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A Hausdorff-Young inequality for the Fourier transform on Riemannian symmetric spaces

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§ 1. Introduction

Let G/K be a Riemannian symmetric space of non-compact type. In [1] spherical Fourier transforms of left K-invariant L^p $(1<p<2)$ functions on G/K are studied and it is shown that the spherical transforms of these functions are extended holomorphically to a certain domain T_p, which is determined only by p, in $\mathfrak{a}_C^*$ and a Hausdorff-Yong inequality holds. We adopt $\pi_\nu(f)=\int_G f(x)\pi_\nu(x)dx$ as the Fourier transform of $f\in C_0^\infty(G/K)$; here π_ν denotes the induced representation of class one from the minimal parabolic subgroup P of G. The purpose of this paper is to show that the Fourier transforms of K-finite L^p functions on G/K also satisfy a Hausdorff-Young type inequality in the domain T_p similar to the spherical case.

§ 2. Notation and Preliminaries

Let G be a connected semisimple Lie group with finite center and $\mathfrak{g}$ its Lie algebra. We denote by $\langle\cdot,\cdot\rangle$ the Killing form of $\mathfrak{g}$. Let $G=KAN$ be an Iwasawa decomposition and $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ the Lie subalgebras of $\mathfrak{g}$ corresponding to K, A and N respectively. Each $x\in G$ can be written uniquely as $x=\kappa(x)\cdot\exp H(x)n(x)$, where $\kappa(x)\in K$, $H(x)\in\mathfrak{a}$ and $n(x)\in N$. Let M' and M be the normalizer and the centralizer of $\mathfrak{a}$ in K respectively and denote by $W=M'/M$ the Weyl group. Throughout this paper, we denote the dual space of a real or complex vector space V by V^* and the complexification of a real vector space V by V_C. We fix an ordering on $\mathfrak{a}^*$ which is compatible with the above Iwasawa decomposition. Let Σ denote the set of all positive roots of $(\mathfrak{g}, \mathfrak{a})$ and $m(\alpha)$ the multiplicity of $\alpha\in\Sigma$. Let Σ_0 be the set of elements in Σ which are not integral multiples of other elements in Σ. We put $a(\alpha)=m(\alpha)+m(2\alpha)$ for $\alpha\in\Sigma_0$ and $\rho=2^{-1}\sum_{\alpha\in\Sigma}m(\alpha)\alpha$. Let $\mathfrak{a}_+^*$ be the positive Weyl chamber of $\mathfrak{a}^*$ and put

$$\mathfrak{a}_+ = \{H\in\mathfrak{a}\,|\,\alpha(H)>0 \text{ for all } \alpha\in\mathfrak{a}_+^*\};\quad A^+ = \exp\mathfrak{a}_+ .$$

For any $\varepsilon\geq 0$, we put

$$C_{\varepsilon\rho} = \{\lambda\in\mathfrak{a}^*\,|\,|(s\lambda)(H)|\leq\varepsilon\rho(H) \text{ for all } H\in\mathfrak{a}_+ \text{ and } s\in W\}.$$

Now we write T_p for the tube domain $\mathfrak{a}^* + \sqrt{-1}\, C_{\varepsilon\rho}$ in $\mathfrak{a}_C^*$, where $\varepsilon = 2/p - 1$ $(1 \leq p < 2)$.

Let $\hat{K}$ be the set of all equivalence classes of irreducible unitary representations of K. For each $\tau \in \hat{K}$, we fix a representative of τ and denote it by the same symbol τ. For each τ, we denote by V_τ, χ_τ and $d(\tau)$ its representation space, character and degree respectively. Let $\hat{K}_M$ be the subset of $\hat{K}$ which consists of all the class one representations with respect to M. We fix a finite subset F of $\hat{K}_M$ and put $\bar{\chi}_F = \sum_{\tau \in F} d(\tau)\bar{\chi}_\tau$. If M is a manifold, $C^\infty(M)$ and $C_0(M)$ denote the set of all $\boldsymbol{C}$-valued C^∞ functions and the set of all $\boldsymbol{C}$-valued continuous functions with compact supports on M respectively and put $C_0^\infty(M) = C^\infty(M) \cap C_0(M)$. Let $C_{0F}^\infty(G/K)$ denote the set of all $f \in C_0^\infty(G)$ which satisfy $f(gk) = f(g)$ $(g \in G,\ k \in K)$ and $\bar{\chi}_F * f = f$, $*$ denoting the convolution on K. Let dk_M denote the invariant measure on K/M such that the total measure equals 1 and da the $(2\pi)^{-l/2}$ -times of the euclidean measure on A $(l = \dim A)$ which is induced by the Killing form. We denote by dx the invariant measure on G/K such that

$$\int_{G/K} f(x)dx = \int_{K/M \times A^+} f(ka)\delta(a)dk_M da,$$

where $\delta(a) = \prod_{\alpha \in \Sigma_0} (\sinh \alpha(\log a))^{m(\alpha)}$.

