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Product formula for nonlinear contraction semigroups in Banach spaces

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1. Introduction

Let H be a Hilbert space and X_0 a closed convex subset of H. Let A and B be maximal dissipative operators in H such that $X_0 \subset \overline{D(A)} \cap \overline{D(B)}$, $(I-\lambda A)^{-1}(X_0) \subset X_0$ and $(I-\lambda B)^{-1}(X_0) \subset X_0$ for $\lambda>0$. We then assume that $A+B$ is also maximal dissipative in H and we write $\{T_A(t)\}_{t\geq 0}$, $\{T_B(t)\}_{t\geq 0}$ and $\{T(t)\}_{t\geq 0}$ for the contraction semigroups generated by A, B and $A+B$, respectively. In the previous paper [7], it was shown that the product formula

$$\lim_{n\to\infty}\left(T_A\left(\frac{t}{n}\right)T_B\left(\frac{t}{n}\right)\right)^n x = T(t)x \tag{1.1}$$

holds for $x \in \overline{D(A+B)} \cap X_0$ and $t \geq 0$. In this paper, we establish the product formula (1.1) in a Banach space whose norm is uniformly Gâteaux differentiable.

Let X be a real Banach space. The norm $\|\cdot\|$ of X is said to be uniformly Gâteaux differentiable if

$$\lim_{a\to 0} a^{-1}(\|x+ay\| - \|x\|)$$

exists for each $y \in X$ and uniformly for $x \in X$ with $\|x\|=1$. Throughout this paper, we assume that the norm of X *is* uniformly Gâteaux differentiable. Let X_0 be a closed convex subset of X. Let A be a dissipative operator in X. We consider the following condition $(R\colon X_0)$ on the operator A:

$(R{:}X_0)$ $\overline{D(A)} = X_0$, $R(A) \subset \overline{\mathrm{sp}}\,(X_0-X_0)$ and $R(I-\lambda A) \supset X_0$ for $\lambda>0$,

where $\overline{\mathrm{sp}}\,(X_0-X_0)$ denotes the closed subspace of X spanned by the set $X_0-X_0=\{x-y;\ x, y \in X_0\}$. If the dissipative operator A satisfies condition $(R\colon X_0)$, then the limit

$$\lim_{n\to\infty}\left(I-\frac{t}{n}A\right)^{-n} x = T(t)x \tag{1.2}$$

exists for $x \in X_0$ and $t \geq 0$ and the family $\{T(t)\}_{t\geq 0}$ becomes a contraction semigroup on X_0. Such a family $\{T(t)\}_{t\geq 0}$ is called a (contraction) semigroup on X_0 generated by A. From the results due to Baillon [1] and Reich [12], it is seen that if in addition X is reflexive and $\{T(t)\}_{t\geq 0}$ is a contraction semigroup on X_0, then there exists a dissipative operator A in X that satisfies $(R: X_0)$ and for which (1.2) holds.

Let A_i, $i=1, 2,\dots, N$, and A be dissipative operators in X. Assume that either of A_i and A satisfies condition $(R: X_0)$ and let $\{T_i(t)\}_{t\geq 0}$, $i=1, 2,\dots, N$, and $\{T(t)\}_{t\geq 0}$ be the semigroups on X_0 generated by A_i, $i=1, 2,\dots, N$ and A, respectively. Our objective of this paper is to show that if A is the closure of $A_1+A_2+\cdots+A_N$, then

$$\lim_{n\to\infty}\left(T_N\left(\frac{t}{n}\right)T_{N-1}\left(\frac{t}{n}\right)\cdots T_1\left(\frac{t}{n}\right)\right)^n x = T(t)x \tag{1.3}$$

and

$$\lim_{n\to\infty}\left(\left(I-\frac{t}{n}A_N\right)^{-1}\left(I-\frac{t}{n}A_{N-1}\right)^{-1}\cdots\left(I-\frac{t}{n}A_1\right)^{-1}\right)^n x = T(t)x \tag{1.4}$$

hold for $x\in X_0$ and $t\geq 0$. Actually, more general formulae will be obtained from which (1.3) and (1.4) follow simultaneously.

