A+1)^{\dim v}\ge(2L_A+1)^{\dim u}2L_A+(2L_A+1)^{\dim u}\ge 2w(N')+w(u)$. Similarly we have the assertion ii).

Let u and v be elements in H such that $v\le u$, and let r be a positive integer. We then denote by $V(u|v^r)$ the k-submodule of A generated by standard monomials N satisfying one of the following conditions:

(a) $\min\operatorname{Supp} N \ngeq v$;

(b) $\min\operatorname{Supp} N = v$, $u\neq v$ and $N(v)>r$;

(c) $N(v)=r$ and $N(z)>0$ for some z in H such that $z\ngeq u$.

LEMMA 2. *Let N be a standrd monomial belonging to $V(u|v^r)$, and suppose that $\sum_{x\in H}N(x)\le L_A$ and $r\le L_A$. Then for every $M\in[uvN]$, $w(M)>w(u^sv^r)+w(uv)$, where $s=L_A-r$.*

PROOF. Suppose first that $\alpha=\min\operatorname{Supp} N\ngeq v$. If $\alpha<v$, then $w(N)>w(u^sv^r)$ by Lemma 1; if α is incomparable with v, then for every $N'\in[\alpha v]$, it follows from Lemma 1 that $w(N')>w(u^sv^r)+w(v)$. Therefore the assertion follows in this case. Suppose next that $\min\operatorname{Supp} N=v$, $u\neq v$ and $N(v)>r$. Since

$w(v) > w(u^s)$ by Lemma 1, $w(v^{r+1}) > w(u^s v^r)$, and hence $w(N) > w(u^s v^r)$; thus the assertion follows. Finally suppose that $N(v) = r$ and $N(z) > 0$ for some z in H with $z \ngeq u$. Then for every $N' \in [uz]$, $w(N') > w(u^s) + w(u)$ by Lemma 1. Therefore for every $N'' \in [uN]$, $w(N'') > w(v^r) + w(u^{s+1})$, and this shows that $w(M) > w(v^{r+1}) + w(u^{s+1})$ for every $M \in [uvN]$. This completes the proof.

§3. Height one prime ideals of a discrete ASL over a field

Let A be a discrete ASL on a poset H over a field k, and let $\Delta_1, \ldots, \Delta_h$ be the maximal chains of H. Let $H_i = H - \Delta_i$ for $i = 1, \ldots, h$. It is well known that $\{H_1 A, \ldots, H_h A\}$ is the set of minimal prime ideals of A.

Let $\mathfrak{p}$ be a height one prime ideal of A. We say that $\mathfrak{p}$ is *of type* 1 if there exists a minimal prime ideal $H_j A$ such that $\mathfrak{p} \supset H_j A$ and $\Delta_j \cap \mathfrak{p} = \emptyset$. If $\mathfrak{p}$ is of type 1, then it is easy to see that $\mathfrak{p}$ contains a unique minimal prime ideal of A and $A_{\mathfrak{p}}$ is a *DVR*. We say that $\mathfrak{p}$ is *of type* 2 if $\mathfrak{p}$ is not of type 1.

Assume that $\mathfrak{p}$ is of type 2 in the rest of this section. Let $H_j A$ be a minimal prime ideal of A contained in $\mathfrak{p}$. By our assumption, $\Delta_j \cap \mathfrak{p} \neq \emptyset$. Since $ht\mathfrak{p} = 1$, $\Delta_j \cap \mathfrak{p}$ consists of one element α. (Note that $A/H_j A \cong k[\Delta_j]$ and $k[\Delta_j]$ is a polynomial ring over k.) Let $\Gamma = \Delta_j - \{\alpha\}$. Then Γ is independent of minimal prime ideals of A contained in $\mathfrak{p}$, and $\mathfrak{p} = (H - \Gamma)A$. Let $u = \min\{x \in \Gamma \mid \alpha < x\}$ and $v = \max\{x \in \Gamma \mid x < \alpha\}$. Let $W = \{x \in H \mid v < x < u\}$. Since $ht\mathfrak{p} = 1$, W is a clutter i.e., no two elements in W are comparable, and moreover $\mathfrak{p}A_{\mathfrak{p}}$ is minimally generated by W; in particular, if $\sharp W = 1$, then $A_{\mathfrak{p}}$ is a *DVR*.

