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Lie algebras in which every 1-dimensional subideal is an ideal

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Introduction

A Lie algebra L is called a t-algebra if every subideal of L is an ideal of L, a T-algebra if any subalgebra of L is a t-algebra and a c-algebra if every nilpotent subideal of L is an ideal of L. We easily see that L is a c-algebra if and only if every 1-dimensional subideal of L is an ideal of L. Recently Varea [14] introduced the concept of C-algebra in Lie algebra: L is a C-algebra if every subalgebra of a nilpotent subalgebra H of L is an ideal in the idealizer of H in L. He investigated the property of finite-dimensional C-algebras, and in [14] he proved the following results:

(a) Let L be an n-dimensional Lie algebra over a field $\mathfrak{k}$ of at least $n-1$ elements. Then the following are equivalent: i) L is a C-algebra. ii) L is a T-algebra. iii) Every subalgebra of L is a c-algebra.

(b) Let L be a finite-dimensional Lie algebra over a field of characteristic zero. Then the following are equivalent: i) L is a c-algebra. ii) L is a t-algebra. iii) $L=R\oplus S$ where R is an ideal of L which is either abelian or almost-abelian and S is a semisimple ideal of L.

The purpose of this paper is to give several generalizations of (a), (b) and other results in [14] without the finite-dimensionality of L and the restriction on the cardinality of $\mathfrak{k}$.

The main results of this paper are as follows.

(1) Let L be a serially finite Lie algebra over a field of characteristic zero. If the locally soluble radical of L belongs to the class $\acute{E}(\mathrm{si})\mathfrak{A}$ of Lie algebras, then the three statements in (b) are equivalent (Theorem 2.3).

(2) Let L be an arbitrary Lie algebra. Then the following are equivalent: i) L is a C-algebra. ii) Every subalgebra of L is a c-algebra. iii) Every 1-dimensional ascendant subalgebra of a subalgebra H of L is an ideal of H (Theorem 3.5).

(3) Let L be a locally finite Lie algebra over any field. Then the following are equivalent: i) L is a C-algebra. ii) L is a T-algebra. iii) Every serial subalgebra of a subalgebra H of L is an ideal of H. iv) Every 1-dimensional serial subalgebra of a subalgebra H of L is an ideal of H (Theorem 3.9).

(4) Over any field there exist a c-algebra which is neither a C-algebra nor a

t-algebra, and a t-algebra which is not a T-algebra (Examples 4.1 and 4.2). Over any field of characteristic zero there exists a C-algebra which is not a T-algebra (Example 4.4).

In this paper we use the terminology $\mathfrak{C}$-algebras, $\mathfrak{C}^*$-algebras, $\mathfrak{T}$-algebras and $\mathfrak{T}^S$-algebras instead of c-algebras, C-algebras, t-algebras and T-algebras in [14] respectively.

1.

Throughout the paper Lie algebras are not necessarily finite-dimensional over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified. We mostly follow [2] for the use of notations and terminology.

Let L be a Lie algebra over $\mathfrak{k}$ and let H be a subalgebra of L. For an ordinal σ, H is a σ-step ascendant (resp. weakly ascendant) subalgebra of L, denoted by $H \lhd^{\sigma} L$ (resp. $H \leq^{\sigma} L$), if there exists an ascending series (resp. chain) $(H_{\alpha})_{\alpha \leq \sigma}$ of subalgebras (resp. subspaces) of L such that

(1) $H_0 = H$ and $H_{\sigma} = L$,

(2) $H_{\alpha} \lhd H_{\alpha+1}$ (resp. $[H_{\alpha+1}, H] \subseteq H_{\alpha}$) for any ordinal $\alpha < \sigma$,

(3) $H_{\lambda} = \bigcup_{\alpha < \lambda} H_{\alpha}$ for any limit ordinal $\lambda \leq \sigma$.