Such product formulae as (1.3) and (1.4) for nonlinear semigroups have been also obtained under different assumptions, for instance, in [2], [6] and [8].

2. The main theorem

To state the product formulae which generalizes (1.3) and (1.4), we introduce a new notion, namely, A-family of contraction operators for a dissipative operator A. Since the norm of X is Gâteaux differentiable, for each $x\in X$, there is a unique $x^*\in X^*$ such that $\|x\|^2=\|x^*\|^2=\langle x^*, x\rangle$. (The symbol $\langle x^*, x\rangle$ stands for the value of x^* at x.) The mapping $x\to x^*$ is called the duality mapping in X and denoted by F. It is known that the mapping F is continuous with respect to the norm topology in X and the weak* topology in X^*.

DEFINITION. Let A be a dissipative operator in X. Let $\{U(t)\}_{t>0}$ be a family of contraction operators of $\overline{D(A)}$ into itself. The family $\{U(t)\}_{t>0}$ is called an *A-family* if, for each $t>0$, there exists a family $\{V_t(s)\}_{0<s<t}$ of contraction operators of $\overline{D(A)}$ into itself with the following three properties (a), (b) and (c):

(a) For each $x\in\overline{D(A)}$, $V_t(s)x$ is strongly measurable on $(0, t)$ as an X-valued function of s.

(b) For each $x\in\overline{D(A)}$ and each $u\in D(A)$,

$$\|V_t(s)u-u\| \leq t|||Au|||, \quad 0<s<t \tag{2.1}$$

and

$$\|U(t)x-u\| \le \left(\frac{1}{t}\int_0^t \|V_t(s)x-u\|^2 ds\right)^{1/2} + t|||Au|||, \tag{2.2}$$

where $|||Au||| = \inf\{\|v\|;\ v \in Au\}$.

(c) For each $x \in \overline{D(A)}$, $u \in D(A)$ and $v \in Au$,

$$\|U(t)x-u\|^2 - \|x-u\|^2 \le 2\int_0^t < F(V_t(s)x-u),\ v>ds. \tag{2.3}$$

REMARKS. Since $V_t(s)$ are contraction operators, (2.1) implies

$$\begin{aligned}\|V_t(s)x-u\| &\le \|V_t(s)x-V_t(s)u\| + \|V_t(s)u-u\| \\ &\le \|x-u\| + t|||Au|||\end{aligned} \tag{2.4}$$

and

$$\left(\frac{1}{t}\int_0^t \|V_t(s)x-u\|^2 ds\right)^{1/2} \le \|x-u\| + t|||Au||| \tag{2.5}$$

for $0<s<t$, $x \in \overline{D(A)}$ and $u \in D(A)$. Hence, (b) yields

$$\|U(t)x-u\| \le \|x-u\| + 2t|||Au||| \tag{2.6}$$

for $x \in \overline{D(A)}$ and $u \in D(A)$.

Let A be a dissipative operator in X which satisfies condition (R: X_0) for a closed convex set $X_0 \subset X$. Set $J(t)x=(I-tA)^{-1}x$ for $t>0$ and $x \in \overline{D(A)}$. As is well known, $J(t)$ is a contraction operator of $\overline{D(A)}$ into itself. Let $\{T(t)\}_{t>0}$ be the semigroup on X_0 generated by A. Several examples of A-familes for the operator A are now in order.

EXAMPLE 2.1. $\{J(t)\}_{t>0}$ is an A-family with $V_t(s)=J(t)$ for $s \in (0, t)$. In fact, (a) is trivially satisfied. Let $x \in \overline{D(A)}$, $u \in D(A)$ and $v \in Au$. Then, $t^{-1}(J(t)x-x) \in AJ(t)x$ and A is dissipative, and so we have

$$\langle F(J(t)x-u),\ t^{-1}(J(t)x-x)-v\rangle \le 0. \tag{2.7}$$

Therefore,

$$\begin{aligned}&t^{-1}\|J(t)x-u\|^2 - \langle F(J(t)x-u),\ v\rangle \\ &\quad = \langle F(J(t)x-u),\ t^{-1}(J(t)x-u)-v\rangle \\ &\quad \le \langle F(J(t)x-u),\ t^{-1}(x-u)\rangle \\ &\quad \le \|J(t)x-u\|\cdot t^{-1}\|x-u\| \\ &\quad \le (2t)^{-1}(\|J(t)x-u\|^2+\|x-u\|^2).\end{aligned}$$