For $f \in C_{0F}^\infty(G/K)$, we put $\|f\|_p = \left(\int_{G/K} |f(x)|^p dx\right)^{1/p}$. And its L^p completion is denoted by $L_F^p(G/K)$. If $\mathscr{H}$ is a complex separable Hilbert space, then $\mathscr{B}(\mathscr{H})$ denotes the space of all bounded linear operators on $\mathscr{H}$. For $B \in \mathscr{B}(\mathscr{H})$, its operator norm is denoted by $\|B\|_\infty$ and the p-norm $\|B\|_p$ is defined by $\|B\|_p = (\mathrm{tr}\,(B^*B)^{p/2})^{1/p}$ $(1 \leq p < \infty)$, where B^* denotes the adjoint operator of B.

§3. The estimate of the norm of $\pi_\nu(f)$

Recall first the definition and properties of the induced representations of the class one from the minimal parabolic subgroup P to G. Let $L^2(K/M)$ be the space of right M-invariant functions in $L^2(K)$ and denote by $(\cdot,\cdot)$ the inner product in $L^2(K/M)$. For each $\nu \in \mathfrak{a}_C^*$ the induced representation π_ν of G on $L^2(K/M)$ is defined by

$$(\pi_\nu(x)\Phi)(k) = e^{(\sqrt{-1}\nu - \rho)(H(x^{-1}k))}\Phi(\kappa(x^{-1}k)),$$
$$(\Phi \in L^2(K/M),\ x \in G,\ k \in K).$$

For $f \in C_0^\infty(G/K)$ the bounded linear operator $\pi_\nu(f)$ on $L^2(K/M)$, which is called the Fourier transform of f, is defined by

$$\pi_\nu(f) = \int_G f(x)\pi_\nu(x)dx.$$

Then the following Parseval equality and the inversion formula are known (cf. [2]):

$$\|f\|_2^2 = [W]^{-1}\int_{\mathfrak{a}^*} \|\pi_\nu(f)\|_2^2 |c(\nu)|^{-2} d\nu;$$

$$f(x) = [W]^{-1}\int_{\mathfrak{a}^*} \mathrm{tr}\,(\pi_\nu(f)\pi_\nu(x^{-1}))\,|c(\nu)|^{-2} d\nu \qquad (f \in C_0^\infty(G/K)),$$

where $c(\nu)$ is the Harish-Chandra c-function for G/K and $[W]$ denotes the order of the Weyl group W. For $f \in C_0^\infty(G/K)$ and $a \in A$ we define a function f^a on K/M by $f^a(kM)=f(ka)$. We fix, for each τ, an orthonormal basis $\{v_1,\dots, v_{d(\tau)}\}$ of V_τ such that $\{v_1,\dots, v_{d_1(\tau)}\}$ is an orthonormal basis in V_τ^M, V_τ^M denoting the subspace of all M fixed vectors in V_τ. Denote by $\{v_1^*,\dots, v_{d(\tau)}^*\}$ the dual basis of $\{v_1,\dots, v_{d(\tau)}\}$. Here we denote by $L_F^2(K/M)$ the closed subspace of $L^2(K/M)$ which is spanned by the set $\{d(\tau)^{1/2}v_j^*(\tau(k^{-1})v_i) | \tau \in F,\ 1 \le i \le d(\tau),\ 1 \le j \le d_1(\tau)\}$ and put $C_F^\infty(K/M) = L_F^2(K/M) \cap C^\infty(K/M)$. Then it is known that the set forms an orthonormal basis of $L_F^2(K/M)$ (cf. [4]). For simplicity, we put $d = \sum_{\tau\in F} d(\tau)d_1(\tau)$ and denote by $\{\Phi_1,\dots, \Phi_d\}$ the above orthonormal basis of $L_F^2(K/M)$. Let $f \in C_{0F}^\infty(G/K)$. If we put $f^i(a)=(f^a, \Phi_i)$, then $f^a \in C_F^\infty(K/M)$ is written as

$$f^a(kM) = \textstyle\sum_{i=1}^d f^i(a)\Phi_i(kM). \tag{3.1}$$

We now put

$$C_0^\infty(A^+, \boldsymbol{C}^d) = \{\varphi=(\varphi^1,\dots, \varphi^d)\colon A^+ \longrightarrow \boldsymbol{C}^d \mid \varphi^i \in C_0^\infty(A^+),\ 1 \le i \le d\},$$

and we denote its L^p completion with the norm $\|\varphi\|_p^p = \int_{A^+} \sum_{i=1}^d |\varphi^i(a)|^p \delta(a)da < \infty$ by $L^p(A^+, \boldsymbol{C}^d)$. By using the decomposition (3.1), we can define a natural linear isomorphism D of $C_{0F}^\infty(G/K)$ into $C_0^\infty(A^+, \boldsymbol{C}^d)$ by $f \mapsto (f^1,\dots, f^d)$.

