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Immersions of real projective spaces into complex projective spaces

Dedicated to Professor Masahiro Sugawara on his 60th birthday

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§1. Introduction

Given two differentiable manifolds M, N and a continuous map f: $M \to N$, we denote by $I[M, N]_f$ the set of regular homotopy classes of immersions of M into N, which are homotopic to f, and by $I[M, N]$ the set of all regular homotopy classes of immersions of M into N.

As for the set $I[M, N]$, some results have so far been obtained when N is not Euclidean space. For example, for the existence of immersions of $P^n(Q)$ into $P^m(C)$ or $P^m(R)$, of $P^n(C)$ into $P^m(C)$, of $P^n(R)$ into $P^m(R)$, and of $L^n(p)$ into $L^m(p)$, see [1], [4] and [15], [8], and [5], respectively, and for the classification of immersions of $P^n(R)$ into $P^m(R)$, see [7]–[9], where $P^k(F)$ is the F-projective space of F-dimension k and $L^k(p)$ is the lens space mod p. But the above results are smaller in number than those when N is Euclidean space.

In this article we shall study the set $I[P^n(R), P^m(C)]$ and $I[P^n(R), P^m(C)]_f$ for any map f: $P^n(R) \to P^m(C)$ $(n \leqq 2m-1)$.

Here we note the fact that $[P^n(R), P^m(C)] = Z_2$ if $n \leqq 2m$ (see (2.5) below).

Let i: $P^n(R) \to P^n(C) \subset P^m(C)$ $(n \leqq m)$ be the natural embedding defined by regarding real numbers as complex numbers and let c: $P^n(R) \to P^m(C)$ be a constant map. Then we shall prove the following theorems:

THEOREM A. *Assume that* $n \geqq 2$. *Then*

(i) *for* $n \leqq m$, *the natural embedding* i *is not null-homotopic*,

(ii) *for* $n > m$, *any immersion of* $P^n(R)$ *into* $P^m(C)$, *if any, is always null-homotopic.*

THEOREM B. *Assume that* $n > 2$. *Then*

(i) *both for* $f = i$ *and* $f = c$,

$$I[P^n(R), P^n(C)]_f = \begin{cases} Z & n \equiv 0 \mod 2, \\ Z_2 & n \equiv 1 \mod 2; \end{cases}$$

(ii) *if* $m < n < 2m-1$, *then*

$$I[P^n(R), P^m(C)] = I[P^n(R), P^m(C)]_c = I[P^n(R), R^{2m}].$$

REMARK. (i) According to Whitney [17], for $n \leqq m$ there exists a null-homotopic immersion of $P^n(R)$ into $P^m(C)$, and for $m > n > 2$ any two immersions of $P^n(R)$ into $P^m(C)$ which are homotopic are regularly homotopic.

(ii) According to Li [6, p. 257], there exists a null-homotopic immersion of $P^n(R)$ into $P^m(C)$ if and only if there exists an immersion of $P^n(R)$ into R^{2m}.

EXAMPLE. For $n \geqq 4$ there exists an immersion of $P^n(R)$ into $P^{n-1}(C)$ if and only if n is not a power of two. If the same condition also holds for $n \geqq 7$, then

$$\begin{aligned} I[P^n(R), P^{n-1}(C)] = I[P^n(R), R^{2n-2}] &= \{0\} && n \equiv 0 \mod 4, \\ &= Z_2 + Z_2 && n \equiv 1 \mod 4, \\ &= Z_2 && n \equiv 2 \mod 4, \\ &= Z_4 + Z_8 && n \equiv 3 \mod 4, \end{aligned}$$

according to the table in [14].

The proof of Theorem A is given in §2. We shall prove Theorem B along the lines of Li and Habegger [11] (cf. [6]–[7]) in §3.

§2. Proof of Theorem A

Let $x \in H^1(P^n(R); Z_2)$ be the first Stiefel-Whitney class $w_1(\xi)$ of the canonical real line bundle ξ over $P^n(R)$, and $y \in H^2(P^m(C); Z)$ the first Chern class $c_1(\zeta)$ of the canonical complex line bundle ζ over $P^m(C)$. Then the following relations hold (see, e.g., [13]):

(2.1) $$H^*(P^n(R); Z_2) = Z_2[x]/(x^{n+1}), \quad H^*(P^m(C); Z) = Z[y]/(y^{m+1}),$$

(2.2) $$\textstyle\sum_{i\geqq 0} w_i(P^n(R)) = (1+x)^{n+1}, \quad \sum_{j\geqq 0} c_j(P^m(C)) = (1-y)^{m+1}.$$

Further we know that

(2.3) $$\rho_2 c_j(P^m(C)) = w_{2j}(P^m(C)) \quad \text{and} \quad \rho_2 y = w_2(\zeta),$$