Hence, we have

(2.8) $$\|J(t)x-u\|^2 - \|x-u\|^2 \leq 2t\langle F(J(t)x-u), v\rangle$$

and (c) is satisfied. By (2.7) with $x=u$,

$$\begin{aligned} t^{-1}\|J(t)u-u\|^2 &\leq \langle F(J(t)u-u), v\rangle \\ &\leq \|F(J(t)u-u)\| \cdot \|v\| \\ &= \|J(t)u-u\| \cdot \|v\| . \end{aligned}$$

Hence,

(2.9) $$\|J(t)u-u\| \leq t\|v\|$$

and (2.1) holds. Finally, it is easily seen that (2.2) is valid.

EXAMPLE 2.2. For any fixed positive integer m, $\{J(t/m)^m\}_{t>0}$ is also an A-family with $V_t(s)=J(t/m)^j$ for $s\in((j-1)t/m, jt/m]$, $j=1, 2,\cdots, m$. First, condition (a) is trivially satisfied. Let $x\in\overline{D(A)}$, $u\in D(A)$ and $v\in Au$. By (2.8),

$$\left\|J\left(\frac{t}{m}\right)^j x - u\right\|^2 - \left\|J\left(\frac{t}{m}\right)^{j-1} x-u\right\|^2 \leq 2\frac{t}{m}\left\langle F\left(J\left(\frac{t}{m}\right)^j x-u\right), v\right\rangle$$

for $j=1, 2,\cdots, m$. Adding these inequalities, we have

$$\left\|J\left(\frac{t}{m}\right)^m x-u\right\|^2 - \|x-u\|^2 \leq 2\frac{t}{m}\sum_{j=1}^m\left\langle F\left(J\left(\frac{t}{m}\right)^j x-u\right), v\right\rangle.$$

From this we obtain (c). Since each $J(t/m)$ is a contraction operator, (2.9) implies

(2.10) $$\begin{aligned} &\left\|J\left(\frac{t}{m}\right)^j u-u\right\| \\ &\leq \sum_{k=1}^j \left\|J\left(\frac{t}{m}\right)^k u-J\left(\frac{t}{m}\right)^{k-1} u\right\| \\ &\leq j\left\|J\left(\frac{t}{m}\right)u - u\right\| \leq \frac{j}{m} t\|v\| \leq t\|v\| \end{aligned}$$

for $j=1, 2,\cdots, m$. Hence, (2.1) holds. Using Minkowski's inequality, we have

$$\begin{aligned} \left\|J\left(\frac{t}{m}\right)^m x-u\right\| &= \left(\frac{1}{m}\sum_{j=1}^m\left\|J\left(\frac{t}{m}\right)^m x-u\right\|^2\right)^{1/2} \\ &\leq \left(\frac{1}{m}\sum_{j=1}^m\left\|J\left(\frac{t}{m}\right)^m x-J\left(\frac{t}{m}\right)^{m-j} u\right\|^2\right)^{1/2} \\ &\quad + \left(\frac{1}{m}\sum_{j=1}^m\left\|J\left(\frac{t}{m}\right)^{m-j} u-u\right\|^2\right)^{1/2}. \end{aligned}$$

Since (2.10) implies

$$\left(\frac{1}{m}\sum_{j=1}^{m}\left\|J\left(\frac{t}{m}\right)^{m-j}u-u\right\|^{2}\right)^{1/2}\leq t\|v\|$$

and each $J(t/m)$ is a contraction operator, the above inequality implies

$$\left\|J\left(\frac{t}{m}\right)^{m}x-u\right\|\leq\left(\frac{1}{m}\sum_{j=1}^{m}\left\|J\left(\frac{t}{m}\right)^{j}x-u\right\|^{2}\right)^{1/2}+t\|v\|,$$

and hence (b) is obtained.