We first show the following lemma.

LEMMA 1. *If $f \in C_{0F}^\infty(G/K)$ then we have*

$$d^{-2+1/p}\|f\|_p \le \|Df\|_p \le d^{1+1/p}\|f\|_p \quad (1 \le p < \infty).$$

PROOF. We shall prove the second inequality. Using the Hölder inequality, we have

$$\begin{aligned}\|Df\|_p^p &= \int_{A^+} (\textstyle\sum_{i=1}^d |f^i(a)|^p)\delta(a)\,da \\ &= \int_{A^+} \textstyle\sum_{i=1}^d |(f^a, \Phi_i)|^p \delta(a)\,da \\ &\le \int_{A^+} \textstyle\sum_{i=1}^d \|f^a\|_p^p \|\Phi_i\|_q^p \delta(a)\,da\,.\end{aligned}$$

where $1/p+1/q=1$. Since $|\Phi_i|\leq d$, we have

$$\begin{aligned}\|Df\|_p^p &\leq d^{p+1}\int_{A^+}\|f^a\|_p^p\delta(a)\,da\\ &= d^{p+1}\int_{K/M\times A^+}|f^a(kM)|^p\delta(a)\,dk_M da\\ &= d^{p+1}\|f\|_p^p.\end{aligned}$$

The first inequality is proved in a way similar to the above.

By this lemma, D can be uniquely extended to a linear isomorphism of $L_F^p(G/K)$ onto $L^p(A^+, \boldsymbol{C}^d)$ and we use the same symbol D for it. Let $f\in C_0^\infty(G/K)$. From the right K-invariantness of f we get

$$\|\pi_\nu(f)\|_\infty = \|\pi_\nu(f)\Phi_0\|_2,$$

where Φ_0 is the constant function on K/M with value 1.

LEMMA 2. *For $f\in C_{0F}^\infty(G/K)$ the following inequality holds.*

$$\|\pi_\nu(f)\|_\infty \leq d^3\|f\|_1 \qquad (\nu\in T_1).$$

PROOF. Using decomposition (3.1), we have

$$\begin{aligned}|(\pi_\nu(f)\Phi_0)(k_1)| &= \left|\int_G f(x)(\pi_\nu(x)\Phi_0)(k_1)dx\right|\\ &= \left|\int_{K\times A^+}\sum_{i=1}^d f^i(a)\,\Phi_i(k)e^{(\sqrt{-1}\nu-\rho)(H(a^{-1}k^{-1}k_1))}\delta(a)\,dk\,da\right|\\ &\leq \int_{K\times A^+}\sum_{i=1}^d |f^i(a)||\Phi_i(k_1kM)||e^{(\sqrt{-1}\nu-\rho)(H(a^{-1}k^{-1}))}|\delta(a)\,dk\,da\\ &\leq d\int_{A^+}\sum_{i=1}^d|f^i(a)|\left(\int_K|e^{(\sqrt{-1}\nu-\rho)(H(a^{-1}k^{-1}))}|\,dk\right)\delta(a)\,da.\end{aligned}$$

Because

$$\int_K|e^{(\sqrt{-1}\nu-\rho)(H(a^{-1}k^{-1}))}|dk\leq 1 \qquad (\nu\in T_1)$$

(cf. [2]) we have

$$\begin{aligned}|(\pi_\nu(f)\Phi_0)(k)| &\leq d\int_{A^+}\sum_{i=1}^d|f^i(a)|\delta(a)da=d\|Df\|_1 \qquad (3.2)\\ &\leq d^3\|f\|_1 \quad \text{(by Lemma 1)}.\end{aligned}$$

This implies $\|\pi_\nu(f)\|_\infty\leq d^3\|f\|_1$.

We see that the Fourier transform can be extended to $L_F^1(G/K)$. If $\varphi\in L^p(A^+, \boldsymbol{C}^d)$ then we write, for simplicity, $\varphi^\dagger$ for $D^{-1}\varphi$. From (3.2) We get

COROLLARY. *If $\varphi \in L^1(A^+, \boldsymbol{C}^d)$ then we have*

$$\|\pi_\nu(\varphi^\dagger)\|_\infty \leq d\|\varphi\|_1 \quad (\nu \in T_1).$$

§ 4. The Hausdorff-Young inequality in real case

To prove the Hausdorff-Young inequality on $\mathfrak{a}^*$, we use the Riesz-Thorin theorem for vector valued functions. Let (X, μ) and (X', μ') be two σ-finite measure spaces. We denote by $\mathscr{S}(X, \boldsymbol{C}^d)$ the set of all compactly supported simple functions on X with values in $\boldsymbol{C}^d$. Namely

$$\mathscr{S}(X, \boldsymbol{C}^d) = \{\varphi=(\varphi^1,\dots, \varphi^d)\colon X \to \boldsymbol{C}^d \mid \varphi^i\text{'s are compactly supported simple functions on } X\}.$$

