

## The convergence of the Poincaré series on the limit set of a discrete group in several dimensions

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### Introduction

In this paper we study the Poincaré series for a discrete group of Möbius transformations acting on the unit ball in the euclidean  $n$ -space, where  $n$  is an integer greater than one. Usually the convergence problem of this series is considered on the open unit ball; but in this paper we consider the convergence on the unit sphere, especially on the limit set.

In case  $n=2$ , Ch. Pommerenke [5] has shown, by using a theorem of O. Frostman [3], that the Poincaré series of dimension one converges at a point on the unit circle if and only if the angular derivative of a Blaschke product exists at the point. He has also studied measure theoretical properties of the set of points at which the Poincaré series of dimension one converges.

We introduce the notion of accumulation index at a limit point (Section 2), which, roughly speaking, expresses how the orbit (of the origin) accumulates at the limit point. We show in Theorem 7 that if the accumulation index is negative (resp. positive) then the Poincaré series of any (resp. some) dimension diverges (resp. converges). We also show in Theorem 4 that the accumulation index is null at Garnett points, non-positive at horospherical limit points, and non-negative at other limit points.

The convergence of the Poincaré series of dimension  $n-1$  is discussed in Section 4. Our result generalizes a theorem of Ch. Pommerenke [5] to higher dimension.

1. Let  $n$  be an integer greater than one. For  $x, y \in \mathbf{R}^n$ , let  $|x|$  be the euclidean norm of  $x$ ,  $(x, y)$  the euclidean inner product and  $[x, y]^2 = 1 + |x|^2|y|^2 - 2(x, y)$ . Let  $B$  be the unit ball in  $\mathbf{R}^n$  and  $S$  the unit sphere in  $\mathbf{R}^n$ . For  $x \in B$  and  $\xi \in S$ , let  $R(x, \xi)$  be the radius of the horosphere at  $\xi$  through  $x$ , i.e.,  $R(x, \xi) = |x - \xi|^2 / (|x - \xi|^2 + 1 - |x|^2)$ .

We use  $B$  as a model of  $n$ -dimensional hyperbolic spaces. Let  $s$  be the hyperbolic distance in  $B$ .  $M(B)$  denotes the group of Möbius transformations preserving  $B$ . We denote by  $J\gamma(x)$  the Jacobian matrix of  $\gamma \in M(B)$  at  $x \in \mathbf{R}^n$  and by  $|J\gamma(x)|$  the positive number such that  $J\gamma(x)/|J\gamma(x)|$  is an orthogonal matrix. It is well-known that

$$(1) \quad |J\gamma(x)| = \frac{1 - |\gamma^{-1}0|^2}{[x, \gamma^{-1}0]^2}, \quad x \in \mathbf{R}^n$$

(cf. Ahlfors [1], p. 27). We remark that for  $\gamma \in M(B)$

$$(2) \quad R(\gamma 0, \xi) = \frac{1}{1 + |J\gamma^{-1}(\xi)|}, \quad \xi \in S.$$

Let  $\Gamma$  be a non-elementary (i.e., the limit set  $A$  is an infinite set) discrete subgroup of  $M(B)$  such that the origin is not fixed by elements of  $\Gamma - \{\iota\}$ , where  $\iota$  is the identity transformation. We decompose  $A$  into the following three sets (cf. Nicholls [4], Sullivan [6]).

$$(3) \quad A_H = \{\xi \in A; \inf_{\gamma \in \Gamma} R(\gamma 0, \xi) = 0\},$$

$$(4) \quad A_G = \{\xi \in A; \inf_{\gamma \in \Gamma} R(\gamma 0, \xi) \text{ is positive and is not attained}\},$$

$$(5) \quad A_E = \{\xi \in A; \min_{\gamma \in \Gamma} R(\gamma 0, \xi) \text{ exists}\}.$$

The points in  $A_H$  (resp.  $A_G$ ) are called horospherical limit points (resp. Garnett points). Using (2), we have

$$(3') \quad A_H = \{\xi \in A; \sup_{\gamma \in \Gamma} |J\gamma(\xi)| = +\infty\},$$

$$(4') \quad A_G = \{\xi \in A; \sup_{\gamma \in \Gamma} |J\gamma(\xi)| \text{ is finite and is not attained}\},$$

$$(5') \quad A_E = \{\xi \in A; \max_{\gamma \in \Gamma} |J\gamma(\xi)| \text{ exists}\}.$$

The following theorem was proved by P. J. Nicholls [4] in case  $n=2$ . It is easily seen that his proof also works for  $n>2$ .