Definition. (Γ, u, v, W) is a *spindle* of H if

(1) Γ and W are non-empty subsets of H and $\sharp W \geq 2$,

(2) $u, v \in H \cup \{+\infty, -\infty\}$,

(3) $W = \{x \in H \mid v < x < u\}$ and

(4) for each $x \in W$, $x \notin \Gamma$ and $\Gamma \cup \{x\}$ is a maximal chain of H.

Summarizing the above arguments, we have the following

Lemma 3. *Let $\mathfrak{p}$ be a height one prime ideal of A. Then $A_{\mathfrak{p}}$ is a DVR or there exists a spindle (Γ, u, v, W) of H such that $\mathfrak{p} = (H - \Gamma)A$; in the latter case, $\mathfrak{p}A_{\mathfrak{p}}$ is minimally generated by W.*

§4. Normality of graded ASL domains over a field

In this section, A is a graded ASL on a poset H over a field k, and we assume that $k[H]$ is Cohen-Macaulay and every maximal chain of H has the same length. Let $R = A[\{q(x) \mid x \in H\}, t]$, where $q(x) = xt^{-w(x)}$. Note that if (Γ, u, v, w) is a spindle of H, then every element x in W has the same weight.

Lemma 4. *A is a normal domain if and only if, for each spindle* (Γ, u, v, M) *of* H, R_p *is normal, where P is the prime ideal of R generated by t and* $q(H-\Gamma)$.

Proof. Let $M=A_+$, $M^*=M[t, t^{-1}]\cap R$ and $N=(M^*, t)R$. If A is a normal domain, then R is a normal domain because R/tR is reduced and R_t is a normal domain, and hence R_N is normal. Conversely, if R_N is normal, then $A[t, t^{-1}]_{M[t,t^{-1}]}=R_{M^*}=(R_N)_{M^*}$ is normal, and hence A_M is normal i.e., A is a normal domain. Since R/tR ($\cong k[H]$) is Cohen-Macaulay, R_N is normal if and only if R_P is normal for each height two prime ideal P of R such that $t\in P$. Let P be a height two prime ideal of R with $t\in P$. Then by Lemma 2, R_P is regular or $P=(t, q(H-\Gamma))R$ for some spindle (Γ, u, v, W) of H. This completes the proof.

Theorem. *Assume that for every spindle* (Γ, u', v', W) *of* H, *one of the following conditions is satisfied*:

(a) *There exist* x, y *in* W $(x\neq y)$ *and* $u^sv^r\in[xy]$ *such that* (a–1) $s, r>0$, $u, v\in\Gamma$, $v\leq v'$, $u'\leq u$ *and* (a–2) *for every* z $(\neq x)$ *in* W, $zx\equiv au^sv^r$ (mod $V(u|v^r)$) *with* $0\neq a\in k$.

(b) *Let* $W=\{x_1,\dots, x_n\}$. *There exist* $v\in\Gamma$ *and* $r>0$ *such that* $v\leq v'$ *and for all* $i\neq j$, $x_ix_j\equiv a_{ij}x_iv^r+b_{ij}x_jv^r+c_{ij}v^{2r}$ (mod $V(v|v^r)$), where a_{ij}, b_{ij}, $c_{ij}\in k$ *and* $a_{ij}b_{ij}+c_{ij}\neq 0$.

(c) $W=\{x, y\}$ *and there exists* $u^sv^r\in[xy]$ such that s, $r>0$, u, $v\in\Gamma$, $v<u\leq v'$ *and* $xy\equiv axv^r+byv^r+cu^sv^r$ (mod $V(u|v^r)$), *where* $a, b, c\in k$, $c\neq 0$.

Then A is a normal domain. Moreover if the condition (a) *is satisfied at a spindle* (Γ, u', v', W), *then* $\#W=2$.