(see, e.g., [2, Theorem 1.4] or [3, (68)]), and hence

(2.4) $$\textstyle\sum_{i\geqq 0} w_i(P^m(C)) = (1+\rho_2 y)^{m+1}.$$

It is also easily verified, e.g., by Feder [4], that for a CW-complex X of dimension less than $2m+1$, the correspondence which associates f^*y with a homotopy class of a map $f\colon X \to P^m(C)$ leads to a bijection between the homotopy set $[X, P^m(C)]$ and the cohomology group $H^2(X; Z)$. In particular we get

(2.5) $$[P^n(R), P^m(C)] = H^2(P^n(R); Z) = Z_2 = \{\beta_2 x, 0\} \quad \text{if} \quad 2 \leqq n \leqq 2m.$$

To prove that the natural embedding i is not null-homotopic, it is sufficient to show that $i^*\rho_2 y = x^2$. Under the above notations, it is easily seen that the induced bundle $i^*\zeta$ over $P^n(R)$ is isomorphic to the complexification of ξ, i.e.,

$$i^*\zeta = \xi \otimes C.$$

As a real 2-plane bundle,

$$\xi \otimes C = \xi \oplus \xi$$

(see [13]). Hence, and because $w_2(\xi \oplus \xi) = x^2$ and $w_2(\zeta) = \rho_2 y$, we have $i^*\rho_2 y = x^2$, which establishes Theorem A(i).

Moreover this, together with (2.5) and Remark (i) above, implies that

$$\text{(2.6)} \qquad [P^n(R), P^m(C)] = \{[i], [c]\} \quad \text{for} \quad 2 \leqq n \leqq m,$$

where c is a null-homotopic immersion.

To prove (ii), assume that $n \geqq 2$ and that there is an immersion f: $P^n(R) \to P^{n-1}(C)$ such that f is not null-homotopic. Then $f^*\rho_2 y = x^2$ by (2.5). Let $w_k(f)$ denote the k-th Stiefel-Whitney class of the normal bundle of this immersion f. Then by the relation $f^*\rho_2 y = x^2$ and both by (2.2) and (2.4), we have a relation

$$\sum_{k \geqq 0} w_k(f) = (1+x^2)^n (1+x)^{-(n+1)} = \left(\sum_{i \geqq 0} \binom{n}{i} x^{2i}\right)\left(\sum_{j \geqq 0} \binom{n+j}{n} x^j\right)$$

and in particular

$$\text{(2.7)} \qquad w_{n-1}(f) = Ax^{n-1}, \quad \text{where} \quad A = \sum_{i=0}^{[(n-1)/2]} \binom{n}{i} \binom{2n-1-2i}{n}.$$

If $n = 2^r$ $(r \geqq 1)$, then

$$\binom{n}{i} \equiv \begin{cases} 0 \bmod 2 & \text{for} \quad 0 < i \leqq 2^{r-1} - 1 = [(n-1)/2], \\ 1 \bmod 2 & \text{for} \quad i = 0, \end{cases}$$

and

$$\binom{2n-1}{n} = \binom{2^{r+1}-1}{2^r} \equiv 1 \bmod 2.$$

Hence we have

$$\text{(2.8)} \qquad A \equiv 1 \bmod 2 \quad \text{if } n = 2^r \ (r \geqq 1).$$

Consider, next, the case when $n \neq 2^r$. Then the diadic expansion of n is given by

$$n = \sum_{s \geqq i \geqq 1} 2^{r_i}, \quad 0 \leqq r_1 < r_2 < \cdots < r_s, \quad 2 \leqq s.$$

Under this expression of n, it is easily seen that $2^{r_{s-1}} \leqq [(n-1)/2] < 2^{r_s}$ and that for $0 \leqq i \leqq [(n-1)/2]$, $\binom{n}{i} \equiv 1 \bmod 2$ if and only if either $i=0$ or $i=\sum_{j\in J} 2^{r_j}$, $J \subset \{1, 2,\dots, s-1\}$, and hence

$$A \equiv \sum_{J\subset\{1,2,\dots,s-1\}} \binom{2n-2\sum_{j\in J} 2^{r_j}-1}{n} + \binom{2n-1}{n} \mod 2.$$

Here

$$\binom{2n-1}{n} = \binom{\sum_{s\geqq j\geqq 3} 2^{r_j+1} + 2^{r_2+1} + 2^{r_1+1} - 1}{\sum_{s\geqq j\geqq 3} 2^{r_j} + 2^{r_2} + 2^{r_1}} \equiv 0 \mod 2.$$

If $J=\{1, 2,\dots, s-1\}$, then $i=2^{r_{s-1}}+\cdots+2^{r_1}$ and

$$\binom{2n-2i-1}{n} = \binom{2^{r_s+1}-1}{n} \equiv 1 \mod 2.$$

If $J \neq \{1, 2,\dots, s-1\}$, let $k=\min\{j \mid j \notin J\}$ $(1 \leqq k \leqq s-1)$. Then

$$\binom{2n-2i-1}{n} = \binom{\sum_{s\geqq j>k,\, j\notin J} 2^{r_j+1} + 2^{r_k+1} - 1}{\sum_{s\geqq j>k} 2^{r_j} + \sum_{k\geqq j\geqq 1} 2^{r_j}} \equiv 0 \mod 2.$$