Let T be a linear mapping of $\mathscr{S}(X, \boldsymbol{C}^d)$ to the space of all μ'-measurable functions on X'. If there exists a positive constant k such that $\|T\varphi\|_q \leq k\|\varphi\|_p$ for all $\varphi \in \mathscr{S}(X, \boldsymbol{C}^d)$, then T is called of type (p, q) and in addition the infimum of such k is called the (p, q)-norm of T.

LEMMA 4. *Suppose that T is simultaneously of type (p_i, q_i) with (p_i, q_i)-norm k_i $(1 \leq p_i, q_i \leq \infty)$ for $i=0, 1$. For each $0<t<1$, define p_t and q_t by*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad \text{and} \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Then T is of type (p_t, q_t) and its (p_t, q_t)-norm k_t satisfies the inequality $k_t \leq d k_0^{1-t} k_1^t$. Namely,

$$\|T\varphi\|_{q_t} \leq d k_0^{1-t} k_1^t \|\varphi\|_{p_t} \qquad (\varphi \in \mathscr{S}(X, \boldsymbol{C}^d)). \tag{4.1}$$

Moreover, if $p_t < \infty$ then T can be extended to an operator on $L^{p_t}(X, \boldsymbol{C}^d)$ and satisfies the same inequality as (4.1).

The proof of this lemma is accomplished by applying the Riesz-Thorin theorem to each component φ^i of $\varphi=(\varphi^1,\dots, \varphi^d) \in \mathscr{S}(X, \boldsymbol{C}^d)$ and so it is omitted. The aim of this section is to prove the following theorem.

THEOREM 1. *If $1<p<2$ and $1/p+1/q=1$, then the Fourier transform can be extended to $L^p_F(G/K)$ and there exists a positive constant C_{pF}, which depends only on p and F, such that*

$$\left(\int_{\mathfrak{a}^*} \|\pi_\nu(f)\|_q^q |c(\nu)|^{-2} d\nu\right)^{1/q} \leq C_{pF}\|f\|_p \qquad (f \in L^p_F(G/K)).$$

To prove the theorem, we need a lemma. We consider two measure spaces $(A^+, \delta(a)da)$ and $(\mathfrak{a}^*, [W]^{-1}|c(\nu)|^{-2}d\nu)$ and define linear mappings T^i $(i=1,\dots, d)$

of $\mathscr{S}(A^+, \boldsymbol{C}^d)$ to the space of $\boldsymbol{C}$-valued functions on $\mathfrak{a}^*$ by

$$T^i(\varphi)(\nu) = (\pi_\nu(\varphi^\dagger)\Phi_0, \Phi_i) \quad (\text{for } \varphi \in \mathscr{S}(A^+, \boldsymbol{C}^d)).$$

Then clearly $T^i(\varphi)$ is measurable on $\mathfrak{a}^*$ and

$$\pi_\nu(\varphi^\dagger)\Phi_0 = \textstyle\sum_{i=1}^d T^i(\varphi)\Phi_i. \tag{4.2}$$

In addition, for $1 \leq p \leq \infty$, we easily have

$$\|\pi_\nu(\varphi^\dagger)\|_p = (\textstyle\sum_{i=1}^d |T^i(\varphi)|^2)^{1/2}. \tag{4.3}$$

LEMMA 5. *Let $1<p<2$ and $1/p+1/q=1$. Then each T^i can be extended to $L^p(A^+, \boldsymbol{C}^d)$ and there exists a positive constant C'_{pF} such taht*

$$\|T^i(\varphi)\|_q \leq C'_{pF}\|\varphi\|_p \qquad (\varphi \in L^p(A^+, \boldsymbol{C}^d)).$$