Proof. Let (Γ, u', v', W) be a spindle of H, and let $P=(t, q(H-\Gamma))R$. It is sufficient to show that R_P is normal. *Consider the case* (a): Choose x, y in W $(x\neq y)$ and $u^sv^r\in[xy]$ satisfying the conditions (a–1) and (a–2). Then by Lemma 2, $q(u)q(v)q(x)q(y)=t^\delta f'$ with $f'\in R-P$, where $\delta=w(u^sv^r)-w(x)-w(y)$; therefore $q(x)q(y)=t^\delta f$ for some $f\in R_P-PR_P$. By replacing y with z $(\neq x)$ in W in the above argument, we have that $q(x)q(z)=t^\delta g$ for some $g\in R_P$. Therefore $q(z)=gf^{-1}q(y)$, and this implies that $W=\{x, y\}$ by Lemma 3. Thus there exists a surjective homomorphism $D=K[[X, Y, T]]/(XY-T^\delta)\to(R_P)\hat{}$, where K is a coefficient field of $(R_P)\hat{}$. Since D is a normal domain of dimension 2, the above homomorphism must be an isomorphism, and hence R_P is normal.

Consider the case (b): Choose $v\in\Gamma$, $r>0$ and a_{ij}, b_{ij}, $c_{ij}\in k$ so that, for all $i\neq j$, $a_{ij}b_{ij}+c_{ij}\neq 0$ and $x_ix_j=a_{ij}x_iv^r+b_{ij}x_iv^r+c_{ij}v^{2r}$ (mod $V(v|v^r)$). Let $S=R_P$ and $F=S[ts^{-1}, q(x_1)s^{-\delta},\dots, q(x_n)s^{-\delta}, s]$, where s is an indeterminate and $\delta=w(v^r)-w(x_i)$. If F/sF is a normal domain, then so is S by filtered version of Theorem 3 in [5], Chapter VIII. Therefore it is sufficient to show that F/sF is a normal domain. By Lemma 2, $q(x_i)q(x_j)=a_{ij}q(x_i)q(v^r)t^\delta+b_{ij}q(x_j)q(v^r)t^\delta+$

$c_{ij}q(v^r)^2t^{2\delta}+t^{2\delta+1}g_{ij}$ for some $g_{ij}\in R_P$. Therefore there exists a surjective homomorphism θ: $D=K[X_1,\dots, X_n, T]/I\to F/sF$ such that $\theta(X_i)=q(x_i)s^{-\delta}/q(v^r)$ for $i=1,\dots, n$ and $\theta(T)=ts^{-1}$, where I is the ideal generated by $X_iX_j-a_{ij}X_iT^\delta-b_{ij}X_jT^\delta-c_{ij}T^{2\delta}$ for all $i\neq j$ and $K=R_P/PR_P$. Since $\dim F/sF=2$, it is sufficient to show that D is a normal domain of dimension 2. Let $I(v)$ be the ideals of A generated by $\{z\in H|z\ngeq v\}$. Then by Proposition 1.2 in [2], $I(v)=V(v|v)$ and $A/I(v)$ is an ASL on $\{z\in H|z\geq v\}$ over k. Let J be the ideal of $K[X_1,\dots, X_n, T^\delta]$ generated by $X_iX_j-a_{ij}X_iT^\delta-b_{ij}X_jT^\delta-c_{ij}T^{2\delta}$ for all $i\neq j$. It then follows from our assumption that there exists a homomorphism σ: $K[X_1,\dots, X_n, T^\delta]/J\to (A/I(v))\otimes_k K$ such that $\sigma(X_i)=x_i$ for $i=1,\dots, n$ and $\sigma(T^\delta)=v^r$. Since $(A/I(v))\otimes_k K$ is an ASL on $\{z\in H|z\geq v\}$ over K, it follows from the axiom (ASL–1) that σ must be injective; hence $K[X_1,\dots, X_n, T^\delta]/J$ is an ASL on $H'=\{x_1,\dots, x_n, v\}$. Since H' is a Cohen-Macaulay poset, $K[X_1,\dots, X_n, T^\delta]/J$ is Cohen-Macaulay, and therefore $D=K[X_1,\dots, X_n, T]/I$ is also Cohen-Macaulay because $I=JK[X_1,\dots, X_n, T]$ and $K[X_1,\dots, X_n, T]$ is free over $K[X_1,\dots, X_n, T^\delta]$. Let $B=K[X_1,\dots, X_n, T]$. We shall now prove that $D=B/I$ is a normal domain of dimension 2. For each e with $2\leq e\leq n$, we put $C_e=K[X_1,\dots, X_e, T]/I_e$, where I_e is the ideal generated by $X_iX_j-a_{ij}X_iT^\delta-b_{ij}X_jT^\delta-c_{ij}T^{2\delta}$ for all i, j with $i<j\leq e$. By arguments similar to the above, C_e is also Cohen-Macaulay for each e, and moreover $C_e\subset C_{e+1}$ for $e=2,\dots, n-1$. We shall now prove that C_e is a normal domain of dimension 2 by induction on e. Assume that $e=2$. Then $C_2\cong K[X, Y, T]/(XY-T^{2\delta})$, and therefore C_2 is a normal domain of dimension 2. Assume that $e\geq 2$ and C_{e-1} is a normal domain of dimension 2. Let $Q'=(X_1-b_{1e}T^\delta,\dots, X_{e-1}-b_{e-1,e}T^\delta)C_{e-1}$ and $Q=(Q', T)C_e$. Note here that $\sqrt{Q'C_e}=Q$. Since $(X_i-b_{ie}T^\delta)(X_e-a_{ie}T^\delta)=(c_{ie}+a_{ie}b_{ie})T^{2\delta}$ in C_e for $i=1,\dots, e-1$, $\operatorname{Spec}(C_{e-1})-V(Q')\cong\operatorname{Spec}(C_e)-V(Q)$. Therefore if $htQ\geq 2$, then C_e is normal. If $htQ=1$, then $X_e-a_{ie}T^\delta\notin Q$, and hence $(C_e)_Q$ is a DVR whose maximal ideal is generated by T; therefore C_e is also normal in this case. Consequently $D=C_n$ is normal. Note here that D is a graded ring such that $\deg X_i=\delta$ for $i=1,\dots, n-1$ and $\deg T=1$. Therefore D is a domain.