EXAMPLE 2.3. $\{T(t)\}_{t>0}$ is an A-family with $V_t(s)=T(s)$ for $s\in(0, t)$. This follows from Example 2.2, since, for $x\in\overline{D(A)}$, $T(t)x=\lim_{m\to\infty}J(t/m)^m x$ uniformly for bounded $t\geq 0$ and $T(t)x$ is continuous in $t\geq 0$. (See Miyadera [10].)

We are now in a position to state our main theorem.

THEOREM. *Let X_0 be a closed convex set of X. Let A_i, $i=1, 2,\cdots, N$, be dissipative operators in X and assume that each A_i satisfies condition $(R\colon X_0)$. Let $\{U_i(t)\}_{t>0}$ be an A_i-family for each $i=1, 2,\cdots, N$ and A the closure of $A_1+A_2+\cdots+A_N$. Suppose that the dissipative operator A satisfies condition $(R\colon X_0)$. Let $\{T(t)\}_{t\geq 0}$ be the semigroup on X_0 generated by A. Then*

$$\lim_{n\to\infty}\left[U_N\left(\frac{t}{n}\right)U_{N-1}\left(\frac{t}{n}\right)\cdots U_1\left(\frac{t}{n}\right)\right]^n x = T(t)x \tag{2.11}$$

for $x\in X_0$ and uniformly for bounded $t\geq 0$.

COROLLARY. *In the above theorem the product formulae* (1.3) *and* (1.4) *hold.*

The proof of the theorem will be given in the next section. We here state a basic lemma which will be used in the proof.

LEMMA. *Let ϕ and ψ be convex Gâteaux differentiable functionals on X. Let X_0 be a convex set of X. If $\phi(x)=\psi(x)$ for $x\in X_0$, then $\langle\phi'(x), v\rangle=\langle\psi'(x), v\rangle$ for $x\in X_0$ and $v\in\overline{\mathrm{sp}}\,(X_0-X_0)$.*

PROOF. Let $x\in X_0$ and $y\in X_0$. Then

$$\phi(x+a(y-x))=\psi(x+a(y-x))\qquad\text{for}\quad a\in(0, 1).$$

Therefore,

$$\begin{aligned}\langle\phi'(x), y-x\rangle &= \lim_{a\downarrow 0}a^{-1}(\phi(x+a(y-x)-\phi(x))\\ &= \lim_{a\downarrow 0}a^{-1}(\psi(x+a(y-x)-\psi(x))\\ &= \langle\psi'(x), y-x\rangle\end{aligned}$$

and

$$\begin{aligned}\langle \phi'(x), y_1-y_2\rangle &= \langle \phi'(x), y_1-x\rangle - \langle \phi'(x), y_2-x\rangle \\ &= \langle \psi'(x), y_1-x\rangle - \langle \psi'(x), y_2-x\rangle \\ &= \langle \phi'(x), y_1-y_2\rangle\end{aligned}$$

for $y_1, y_2 \in X_0$. Since $\phi'(x)$ and $\psi'(x)$ are bounded linear on X, we obtain the required result. Q.E.D.

3. Proof of Theorem

To establish (2.11), it is sufficient to show that

(3.1) $$\lim_{t\downarrow 0}(I-\lambda t^{-1}(U_N(t)\cdots U_1(t)-I))^{-1}x = (I-\lambda A^{-1}x)$$

for $\lambda>0$ and $x\in X_0$. In fact, by the approximation theorem for nonlinear semigroups due to Brezis and Pazy [2], (3.1) implies (2.11). Furthermore, $(I-\lambda t^{-1}(U_N(t)\cdots U_1(t)-I))^{-1}$ and $(I-\lambda A)^{-1}$ are contraction operators on X_0 and $D(A)$ is dense in X_0, so that it is sufficient to prove (3.1) only for $x\in D(A)$ and and $\lambda>0$. To this end, fix any $x\in D(A)$ and $\lambda>0$ and set