H is an ascendant (resp. a weakly ascendant) subalgebra of L, denoted by H asc L (resp. H wasc L), if $H \lhd^{\sigma} L$ (resp. $H \leq^{\sigma} L$) for some ordinal σ. When σ is finite, H is a subideal (resp. weak subideal) of L and denoted by H si L (resp. H wsi L). For a totally ordered set Σ, a series from H to L of type Σ is a collection $\{\Lambda_{\sigma}, V_{\sigma}: \sigma \in \Sigma\}$ of subalgebras of L such that

(1) $H \subseteq V_{\sigma} \subseteq \Lambda_{\sigma}$ for all $\sigma \in \Sigma$,

(2) $L \setminus H = \bigcup_{\sigma \in \Sigma} (\Lambda_{\sigma} \setminus V_{\sigma})$,

(3) $\Lambda_{\tau} \subseteq V_{\sigma}$ if $\tau < \sigma$,

(4) $V_{\sigma} \lhd \Lambda_{\sigma}$ for all $\sigma \in \Sigma$.

H is a serial subalgebra of L, denoted by H ser L, if there exists a series from H to L of type Σ for some Σ.

Let $\mathfrak{X}$ be a class of Lie algebras and let Δ be any of the relations $\leq$, $\lhd$, si, asc, ser. A Lie algebra L is said to lie in $\mathrm{L}(\Delta)\mathfrak{X}$ if for any finite subset X of L there exists an $\mathfrak{X}$-subalgebra K of L such that $X \subseteq K \Delta L$. In particular we write $\mathrm{L}\mathfrak{X}$ for $\mathrm{L}(\leq)\mathfrak{X}$. When $L \in \mathrm{L}\mathfrak{X}$ (resp. $\mathrm{L}(\mathrm{ser})\mathfrak{X}$), L is called a locally (resp. serially) $\mathfrak{X}$-algebra. $\mathfrak{F}$, $\mathfrak{A}$ and $\mathfrak{N}$ are the classes of Lie algebras which are finite-dimensional, abelian and nilpotent respectively. For an ordinal σ, $\acute{\mathrm{E}}_{\sigma}(\Delta)\mathfrak{X}$ is the class of Lie algebras L having an ascending series $(L_{\alpha})_{\alpha \leq \sigma}$ of Δ-subalgebras such that

(1) $L_0 = 0$ and $L_{\sigma} = L$,

(2) $L_{\alpha} \lhd L_{\alpha+1}$ and $L_{\alpha+1}/L_{\alpha} \in \mathfrak{X}$ for any ordinal $\alpha < \sigma$,

(3) $L_{\lambda} = \bigcup_{\alpha < \lambda} L_{\alpha}$ for any limit ordinal $\lambda \leq \sigma$.

We define $\acute{\mathrm{E}}(\Delta)\mathfrak{X}=\bigcup_{\sigma>0}\acute{\mathrm{E}}_{\sigma}(\Delta)\mathfrak{X}$, $\mathrm{E}(\Delta)\mathfrak{X}=\bigcup_{n<\omega}\acute{\mathrm{E}}_{n}(\Delta)\mathfrak{X}$. In particular we write $\acute{\mathrm{E}}\mathfrak{X}$ and $\mathrm{E}\mathfrak{X}$ for $\acute{\mathrm{E}}(\leq)\mathfrak{X}$ and $\mathrm{E}(\leq)\mathfrak{X}$ respectively. Thus $\mathrm{E}\mathfrak{A}$ is the class of soluble Lie algebras. When $L\in\acute{\mathrm{E}}(\lhd)\mathfrak{X}$, L is called a hyper $\mathfrak{X}$-algebra. $\mathrm{Q}\mathfrak{X}$ is the class of Lie algebras consisting of all homomorphic images of $\mathfrak{X}$-algebras. We say that $\mathfrak{X}$ is A-closed if $\mathfrak{X}=\mathrm{A}\mathfrak{X}$, where A is either L, E or Q. Let Δ be any of the relations si, asc, ser. $\mathfrak{T}(\Delta)$ is the class of Lie algebras L in which every Δ-subalgebra of L is an ideal of L. In particular we write $\mathfrak{T}$ for $\mathfrak{T}(\mathrm{si})$. $\mathfrak{T}(\mathrm{asc})$ (resp. $\mathfrak{T}(\mathrm{ser})$) is denoted by $\mathfrak{M}'$ (resp. $\mathfrak{X}_3$) in [13] (resp. [6]).