Finally consider the case (c): Write $xy=axv^r+byv^r+cu^sv^r \pmod{V(u|v)}$ with $a, b, c\in k$, $c\neq 0$. By Lemma 2, $q(x)q(y)=aq(x)q(v^r)t^{w(v^r)-w(y)}+bq(y)q(v^r)\cdot t^{w(v^r)-w(x)}+cq(u^s)q(v^r)t^\delta+t^{\delta+1}f$ for some $f\in R_P$, where $\delta=w(u^s)+w(v^r)-w(x)-w(y)$. Let $S=R_P$ and $F=S[tT^{-1}, q(x)T^{-w(u^s)+w(x)}, q(y)T^{-W(v^r)+w(y)}, T]$, where T is an indeterminate. It is sufficient to show that F/TF is a normal domain. Let $\delta'=w(v^r)-w(y)$ and $\delta''=w(u^s)-w(x)$. Note that $\delta'>\delta''$. Since $(q(x)T^{-\delta''})\cdot(q(y)T^{-\delta'})=aq(v^r)(q(x)T^{-\delta''})(tT^{-1})^{\delta'}+cq(u^s)q(v^r)(tT^{-1})^{\delta'+\delta''}+Tg$ for some $g\in F$, there exists a surjective homomorphism $D=K[X, Y, T]/(XY-a'XT^{\delta'}-T^\delta)\to F/TF$, where $K=R_P/PR_P$. Since D is a normal domain and both D and F/TF are two-dimensional, F/TF must be a normal domain. This completes the proof.

If the poset H satisfies the additional condition that *for every spindle* (Γ, u, v, W), $\sharp W=2$ (e.g., H is a distributive lattice), then we have the following

COROLLARY. *Assume that for every spindle* $(\Gamma, u, v, \{x, y\})$, *one of the following conditions is satisfied*:

(a) $xy \equiv uv \pmod{V(v|v)}$;

(b) $xy \equiv auv + byv + cv^2 \pmod{V(v|v)}$ *and* $ab + c \neq 0$.

Then A is a normal domain.

It is known that the homogeneous coordinate ring of a Grassmann variety is a homogeneous ASL domain on a distributive lattice satisfying the condition (a) in the above Corollary (cf. [1]); therefore it is normal by the above corollary and the fact that distributive lattices are Cohen-Macaulay posets.

Note that if A satisfies the conditions in the above corollary, then, for every w in H, $A/I(w)$ $(I(w) = V(w|w))$ also satisfies the same conditions; therefore $A/I(w)$ is also a normal domain. Thus we have an alternative proof of the fact that Schubert varieties are projectively normal.

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