(3.2) $$y_0(t) = (I-\lambda t^{-1}(U_N(t)\cdots U_1(t)-I))^{-1}x$$

(3.3) $$y_j(t) = U_j(t)\cdots U_1(t)y_0(t), \quad j = 1, 2,\cdots, N$$

and

(3.4) $$z_j(s, t) = V_{j,t}(s)y_{j-1}(t), \quad j=1, 2,\cdots, N$$

for $t>0$ and $s\in(0, t)$. Here, for each j, $\{V_{j,t}(s)\}_{0<s<t}$ denotes the family of contraction operators of $\overline{D(A_j)}=X_0$ into itself satisfying conditions (a), (b) and (c) with $U(t)$, $V_t(s)$ and A, replaced respectively by $U_j(t)$, $V_{j,t}(s)$ and A_j. We observe that (3.2) and (3.3) together imply

$$y_0(t) - \lambda t^{-1}(y_N(t)-y_0(t)) = x$$

or

(3.5) $$t^{-1}(y_N(t)-y_0(t)) = \lambda^{-1}(y_0(t)-x).$$

PROPOSITION 3.1. *For each* $j=1, 2,\cdots, N$, $\|y_0(t)\|$, $\|y_j(t)\|$ *and* $\sup_{0<s<t}\|z_j(s, t)\|$ *are bounded as* $t\downarrow 0$.

PROOF. We first note that

(3.6) $$\|U_k(t)\cdots U_1(t)x-x\| \le 2t\sum_{j=1}^{k}|||A_jx|||, \quad k = 1, 2,\cdots, N.$$

In fact, since $U_j(t)$ are contraction operators on $X_0=\overline{D(A_j)}$, we have

$$\begin{aligned}&\|U_k(t)\cdots U_1(t)x-x\|\\&\quad\leq \textstyle\sum_{j=1}^{k}\|U_k(t)\cdots U_{j+1}(t)U_j(t)x-U_k(t)\cdots U_{j+1}(t)x\|\\&\quad\leq \textstyle\sum_{j=1}^{k}\|U_j(t)x-x\|\end{aligned}$$

and so (3.6) follows from (2.6) with $U(t)=U_j(t)$, $A=A_j$ and $u=x$.

Since $(I-\lambda t^{-1}(U_N(t)\cdots U_1(t)-I))^{-1}$ is a contraction operator on X_0, we have

$$\begin{aligned}\|y_0(t)-x\| &\leq \|x-(x-\lambda t^{-1}(U_N(t)\cdots U_1(t)x-x))\|\\&= \|\lambda t^{-1}(U_N(t)\cdots U_1(t)x-x)\|.\end{aligned}$$

Hence (3.6) implies

(3.7) $$\|y_0(t)-x\| \leq 2\lambda \textstyle\sum_{j=1}^{N}\|\|A_jx\|\|,$$

which shows that $\|y_0(t)\|$ is bounded for $t>0$.

Since each $U_j(t)$ as is a contraction operator on X_0, we have

$$\|y_j(t)-U_j(t)\cdots U_1(t)x\| \leq \|y_0(t)-x\|$$

and so

$$\|y_j(t)-x\| \leq \|U_j(t)\cdots U_1(t)x-x\| + \|y_0(t)-x\|$$

for $j=1, 2,\cdots, N$. Thus, it follows from (3.6) and (3.7) that, for $j=1, 2,\cdots, N$, $\|y_j(t)\|$ is bounded as $t\downarrow 0$.

Since $V_{j,t}(s)$ is also a contraction operator,

$$\|z_j(s, t)-V_{j,t}(s)x\| \leq \|y_j(t)-x\|$$

by (3.4) and

$$\|z_j(s, t)-x\| \leq \|V_{j,t}(s)x-x\| + \|y_j(t)-x\|.$$

But

$$\|V_{j,t}(s)x-x\| \leq t\|\|A_jx\|\|$$

for $0<s<t$ by (2.1) and we conclude that $\sup_{0<s<t}\|z_j(s, t)-x\|$ is bounded as $t\downarrow 0$. Q.E.D.

Let $\{t(n)\}_{n=1}^{\infty}$ be a null sequence of positive numbers. For each $y\in X$, we set

$$\phi_j(y) = \mathrm{LIM}_{n\to\infty}\|y_j(t(n))-y\|^2, \quad j = 0, 1,\cdots, N$$

and

$$\psi_j(y) = \mathrm{LIM}_{n\to\infty}\, t(n)^{-1}\int_0^{t(n)}\|z_j(s, t(n))-y\|^2 ds \quad j = 1,\cdots, N,$$

where $\mathrm{LIM}_{n\to\infty} a_n$ denotes the Banach limit of a bounded sequence $\{a_n\}_{n=1}^{\infty}$. As is easily seen, ϕ_j and ψ_j define convex continuous functionals on X.