PROOF. We fix an i. Using (4.3) and the corollary to Lemma 2, we have for $\varphi \in \mathscr{S}(A^+, \boldsymbol{C}^d)$

$$\|T^i(\varphi)\|_\infty = \sup_{\nu \in \mathfrak{a}^*} |T^i(\varphi)(\nu)| \leq \sup_{\nu \in \mathfrak{a}^*} \|\pi_\nu(\varphi^\dagger)\|_\infty \leq d\|\varphi\|_1 .$$

and so T^i is of type $(1, \infty)$. On the other hand, using (4.3), the Parseval equality and Lemma 1, we have for $\varphi \in \mathscr{S}(A^+, \boldsymbol{C}^d)$

$$\begin{aligned} \|T^i(\varphi)\|_2 &= \Big([W]^{-1}\int_{\mathfrak{a}^*} |T^i(\varphi)(\nu)|^2 |c(\nu)|^{-2} d\nu\Big)^{1/2} \\ &\leq \Big([W]^{-1}\int_{\mathfrak{a}^*} \|\pi_\nu(\varphi^\dagger)\|_2^2 |c(\nu)|^{-2} d\nu\Big)^{1/2} \\ &= \|\varphi^\dagger\|_2 \leq d^{3/2}\,\|\varphi\|_2 , \end{aligned}$$

and so T^i is of type (2, 2). Applying Lemma 4 to our case, we can find a positive constant C'_{pF} which satisfies the desired inequality.

PROOF OF THEOREM 1. From (4.3), we have

$$\begin{aligned} \|\pi_\nu(\varphi^\dagger)\|_q^q = (\textstyle\sum_{i=1}^d |T^i(\varphi)|^2)^{q/2} &= (\textstyle\sum_{i=1}^d (|T^i(\varphi)|^q)^{2/q})^{q/2} \\ &\leq d^{q/2} \textstyle\sum_{i=1}^d |T^i(\varphi)|^q . \end{aligned}$$

Therefore using the Minkowski inequality, we get

$$\begin{aligned} &\Big(\int_{\mathfrak{a}^*} \|\pi_\nu(\varphi^\dagger)\|_q^q |c(\nu)|^{-2} d\nu\Big)^{1/q} \\ &\quad \leq d^{1/2}\Big(\int_{\mathfrak{a}^*} \textstyle\sum_{i=1}^d |T^i(\varphi)|^q |c(\nu)|^{-2} d\nu\Big)^{1/q} \\ &\quad \leq d^{1/2} \textstyle\sum_{i=1}^d \|T^i(\varphi)\|_q \leq d^{3/2} C'_{pF}\|\varphi\|_p . \end{aligned}$$

Because D is a bijection, the proof is completed.

§ 5. The Hausdorff- Young inequality in general case

To prove the Hausdorff-Young inequality on $\mathfrak{a}_C^*$, we use the Kunze-Stein interpolation theorem (cf. [3]). The function space which we consider is not $L^p(A^+)$ but $L^p(A^+, \boldsymbol{C}^d)$ and so we need a slight extension of the theorem. Let (X, μ) be a σ-finite measure space. $\mathscr{S}(X, \boldsymbol{C}^d)$ and $L^p(X, \boldsymbol{C}^d)$ are the same as in section 4 and section 3 respectively. Moreover let Y be a locally compact space satisfying second countability axiom with a regular measure ω and let $\mathscr{H}$ be a complex separable Hilbert space. If F is a $\mathscr{B}(\mathscr{H})$-valued measurable function on Y, then we define p-norms $\|F\|_p$ $(1 \leq p \leq \infty)$ by

$$\|F\|_p = \left(\int_Y \|F(y)\|_p^p \, d\omega(y)\right)^{1/p} \quad (1 \leq p < \infty),$$

$$\|F\|_\infty = \text{ess.}\sup_{y \in Y} \|F(y)\|_\infty .$$

For a, $b \in \boldsymbol{R}$, $b > a$, we put $D = D(a, b) = \{z \in \boldsymbol{C} | a \leq \operatorname{Im} z \leq b\}$. And suppose T_z $(z \in D)$ is a linear operator from $\mathscr{S}(X, \boldsymbol{C}^d)$ to the space of all $\mathscr{B}(\mathscr{H})$-measurable functions on Y. The family $\{T_z | z \in D\}$ is called admissible on D if (i) for any Φ, $\Psi \in \mathscr{H}$ and $\varphi \in \mathscr{S}(X, \boldsymbol{C}^d)$, the $\boldsymbol{C}$-valued function $(T_z(\varphi)(y)\Phi, \Psi)$ is locally integrable on Y; and (ii) for any measurable relatively compact subset Y' of Y, the function

$$\phi(z) = \int_{Y'} (T_z(\varphi)(y)\Phi, \Psi) \, d\omega(y)$$

is admissible on D. Here we say that a $\boldsymbol{C}$-valued function ϕ on D is admissible if (i) ϕ is holomorphic in the interior of D and is continuous on D; and (ii) ϕ is of admissible growth, that is, ϕ satisfies

$$\sup_{a \leq y \leq b} \log |\phi(x + \sqrt{-1}y)| = O(e^{c|x|})$$

for some $c < \pi/(b-a)$.