Let H be a subalgebra of L. We denote by $C_L(H)$ (resp. $I_L(H)$) the centralizer (resp. idealizer) of H in L. For $x\in L$ we put $H^x=\sum_{n\geq 0}[H, {}_nx]$, where $[H, {}_nx]=[H, \underbrace{x, x,\ldots, x}_{n}]$. The Fitting radical $\nu(L)$ of L is the sum of all nilpotent ideals of L. The Hirsch-Plotkin radical $\rho(L)$ of L is the unique maximal locally nilpotent ideal of L. The Baer radical $\beta(L)$ of L is the subalgebra generated by all nilpotent subideals of L and the Gruenberg radical $\gamma(L)$ of L is the subalgebra generated by all nilpotent ascendant subalgebras of L. For a locally finite Lie algebra L the locally soluble radical $\sigma(L)$ of L is the unique maximal locally soluble ideal of L.

Now we begin with an elementary result whose proof is easy.

LEMMA 1.1. *Let L be a Lie algebra and let N be a subspace of L. Then the following are equivalent*:

(1) *Every* 1-*dimensional subspace of N is an ideal of L.*

(2) *For every* $x\in L$, $\operatorname{ad}x$ *is a scalar transformation on N.*

Moreover, if either (1) *or* (2) *holds, then N is an abelian ideal of L and* $\dim L/C_L(N)\leq 1$.

$\mathrm{J}(\mathrm{wasc})\mathrm{L}\mathfrak{F}$ is the class of Lie algebras generated by a set of weakly ascendant locally finite subalgebras. We know that in any Lie algebra (resp. $\mathrm{J}(\mathrm{wasc})\mathrm{L}\mathfrak{F}$-algebra) L over a field of characteristic 0 $\beta(L)$ (resp. $\gamma(L)$) is an ideal of L owing to [10, Theorem 10.7] (resp. [12, Theorem 3.3.4]). For our arguments we need the following result, which may be shown by using [2, Lemma 9.1.2(c)] and [4, Lemma 4.1].

LEMMA 1.2. *Let L be a Lie algebra over a field* $\mathfrak{k}$ *and let N be a subalgebra of L. Then we have* $C_L(N)\leq N$ *if L, N and* $\mathfrak{k}$ *satisfy one of the following statements*:

(1) $L\in\acute{\mathrm{E}}(\lhd)\mathfrak{A}$, $N=\nu(L)$ *and* $\mathfrak{k}$ *has any characteristic.*

(2) $L\in\acute{\mathrm{E}}(\lhd)\mathrm{L}\mathfrak{N}$, $N=\rho(L)$ *and* $\mathfrak{k}$ *has any characteristic.*

(3) $L\in\acute{\mathrm{E}}(\mathrm{si})\mathfrak{A}$, $N=\beta(L)$ *and* $\mathfrak{k}$ *has characteristic* 0.

(4) $L\in\acute{\mathrm{E}}\mathfrak{A}\cap\mathrm{J}(\mathrm{wasc})\mathrm{L}\mathfrak{F}$, $N=\gamma(L)$ *and* $\mathfrak{k}$ *has characteristic* 0.

2.

In this section we shall consider three new classes $\mathfrak{C}$, $\mathfrak{C}(\mathrm{asc})$ and $\mathfrak{C}(\mathrm{ser})$ of Lie algebras containing the classes $\mathfrak{T}$, $\mathfrak{T}(\mathrm{asc})$ and $\mathfrak{T}(\mathrm{ser})$ respectively. Let Δ be one of the relations si, asc, ser. $\mathfrak{C}(\Delta)$ is the class of Lie algebras L in which every 1-dimensional Δ-subalgebra of L is an ideal of L. In particular we write $\mathfrak{C}$ for $\mathfrak{C}(\mathrm{si})$. A $\mathfrak{C}$-algebra is equal to a c-algebra in [14], that is, a Lie algebra in which every nilpotent subideal is an ideal. A Lie algebra L is said to be almost-abelian if L is the split extension of an abelian algebra by the 1-dimensional algebra of scalar multiplications. The following result generalizes [14, Corollary 3.3], [6, Theorem 6.4] and [3, Proposition 2.8].