PROPOSITION 3.2. *$\phi_0(y)=\psi_j(y)=\phi_j(y)$ for $y\in X_0$ and $j=1, 2,\cdots, N$.*

PROOF. Using (3.5) and (3.7), we have

$$\|y_N(t)-y_0(t)\| \leq 2t\sum_{j=1}^{N}|||A_jx|||$$

and so

$$\begin{aligned} &|\,\|y_N(t)-y\|^2-\|y_0(t)-y\|^2\,| \\ &\quad\leq (\|y_N(t)-y\|+\|y_0(t)-y\|)\cdot\|y_N(t)-y_0(t)\| \\ &\quad\leq 2t(\|y_N(t)-y\|+\|y_0(t)-y\|)\sum_{j=1}^{N}|||A_jx||| \end{aligned}$$

for $y\in X$. Since $\|y_N(t)\|$ and $\|y_0(t)\|$ are bounded as $t\downarrow 0$, this implies that $\phi_N(y)=\phi_0(y)$ for $y\in X$. Let $y\in D(A)$ be fixed. By the inequality (2.2) with $U(t)=U_j(t)$, $V_t(s)=V_{j,t}(s)$, $A=A_j$, $x=\phi_{j-1}(t)$ and $u=y$, we have

$$\|y_j(t)-y\| \leq \left(t^{-1}\int_0^t \|z_j(s, t)-y\|^2 ds\right)^{1/2} + t|||A_jy|||$$

for $j=1, 2,\cdots, N$, since $y_j(t)=U_j(t)y_{j-1}(t)$ and $z_j(s, t)=V_{j,t}(s)y_{j-1}(t)$. This implies

$$\phi_j(y) \leq \psi_j(y) \qquad \text{for} \quad j = 1, 2,\cdots, N, \tag{3.9}$$

since $\sup_{0<s<t}\|z_j(s, t)\|$ is bounded as $t\downarrow 0$. On the other hand, applying (2.5) for $V_t(s)=V_{j,t}(s)$, $A=A_j$, $x=\phi_{j-1}(t)$ and $u=y$, we have

$$\left(t^{-1}\int_0^t \|z_j(s, t)-y\|^2 ds\right)^{1/2} \leq \|y_{j-1}(t)-y\| + t|||A_jy|||.$$

Since $\|y_{j-1}(t)\|$ is bounded as $t\to 0$, the above estimate implies

$$\psi_j(y) \leq \phi_{j-1}(y) \qquad \text{for} \quad j = 1, 2,\cdots, N. \tag{3.10}$$

Combining (3.8), (3.9) and (3.10), we obtain $\phi_j(y)=\psi_j(y)=\phi_0(y)$ for $y\in D(A)$ and $j=1, 2,\cdots, N$. Since $\overline{D(A)}=X_0$ and ϕ_j and ψ_j are continuous on X, we get the required result. Q.E.D.

PROPOSITION 3.3. *The functionals $\phi_j, j=0, 1,\cdots, N$ and $\psi_j, j=1, 2,\cdots, N$ are Gâteaux differentiable on X and*

$$\langle\phi_j'(y), v\rangle = -2\cdot\mathrm{LIM}_{n\to\infty}\langle F(y_j(t(n)))-y), v\rangle, \quad j=0, 1,\cdots, N$$

and

$$\langle\psi'_j(y),\ v\rangle = -2\cdot \mathrm{LIM}_{n\to\infty}\, t(n)^{-1}\int_0^{t(n)} \langle F(z_j(s,\ t(n))-y),\ v\rangle ds,$$

$j=1,\dots, N$, for y, $v\in X$.