Let $1 \leq p_0$, $p_1 \leq \infty$ and $1 \leq q_0$, $q_1 \leq \infty$. If $t \in \boldsymbol{R}$, $a < t < b$, then we put $\tau = (t-a)/(b-a)$,

$$\frac{1}{p} = \frac{1-\tau}{p_0} + \frac{\tau}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\tau}{q_0} + \frac{\tau}{q_1}.$$

Let A_i $(i=0, 1)$ be positive functions on $\boldsymbol{R}$ which satisfy, for some $C > 0$ and $c < \pi/(b-a)$, the inequality

$$\log A_i(x) \leq Ce^{c|x|} \qquad \text{for} \quad i = 0, 1$$

simultaneously. The following lemma is an easy consequence of Kunze-Stein [3].

LEMMA 6. *Let* $\{T_z | z \in D\}$ *be an admissible family on* D *of linear operators of* $\mathscr{S}(X, \boldsymbol{C}^d)$ *to the space of* $\mathscr{B}(\mathscr{H})$*-valued measurable functions on* Y *such that*

$$\|T_{x+\sqrt{-1}a}(\varphi)\|_{q_0} \leq A_0(x)\|\varphi\|_{p_0}$$
$$\|T_{x+\sqrt{-1}b}(\varphi)\|_{q_1} \leq A_1(x)\|\varphi\|_{p_1}$$

for all $\varphi \in \mathscr{S}(X, \boldsymbol{C}^d)$. *Then we have*

$$\|T_{\sqrt{-1}t}(\varphi)\|_q \leq dC_t\|\varphi\|_p \quad (\varphi \in \mathscr{S}(X, \boldsymbol{C}^d))$$

for a positive constant C_t *which is given by*

$$\log C_t = \int_{-\infty}^{\infty} \chi(1-\tau, x)\log A_0((b-a)x)dx + \int_{-\infty}^{\infty} \chi(\tau, x)\log A_1((b-a)x)dx, \tag{5.1}$$

where

$$\chi(\tau, x) = \tan\left(\frac{\pi\tau}{2}\right)\operatorname{sech}^2\left(\frac{\pi\tau}{2}\right)\Big/2\left(\tan^2\left(\frac{\pi\tau}{2}\right) + \tanh^2\left(\frac{\pi\tau}{2}\right)\right).$$

To prove the Hausdorff-Young inequality, we need another lemma.

LEMMA 7 (cf. [1]). *There exist positive constants* B_1 *and* B_2 *such that*

$$B_1|c(\nu)|^{-2} \leq \textstyle\prod_{\alpha\in\Sigma_0}|\langle\nu, \alpha\rangle|^2(1+|\langle\nu, \alpha\rangle|)^{a(\alpha)-2} \leq B_2|c(\nu)|^{-2}$$

for all $\nu \in \mathfrak{a}^*$.

We take two measure spaces $(A^+, \delta(a)da)$ and $(\mathfrak{a}^*, d\omega(\nu)=\prod_{\alpha\in\Sigma_0}(1+|\langle\nu, \alpha\rangle|)^{a(\alpha)}d\nu)$ as (X, μ) and (Y, ω) respectively. Let $1<p<2$, $1/p+1/q=1$ and $\varepsilon=2/p-1$. We fix $\eta \in C_{\varepsilon\rho}$ $(\eta \neq 0)$ and choose an orthonormal basis $\mu_1,\dots,\mu_l$ of $\mathfrak{a}^*$ so that $\mu_1=\eta/|\eta|$. We then put $D=D(0, |\eta|/\varepsilon)$. For $\varphi \in \mathscr{S}(A^+, \boldsymbol{C}^d)$ we put

$$F_{z\mu_1}(\varphi)(\nu) = \pi_{z\mu_1+\nu}(\varphi^\dagger)\textstyle\prod_{\alpha\in\Sigma_0}(1+|\langle\nu, \alpha\rangle|)^{-1}|\langle z\mu_1+\nu, \alpha\rangle| \quad (z \in D,\ \nu \in \mathfrak{a}^*)$$

and define a family $\{T_z | z \in D\}$ by

$$T_z\colon \varphi \longrightarrow F_{z\mu_1}(\varphi) \quad (\varphi \in \mathscr{S}(A^+, \boldsymbol{C}^d)).$$

From the relation $(\pi_\nu(\varphi^\dagger)\Phi, \Psi)=\sum_{i=1}^d T^i(\varphi)(\nu)(\Phi, \Phi_0)(\Phi_i, \Psi)$ and the fact that if

$z\in D$ then $z\mu_1+\nu\in T_1$, it follows that $\{T_z|z\in D\}$ is an admissible family on D. For $\xi=x+\sqrt{-1}|\eta|/\varepsilon$ $(x\in \boldsymbol{R})$, by the corollary to Lemma 2 and the inequality $|\langle\nu+\xi\mu_1, \alpha\rangle|\leq(1+|\langle\nu, \alpha\rangle|)(1+|\langle\xi\mu_1, \alpha\rangle|)$, we have

$$\|T_\xi\|_\infty \leq d\|\varphi\|_1 \prod_{\alpha\in\Sigma_0}(1+|\langle\xi\mu_1, \alpha\rangle|).$$