**THEOREM 1.** *Let  $D$  be the Ford region for  $\Gamma$ , i.e.,*

$$D = \{x \in B; |J\gamma(x)| < 1 \text{ for all } \gamma \in \Gamma - \{\iota\}\},$$

*and let  $D^*$  be the intersection of the euclidean closure of  $D$  with  $A$ . Then*

$$A_E = \bigcup_{\gamma \in \Gamma} \gamma(D^*).$$

**2.** We define the *accumulation index*  $k(\xi) = k_{B, \Gamma}(\xi)$  at  $\xi \in A$  by

$$k(\xi) = \liminf_{\gamma \in \Gamma} \frac{\log |J\gamma(\xi)|}{\log |J\gamma(0)|}.$$

We have  $-1 \leq k(\xi) \leq 1$  since  $1 - |x| \leq |\xi - x| < 2$  for  $x \in B$ . The following theorem gives a geometrical meaning of  $k(\xi)$ .

**THEOREM 2.** *For any  $\xi \in A$*

$$1 - k(\xi) = 2 \limsup_{\gamma \in \Gamma} \frac{\log |\xi - \gamma 0|}{\log (1 - |\gamma 0|)}.$$

PROOF. By (1) we have

$$1 - \frac{\log |J\gamma^{-1}(\xi)|}{\log |J\gamma^{-1}(0)|} = 2 \frac{\log |\xi - \gamma 0|}{\log (1 - |\gamma 0|^2)}.$$

Since  $1 - |x| \leq 1 - |x|^2 < 2(1 - |x|)$  for  $x \in B$ , the desired equality immediately follows.

Now we study a relation of the accumulation index with the decomposition of  $A$  in Section 1.

LEMMA 3. (i) *If  $\limsup_{\gamma \in \Gamma} |J\gamma(\xi)| > 0$ , then  $k(\xi) \leq 0$ .*  
 (ii) *If  $\{|J\gamma(\xi)|; \gamma \in \Gamma\}$  is bounded, then  $k(\xi) \geq 0$ .*

PROOF. (i) By the hypothesis there is  $t > 0$  such that  $|J\gamma(\xi)| > t$  for infinitely many  $\gamma \in \Gamma$ . For such  $\gamma$  we have

$$\frac{\log |J\gamma(\xi)|}{\log |J\gamma(0)|} < \frac{\log t}{\log |J\gamma(0)|},$$

since  $|J\gamma(0)| < 1$ . Therefore  $k(\xi) \leq 0$ .

(ii) is similarly proved.

From this lemma, we obtain

THEOREM 4. (i) *If  $\xi \in A_H$ , then  $k(\xi) \leq 0$ .*  
 (ii) *If  $\xi \in A_G$ , then  $k(\xi) = 0$ .*  
 (iii) *If  $\xi \in A_E$ , then  $k(\xi) \geq 0$ .*

3. We can now consider the convergence problem of the Poincaré series  $\Theta^\alpha = \Theta_{B, \Gamma}^\alpha$  of dimension  $\alpha$ :

$$\Theta^\alpha(x) = \sum_{\gamma \in \Gamma} |J\gamma(x)|^\alpha, \quad x \in B \cup S.$$

By (3') and (4') we have

LEMMA 5.  $\Theta^\alpha$  diverges on  $A_H \cup A_G$  for any  $\alpha > 0$ .

We need one more lemma.

LEMMA 6. *Let  $\delta$  be the exponent of convergence of  $\Gamma$  (i.e.,  $\delta = \delta_\Gamma = \inf \{\alpha > 0; \Theta^\alpha(0) \text{ converges}\}$ ). If  $k = k(\xi) > 0$ , then  $\Theta^\alpha(\xi)$  converges for  $\alpha > \delta/k$ .*

PROOF. Take  $\varepsilon > 0$  with  $\alpha(k - \varepsilon) > \delta + \varepsilon$ . Since  $|J\gamma(0)| < 1$ , we have

$$|J\gamma(\xi)|^\alpha < |J\gamma(0)|^{\alpha(k - \varepsilon)} < |J\gamma(0)|^{\delta + \varepsilon}$$

except for finitely many  $\gamma \in \Gamma$ , so that the lemma follows.