PROOF. Since the norm of X is uniformly Gâteaux differentiable,

$$\lim_{a\to 0} a^{-1}(\|u+av\|^2-\|u\|^2) = 2\langle F(u),\ v)$$

for $v\in X$ and uniformly for bounded $u\in X$. Let y, $v\in X$ and let $\varepsilon>0$. Since $\|y_j(t)\|$ is bounded as $t\downarrow 0$, $\|y_j(t(n))\|$ is bounded with respect to n. Therefore, there exists a positive number δ such that

$$|a^{-1}(\|y_j(t(n))-y-av\|^2 - \|y_j(t(n))-y\|^2) - 2\cdot\langle F(y_j(t(n))-y),\ -v\rangle| < \varepsilon$$

for $|a|<\delta$ and $n=1, 2,\cdots$. Taking the Banach limits of each term on the left hand side, we obtain

$$|a^{-1}(\phi_j(y+av)-\phi_j(y))-2\cdot \mathrm{LIM}_{n\to\infty}\langle F(y_j(t(n))-y),\ -v\rangle| \le \varepsilon.$$

Thus, ϕ_j is Gâteaux differentiable at y and $\langle\phi'_j(y),\ v\rangle = -2\cdot\mathrm{LIM}_{n\to\infty}\langle F(y_j(t(n))-y),\ v\rangle$, $j=0, 1,\cdots, N$.

Since $\sup_{0<s<t}\|z_j(s,\ t\|)$ is bounded as $t\downarrow 0$, $\sup_{0<s<t}\|z_j(t(n))\|$ is bounded with respect to n. Therefore, there exists a positive number δ such that

$$|a^{-1}(\|z_j(s,\ t(n))-y-av\|^2-\|z_j(s,\ t(n))-y\|^2) \\ -2\cdot\langle F(z_j(s,\ t(n))-y),\ -v\rangle| < \varepsilon$$

for $|a|<\delta$, $0<s<t(n)$ and $n=1, 2,\cdots$. Hence,

$$\left| a^{-1}t(n)^{-1}\int_0^{t(n)} (\|z_j(s,\ t(n))-y-av\|^2-\|z_j(s,\ t(n))-y\|^2)ds \right. \\ \left. -2\cdot t(n)^{-1}\int_0^{t(n)} \langle F(z_j(s,\ t(n))-y),\ -v\rangle ds \right| \le \varepsilon$$

for $n=1, 2,\cdots$ and $|a|<\delta$; and consequently

$$\left| a^{-1}(\psi_j(y+av)-\psi_j(y)) \right. \\ \left. -2\cdot \mathrm{LIM}_{n\to\infty}\, t(n)^{-1}\int_0^{t(n)} \langle F(z_j(s,\ t(n))-y),\ -v\rangle ds \right| \le \varepsilon$$

for $|a|<\delta$. Thus, ψ_j is also Gâteaux differentiable at y and $\langle\psi'_j(y),\ v\rangle = -2\cdot \mathrm{LIM}_{n\to\infty}\, t(n)^{-1}\int_0^{t(n)}\langle F(z_j(s,\ t(n))-y),\ v\rangle ds$. Q. E. D.

In view of the Lemma stated in the end of Section 2, Propositions 3.2 and

3.3 together imply the following Corollary.

COROLLARY. *For* $y \in X_0$, $v \in \overline{\mathrm{sp}}(X_0 - X_0)$ *and* $j = 1, 2, \cdots, N$,

$$\begin{aligned} &\mathrm{LIM}_{n\to\infty} \langle F(y_j(t(n)) - y), v\rangle \\ &= \mathrm{LIM}_{n\to\infty} t(n)^{-1} \int_0^{t(n)} \langle F(z_j(s, t(n)) - y), v\rangle ds \\ &= \mathrm{LIM}_{n\to\infty} \langle F(y_0(t(n)) - y), v\rangle. \end{aligned}$$

PROPOSITION 3.4. $y_0(t) \to (I - \lambda A)^{-1} x$ as $t \downarrow 0$. *Therefore* (3.1) *is obtained for* $\lambda > 0$ *and* $x \in D(A)$.