If we put $A_1(x)=d\prod_{\alpha\in\Sigma_0}(1+|\langle\xi\mu_1, \alpha\rangle|)$, then for any c, $0<c<\varepsilon\pi/|\eta|$, we can choose a positive constant C_1 such that

$$\log A_1(x) \leq C_1 e^{c|x|} \qquad (x\in \boldsymbol{R}).$$

On the other hand, for $\xi=x$ $(x\in \boldsymbol{R})$

$$\|T_\xi\|_2^2 = \int_{\mathfrak{a}^*} \|T_{\xi\mu_1+\nu}(\varphi^\dagger)\|_2^2 \prod_{\alpha\in\Sigma_0}(1+|\langle\nu, \alpha\rangle|)^{a(\alpha)-2}|\langle\xi\mu_1+\nu, \alpha\rangle|^2 d\nu$$

$$\leq \int_{\mathfrak{a}^*} \|\pi_{\xi\mu_1}(\varphi^\dagger)\|_2^2 \prod_{\alpha\in\Sigma_0}(1+|\langle\xi\mu_1+\nu, \alpha\rangle|)^{a(\alpha)-2}|\langle\xi\mu_1+\nu, \alpha\rangle|^2 d\nu$$

$$\cdot \prod_{\alpha\in\Sigma_0}\sup_{\nu\in\mathfrak{a}^*}(1+|\langle\xi\mu_1+\nu, \alpha\rangle|)^{2-a(\alpha)}(1+|\langle\nu, \alpha\rangle|)^{a(\alpha)-2}.$$

Because there exists a positive constant k such that

$$(1+|\langle\xi\mu_1+\nu, \alpha\rangle|)^{2-a(\alpha)}\leq k(1+|\langle\nu, \alpha\rangle|)^{2-a(\alpha)}(1+|\langle\xi\mu_1, \alpha\rangle|)^{|a(\alpha)-2|}$$

for all $\nu\in\mathfrak{a}^*$, we have by Lemma 7 and Lemma 1

$$\|T_\xi\|_2^2\leq k\prod_{\alpha\in\Sigma_0}(1+|\langle\xi\mu_1, \alpha\rangle|)^{|2-a(\alpha)|}$$

$$\cdot\int_{\mathfrak{a}^*} \|\pi_{\xi\mu_1}(\varphi^\dagger)\|_2^2 \prod_{\alpha\in\Sigma_0}(1+|\langle\xi\mu_1+\nu, \alpha\rangle|)^{a(\alpha)-2}|\langle\xi\mu_1+\nu, \alpha\rangle|^2 d\nu$$

$$\leq B_2 k\prod_{\alpha\in\Sigma_0}(1+|\langle\xi\mu_1, \alpha\rangle|)^{|2-a(\alpha)|}d\|\varphi\|_2^2.$$

If we put $A_0(x)=(dB_2k)^{1/2}\prod_{\alpha\in\Sigma_0}(1+|\langle\xi\mu_1, \alpha\rangle|)^{|a(\alpha)/2-1|}$ then for any c, $0<c<\varepsilon\pi/|\eta|$, we can choose a positive constant C_0 such that

$$\log A_0(x) \leq C_0 e^{c|x|} \qquad (x\in \boldsymbol{R}),$$

and we have

$$\|T_\xi(\varphi)\|_2 \leq A_0(x)\|\varphi\|_2 \qquad (\varphi\in\mathscr{S}(A^+, \boldsymbol{C}^d)).$$

Applying Lemma 6 to our case, we can find a constant $C>0$ such that

$$\|T_{\sqrt{-1}\eta}(\varphi)\|_q \leq dC\|\varphi\|_p.$$

Therefore, the following inequality holds:

$$\Big(\int_{\mathfrak{a}^*} \|\pi_{\nu+\sqrt{-1}\eta}(\varphi^\dagger)\|_q^q \prod_{\alpha\in\Sigma_0} (1+|\langle\nu, \alpha\rangle|)^{a(\alpha)-q}$$

$$\cdot|\langle\nu+\sqrt{-1}\eta, \alpha\rangle|^q d\nu\Big)^{1/q} \leq C_{pF\eta}\|\varphi\|_p .$$

Thus we obtain the following proposition.