These lemmas and Theorem 4 imply

**THEOREM 7.** (i) *If  $k(\xi) < 0$ , then  $\Theta^\alpha(\xi)$  diverges for any  $\alpha > 0$ .*  
 (ii) *If  $k(\xi) > 0$ , then  $\Theta^\alpha(\xi)$  converges for some  $\alpha > 0$ .*

Neither the converse of (i) nor that of (ii) is valid (see Section 5).

**4.** We now deal with the special case  $\alpha = n - 1$ .

**THEOREM 8.** *Let  $\sigma$  be the rotation-invariant probability measure on  $S$ . Then  $\Theta^{n-1}$  converges  $\sigma$ -a.e. on  $\Lambda_+ = \{\xi \in \Lambda; k(\xi) > 0\}$ .*

This theorem follows from Theorem 8' below because  $\Lambda_+ \subset \Lambda_E$  by Theorem 4.

**THEOREM 8'.**  *$\Theta^{n-1}$  converges  $\sigma$ -a.e. on  $\Lambda_E$ .*

**PROOF.** By Theorem 1 it suffices to show that  $\Theta^{n-1}$  converges  $\sigma$ -a.e. on  $D^*$ .

We claim that  $\gamma(D^*)$ ,  $\gamma \in \Gamma$ , are mutually disjoint up to null sets. In fact, if  $\eta \in D^*$  and  $\varphi\eta \in D^*$  for some  $\varphi \in \Gamma - \{t\}$ , then  $|J\gamma(\eta)| \leq 1$  and  $|J\gamma(\varphi\eta)| \leq 1$  for all  $\gamma \in \Gamma$ . Therefore  $|J\varphi(\eta)| = 1$ , so that  $\eta$  lies on the isometric sphere of  $\varphi$ . Since the intersection of the isometric sphere with  $S$  is a null set, the claim follows.

Hence

$$\sum_{\gamma \in \Gamma} \int_{D^*} |J\gamma(\xi)|^{n-1} d\sigma(\xi) = \sum_{\gamma \in \Gamma} \int_{\gamma(D^*)} d\sigma(\xi) \leq 1,$$

from which we conclude that  $\Theta^{n-1}$  converges  $\sigma$ -a.e. on  $D^*$ .

**5.** As stated at the end of Section 3, the converses of Theorem 7 (i) and (ii) are false. To see this we give examples. For technical reasons, we use the upper half space  $U = \{x = (x_1, \dots, x_n) \in \mathbf{R}_n; x_n > 0\}$  instead of  $B$ . We write  $h(x) = x_n$ . Take a Möbius transformation  $\psi$  which maps  $U$  to  $B$  and  $a = (0, \dots, 0, 1)$  to the origin. Let  $M(U)$  be the group of Möbius transformations preserving  $U$  and let  $\Gamma$  be a discrete subgroup of  $M(U)$ . Then the Poincaré series of dimension  $\alpha$  is translated to

$$\Theta_U^\alpha(x) = \Theta_{U, \Gamma}^\alpha(x) = \Theta_{B, \psi\Gamma\psi^{-1}}^\alpha(\psi x), \quad x \in U \cup \hat{\mathbf{R}}^{n-1}$$

where  $\hat{\mathbf{R}}^{n-1} = \mathbf{R}^{n-1} \cup \{\infty\}$  is the one-point compactification of  $\mathbf{R}^{n-1}$ , and the accumulation index is translated to

$$k_U(\xi) = k_{U, \Gamma}(\xi) = k_{B, \psi\Gamma\psi^{-1}}(\psi\xi), \quad \xi \in \Lambda.$$

Specifically we have

$$\Theta_U^\alpha(\infty) = \sum_{\gamma \in \Gamma} h(\gamma a)^\alpha,$$

and, when  $\infty$  is a limit point of  $\Gamma$ , we have

$$k_U(\infty) = \liminf_{\gamma \in \Gamma} \frac{\log h(\gamma a)}{\log \{4h(\gamma a)/|\gamma a + a|^2\}}.$$

Remark that the denominator is negative since  $4h(x)/|x+a|^2 < 1$  if  $x \in U - \{a\}$ .