PROOF. Let $u \in D(A)$ and $v_j \in A_j u$, $j = 1, 2, \cdots, N$. The inequality (2.3) with $U(t) = U_j(t)$, $V_t(s) = V_{j,t}(s)$, $x = y_{j-1}(t)$ and $v = v_j$ implies

$$\|y_j(t) - u\|^2 - \|y_{j-1}(t) - u\|^2 \le 2\int_0^t \langle F(z_j(s, t) - y), v_j\rangle ds, \tag{3.11}$$

since $y_j(t) = U_j(t) y_{j-1}(t)$ and $z_j(s, t) = V_{j,t}(s) y_{j-1}(t)$. Summing the relations (3.11) over $j = 1, 2, \cdots, N$, we obtain

$$\|y_N(t) - u\|^2 - \|y_0(t) - u\|^2 \le 2\sum_{j=1}^N \int_0^t \langle F(z_j(s, t) - y), v_j\rangle ds. \tag{3.12}$$

On the other hand, we have

$$\begin{aligned} &\|y_N(t) - u\|^2 - \|y_0(t) - u\|^2 \\ &\ge 2\|y_N(t) - u\| \cdot \|y_0(t) - u\| - 2\|y_0(t) - u\|^2 \\ &\ge 2\langle F(y_0(t) - u), y_N(t) - u\rangle - 2\langle F(y_0(t) - u), y_0(t) - u\rangle \\ &= 2\langle F(y_0(t) - u), y_N(t) - y_0(t)\rangle. \end{aligned}$$

Thus, using (3.5), we have

$$\begin{aligned} &\|y_N(t) - u\|^2 - \|y_0(t) - u\|^2 \\ &\ge 2\langle F(y_0(t) - u), t\lambda^{-1}(y_0(t) - x)\rangle \\ &= 2t\lambda^{-1}(\|y_0(t) - u\|^2 - \langle F(y_0(t) - u), x - u\rangle). \end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13), we obtain

$$\|y_0(t) - u\|^2 \le \langle F(y_0(t) - u), x - u\rangle + \lambda \sum_{j=1}^N t^{-1} \int_0^t \langle F(z_j(s, t) - y), v_j\rangle ds$$

and it follows that

$$\mathrm{LIM}_{n\to\infty} \|y_0(t(n)) - u\|^2 \le \mathrm{LIM}_{n\to\infty} \langle F(y_0(t(n) - u), x - u\rangle$$

$$+\lambda\sum_{j=1}^{N}\mathrm{LIM}_{n\to\infty}\,t(n)^{-1}\int_0^{t(n)}\langle F(z_j(s,\,y(n))-y),\,v_j\rangle ds.$$

Note that $x-u\in X_0-X_0$ and $v_j\in\mathrm{sp}\,(X_0-X_0)$ for $j=1, 2,\cdots, N$, since each A_j satisfies condition $(R\colon X_0)$. So, in view of the Corollary before Proposition 3.4, we have

$$\begin{aligned}&\mathrm{LIM}_{n\to\infty}\,\|y_0(t(n))-u\|^2\\&\le \mathrm{LIM}_{n\to\infty}\,\langle F(y_0(t(n))-u),\,x-u+\lambda\textstyle\sum_{j=1}^{N}v_j\rangle\\&\le \sup_n\|y_0(t(n))-u\|\cdot\|x-u+\lambda\textstyle\sum_{j=1}^{N}v_j\|\,.\end{aligned}$$

Since A is the closure of $A_1+A_2+\cdots+A_N$, it turns out that

$$\begin{aligned}&\mathrm{LIM}_{n\to\infty}\|y_0(t(n))-u\|^2\\&\le \sup_n\|y_0(t(n))-u\|\cdot\|x-u+\lambda v\|\end{aligned}\tag{3.14}$$

for $u\in D(A)$ and $v\in Au$. Putting $u=(I-\lambda A)^{-1}x$ and $v=\lambda^{-1}((I-\lambda A)^{-1}x-x)$ in (3.14), we have

$$\mathrm{LIM}_{n\to\infty}\,\|y_0(t(n))-(I-\lambda A)^{-1}x\|^2=0.$$

This shows that there exists a subsequence $\{t(n(k))\}$ of $\{t(n)\}$ such that

$$\lim_{k\to\infty}\,y_0(t(n(k)))=(I-\lambda A)^{-1}x.$$

Thus, it is concluded that $y_0(t)$ converges to $(I-\lambda A)^{-1}x$ as $t\downarrow 0$, as required.

Q. E. D.

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