Therefore

$$A \equiv 1 \mod 2 \quad \text{if} \quad n \neq 2^r. \tag{2.9}$$

From (2.7)–(2.9), we have

$$w_{n-1}(f) \neq 0,$$

which is a contradiction because the normal bundle of this immersion f is an $(n-2)$-plane bundle. This implies that immersions of $P^n(R)$ into $P^{n-1}(C)$ are always null-homotopic, which is the case for any immersion of $P^n(R)$ into $P^m(C)$ for $m<n$.

§3. Proof of Theorem B

Let ξ and ζ be m- and n-dimensional real vector bundles $(m<n)$ over the spaces X and Y, respectively, and let Mono(ξ, ζ) be the space of all vector bundle maps of ξ to ζ which are monomorphisms on each fiber. Moreover for a map $f\colon X \to Y$, let $\text{Mono}_f(\xi, \zeta)$ and $\text{Mono}_{[f]}(\xi, \zeta)$ be the subspaces of Mono(ξ, ζ) consisting of monomorphisms covering, respectively, f and maps homotopic to f. Then Li and Habegger [11, 3.1] have shown that there is a bijection

$$(3.1)\qquad \pi_0(\mathrm{Mono}_f(\xi, \zeta)) = \pi_0(\mathrm{Mono}_{[f]}(\xi, \zeta)) \quad \text{if} \quad \pi_1(Y^X, f) = 0.$$

For a manifold M, let τ_M denote its tangent bundle. The Smale-Hirsch theorem says that if $\dim M < \dim N$, then there is a bijection between the sets $I[M, N]$ and $\pi_0(\mathrm{Mono}(\tau_M, \tau_N))$. In particular we get

$$(3.2)\qquad I[M, N]_f = \pi_0(\mathrm{Mono}_{[f]}(\tau_M, \tau_N)).$$

If $N = P^m(C)$ and $\dim M < 2m$, then the Eilenberg classification theorem (see [16, p. 243]) says that

$$\pi_1(P^m(C)^M, f) = H^2(M \times (I, \dot{I}); \pi_2(P^m(C))) = H^1(M; Z).$$

Hence for any f: $P^n(R) \to P^m(C)$,

$$\pi_1(P^m(C)^{P^n(R)}, f) = 0 \quad \text{if} \quad n < 2m.$$

Therefore for any f: $P^n(R) \to P^m(C)$, there is a bijection

$$(3.3)\qquad I[P^n(R), P^m(C)]_f = \pi_0(\mathrm{Mono}_f(\tau_{P^n(R)}, \tau_{P^m(C)})) \quad \text{if } n < 2m.$$

If $n = m$, we have

$$\pi_0(\mathrm{Mono}_f(\tau_{P^n(R)}, \tau_{P^n(C)})) = \begin{cases} H^n(P^n(R); Z[f^*w_1(P^n(C)) - w_1(P^n(R))]) & \text{for even } n, \\ H^n(P^n(R); Z_2) & \text{for odd } n, \end{cases}$$

where $Z[a]$ stands for the integers twisted by $a \in H^1(P^n(R); Z_2)$ (see, e.g., [11, Proposition 4.7.1]). Hence

$$\pi_0(\mathrm{Mono}_f(\tau_{P^n(R)}, \tau_{P^n(C)})) = \begin{cases} Z & \text{if } n \text{ is even}, \\ Z_2 & \text{if } n \text{ is odd}. \end{cases}$$

This, together with (3.3), leads to Theorem B(i).

For a constant map c of $P^n(R)$ to either $P^m(C)$ or R^{2m}, we have

$$\begin{aligned} \pi_0(\mathrm{Mono}_c(\tau_{P^n(R)}, \tau_{P^m(C)})) &= \pi_0(\mathrm{Mono}_1(\tau_{P^n(R)}, P^n(R) \times R^{2m})) \\ &= \pi_0(\mathrm{Mono}_c(\tau_{P^n(R)}, \tau_{R^{2m}})), \end{aligned}$$

because the space $\mathrm{Mono}_f(\xi, \zeta)$ is isomorphic to $\mathrm{Mono}_1(\xi, f^*\zeta)$. Hence using (3.1)–(3.3) and the fact that $\pi_1((R^{2m})^{P^n(R)}, c) = 0$, we have

$$I[P^n(R), P^m(C)]_c = I[P^n(R), R^{2m}]_c = I[P^n(R), R^{2m}].$$

This, together with Theorem A(ii), completes the proof of Theorem B(ii).

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