For three sequences  $(r_v)_{v=1}^\infty$ ,  $(p_v)_{v=1}^\infty$  and  $(q_v)_{v=1}^\infty$  of positive numbers, we denote by  $\sigma_v$  the inversion in the sphere of radius  $r_v$  centered at  $P_v = (-p_v, 0, \dots, 0)$  and by  $\tau_v$  the reflection in the euclidean bisector of  $P_v$  and  $Q_v = (q_v, 0, \dots, 0)$ . Set  $\gamma_v = \tau_v \sigma_v$  and let  $\Gamma$  be the group generated by  $\gamma_1, \gamma_2, \dots$ .

(i) We choose  $r_v, p_v$  and  $q_v$  so that

$$\frac{r_v^2}{p_v^2 + 1} = \frac{1}{2} + \frac{1}{4v}, \quad v = 1, 2, \dots,$$

$$p_1 = q_1 = r_1, \quad p_v = q_v = 2 \sum_{j=1}^{v-1} r_j + r_v, \quad v = 2, 3, \dots$$

Then clearly  $h(\gamma_v a) = 1/2 + 1/4v$ . Furthermore  $v_v a \rightarrow \infty$  as  $v \rightarrow +\infty$ . Thus  $k_U(\infty) = 0$  and  $\Theta_U^\alpha(\infty)$  diverges for any  $\alpha > 0$ .

(ii) We take  $\alpha > 0$  and  $c$  with  $0 < c < \min(1/2, 2^{-1/\alpha})$ . We choose  $r_v, p_v$  and  $q_v$  so that

$$r_v = cv^{-1/\alpha}, \quad p_v = 2 \sum_{j=1}^{v-1} r_j + r_v + v, \quad q_v = 2^v.$$

Then  $\Gamma$  is a free group since  $c < 1/2$ . Let  $\Gamma_0 = \{e\}$  and  $\Gamma_\mu$  the set of elements of  $\Gamma$  of word-length  $\mu$  ( $\mu = 1, 2, \dots$ ). Take  $\gamma \in \Gamma_\mu$  ( $\mu \geq 1$ ). Then either  $\gamma = \gamma_v \varphi$  or  $\gamma = \gamma_v^{-1} \varphi$  for some  $v \geq 1$  and for some  $\varphi \in \Gamma_{\mu-1}$ . If  $\gamma = \gamma_v \varphi$ , then

$$h(\gamma a) = \frac{r_v^2}{|P_v - \varphi a|^2} h(\varphi a).$$

Since  $|P_v - \varphi a| > 1$ , we have

$$h(\gamma a) \leq r_v^2 h(\varphi a).$$

If  $\gamma = \gamma_v^{-1} \varphi$ , then, since  $c < 1/2$ , we have the same inequality again. Hence

$$\begin{aligned} \sum_{\gamma \in \Gamma_\mu} h(\gamma a)^\alpha &\leq 2 \sum_{v=1}^\infty \sum_{\varphi \in \Gamma_{\mu-1}} (r_v^2 h(\varphi a))^\alpha \\ &= 2c^{2\alpha} \frac{\pi^2}{6} \sum_{\varphi \in \Gamma_{\mu-1}} h(\varphi a)^\alpha \leq \left( c^{2\alpha} \frac{\pi^2}{3} \right)^\mu. \end{aligned}$$

Therefore  $\Theta_U^\alpha(\infty) < +\infty$  for  $c < 2^{-1/\alpha}$ .

On the other hand, we have  $k_U(\infty) = 0$  since  $h(\gamma_v a) = O(v^{-2-2/\alpha})$  and  $|\gamma_v a| = O(2^v)$  as  $v \rightarrow +\infty$ .

### References

[1] Ahlfors, L. V.: Möbius Transformations in Several Dimensions, Ordway Professorship Lectures in Math., Univ. of Minnesota, Minneapolis, 1981.

- [2] Beardon, A. F.: *The Geometry of Discrete Groups*, Graduate Texts in Math. **91**, Springer-Verlag, New York-Berlin, 1983.
- [3] Frostman, O.: Sur les produits de Blaschke, *Proc. Roy. Physiog. Soc. Lund*, **12** (1942), 169–182.
- [4] Nicholls, P. J.: Garnett points for Fuchsian groups, *Bull. London Math. Soc.*, **12** (1980), 216–218.
- [5] Pommerenke, Ch.: On the Green's function of Fuchsian groups, *Ann. Acad. Sci. Fenn. Ser. AI Math.*, **2** (1976), 409–427.
- [6] Sullivan, D.: On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions, *Ann. of Math. Stud.*, **97** (1981), 465–496.

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