LEMMA 8. *Let* $1<p<2$, $1/p+1/q=1$ *and* $\varepsilon=2/p-1$. *If we fix an* $\eta\in C_{\varepsilon\rho}$, *then there exists a positive constant* $C_{pF\eta}$ *which depends only on p, F and* η *such that*

$$\Big(\int_{\mathfrak{a}^*} \|\pi_{\nu+\sqrt{-1}\eta}(\varphi^\dagger)\|_q^q |c(\nu)|^{-2} d\nu\Big)^{1/q} \leq C_{pF\eta}\|\varphi\|_p$$

for all $\varphi\in\mathscr{S}(A^+, \boldsymbol{C}^d)$.

From this lemma we get the following theorem.

THEOREM 2. *Let p, q and* ε *be in Lemma* 8. *If* $f\in L_F^p(G/K)$ *then the Fourier transform* $\pi_\nu(f)$ *can be holomorphically extended to the tube domain* T_p *and for any* $\eta\in C_{\varepsilon\rho}$, *there exists a positive constant* $C_{pF\eta}$ *such that*

$$\Big(\int_{\mathfrak{a}^*} \|\pi_{\nu+\sqrt{-1}\eta}(f)\|_q^q |c(\nu)|^{-2} d\nu\Big)^{1/q} \leq C_{pF\eta}\|f\|_p \qquad (f\in L_p^F(G/K)) .$$

§6. The Hausdorff-Young inequality for the Radon-Fourier transform

The Radon-Fourier tansform on the Riemannian symmetric space G/K is defined as follows. Let $f\in C_0^\infty(G/K)$. Then

$$\tilde{f}(kM, \nu) = \int_{A\times N} f(kan)e^{(-\sqrt{-1}\nu+\rho)(H(a))}da\,dn, \quad (kM\in K/M,\ \nu\in\mathfrak{a}^*).$$

Concerning this transform, the Parseval equality and the inversion formula are known: for $f\in C_0^\infty(G/K)$

$$\|f\|_2^2 = [W]^{-1}\int_{K/M\times\mathfrak{a}^*} |\tilde{f}(kM, \nu)|^2|c(\nu)|^{-2}dk_M d\nu,$$

$$f(x) = [W]^{-1}\int_{K/M\times\mathfrak{a}^*} \tilde{f}(kM, \nu)e^{(\sqrt{-1}\nu-\rho)(H(x^{-1}k))}|c(\nu)|^{-2}dk_M d\nu .$$

The aim of this section is to give the Hausdorff-Young inequality for the Radon-Fourier transform. If $\nu\in\mathfrak{a}_C^*$ and $f\in C_0^\infty(G/K)$ then, by a simple calculation using integral formula for the Iwasawa decomposition, we have

$$(\pi_\nu(f)\Phi_0)(kM) = \int_G f(x)e^{(\sqrt{-1}\nu-\rho)(H(x^{-1}k))}dx = \tilde{f}(kM, \nu).$$

Therefore, using (4.2), (4.3) and the Schwarz inequality, we have for any $q>1$,

$$\begin{aligned}|\tilde{f}(kM, \nu)|^q &= |\pi_\nu(f)\Phi_0|^q = |\textstyle\sum_{i=1}^d T^i(\varphi)\Phi_i|^q \\ &\leq (\textstyle\sum_{i=1}^d |T^i(\varphi)|^2)^{q/2}(\sum_{i=1}^d |\Phi_i|^2)^{q/2} \leq d^{q/2}\|\pi_\nu(f)\|_q^q.\end{aligned}$$

Let $1<p<2$, $1/p+1/q=1$ and $\varepsilon=2/p-1$. If $\eta\in C_{\varepsilon\rho}$ then we have from the above,

$$\begin{aligned}&\left(\int_{K/M\times\mathfrak{a}^*}|\tilde{f}(kM, \nu+\sqrt{-1}\eta)|^q|c(\nu)|^{-2}d\nu\right)^{1/q} \\ &\leq \left(\int_{K/M\times\mathfrak{a}^*} d^{q/2}\|\pi_{\nu+\sqrt{-1}\eta}(f)\|_q^q|c(\nu)|^{-2}d\nu\right)^{1/q} \\ &= d^{1/2}\left(\int_{\mathfrak{a}^*}\|\pi_{\nu+\sqrt{-1}\eta}(f)\|_q^q|c(\nu)|^{-2}d\nu\right)^{1/q} \\ &\leq d^{1/2}C_{pF\eta}\|f\|_p.\end{aligned}$$

Thus we obtain the following theorem.

THEOREM 3. *Let $1<p<2$, $1/p+1/q=1$ and $\varepsilon=2/p-1$. If $f\in L_F^p(G/K)$ and $\eta\in C_{\varepsilon\rho}$ then there exists a positive constant $C_{pF\eta}$ such that*

$$\left(\int_{K/M\times\mathfrak{a}^*}|\tilde{f}(kM, \nu+\sqrt{-1}\eta)|^q|c(\nu)|^{-2}d\nu\right)^{1/q} \leq C_{pF\eta}\|f\|_p.$$

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