PROPOSITION 2.1. *Let L be a hyperabelian Lie algebra over a field of any characteristic (resp. an* $\acute{E}(\mathrm{si})\mathfrak{A}$*-algebra over a field of characteristic zero). Then the following are equivalent*:

(1) *Every serial subalgebra of L is an ideal of L.*
(2) *Every 1-dimensional serial subalgebra of L is an ideal of L.*
(3) *Every ascendant subalgebra of L is an ideal of L.*
(4) *Every 1-dimensional ascendant subalgebra of L is an ideal of L.*
(5) *Every subideal of L is an ideal of L.*
(6) *Every 1-dimensional subideal of L is an ideal of L.*
(7) *L is either abelian or almost-abelian.*

PROOF. Evidently we have the following diagram of implications:

$$\begin{array}{ccccc} (1) & \Rightarrow & (3) & \Rightarrow & (5) \\ \Downarrow & & \Downarrow & & \Downarrow \\ (2) & \Rightarrow & (4) & \Rightarrow & (6) \end{array}$$

$(6) \Rightarrow (7)$: Let $N = \nu(L)$ (resp. $N = \beta(L)$). Then for any $y \in N$ we have $\langle y \rangle \operatorname{si} L$ by [2, Lemma 1.3.7 (resp. Theorem 6.2.1)]. Therefore $\langle y \rangle \lhd L$. It follows from Lemmas 1.1 and 1.2 that N is abelian, $\dim L/N \leq 1$ and $\operatorname{ad} x$ is a scalar transformation on N for any $x \in L$. Thus L is either abelian or almost-abelian.

$(7) \Rightarrow (1)$ is clear by [6, Lemma 6.3].

Furthermore we have the following proposition whose proof is similar to that of Proposition 2.1.

PROPOSITION 2.2. *Let L be a Lie algebra over a field of characteristic zero. If* $L \in \acute{E}\mathfrak{A} \cap \mathrm{J}(\mathrm{wasc})\mathrm{L}\mathfrak{F}$, *then the following are equivalent*:

(1) *Every serial subalgebra of L is an ideal of L.*
(2) *Every 1-dimensional serial subalgebra of L is an ideal of L.*

(3) *Every ascendant subalgebra of L is an ideal of L.*
(4) *Every* 1*-dimensional ascendant subalgebra of L is an ideal of L.*
(5) *L is either abelian or almost-abelian.*

Now we shall give the main theorem in this section, which generalizes [14, Corollary 3.5].

THEOREM 2.3. *Let L be a serially finite Lie algebra over a field of characteristic zero. If $\sigma(L)\in \acute{\mathrm{E}}(\mathrm{si})\mathfrak{A}$, then the following are equivalent:*
(1) *Every* 1*-dimensional serial subalgebra of L is an ideal of L.*
(2) *Every* 1*-dimensional ascendant subalgebra of L is an ideal of L.*
(3) *Every* 1*-dimensional subideal of L is an ideal of L.*
(4) *Every subideal of L is an ideal of L.*
(5) *$L=R\oplus S$, where R is an ideal of L which is either abelian or almost-abelian and S is a semisimple ideal of L.*

PROOF. Implications $(1)\Rightarrow(2)\Rightarrow(3)$ and $(4)\Rightarrow(3)$ are trivial.

$(3)\Rightarrow(5)$: Let $R=\sigma(L)$, $N=\beta(R)$. For any $y\in N$ we have $\langle y\rangle\lhd L$ since $\langle y\rangle \operatorname{si} L$. By Lemma 1.1 N is an abelian ideal of L and $\dim L/C_L(N)\leq 1$. On the other hand [11, Theorem 2] and [2, Theorem 13.5.7] show that there exists a Levi factor S of L. Furthermore $S=S^2\leq L^2\leq C_L(N)$ by [2, Theorem 13.4.2] and $C_R(N)=N$ by Lemma 1.2. Hence we have

$$C_L(N) = C_L(N) \cap (R\dotplus S) = C_R(N) \dotplus S = N \oplus S.$$

Suppose that $\langle x\rangle \operatorname{si} R$. Then $\langle x\rangle\lhd L$ and so $\langle x\rangle\lhd R$. By making use of Proposition 2.1, we see that R is either abelian or almost-abelian. Finally since $S=S^2=C_L(N)^2 \operatorname{ch} C_L(N)\lhd L$ it follows that $S\lhd L$.

$(5)\Rightarrow(1)$: Suppose that $\langle x\rangle \operatorname{ser} L$. Then $\langle x\rangle\leq\rho(L)\leq R$ by [2, Theorem 13.3.7]. Since $\langle x\rangle \operatorname{ser} R$, we have $\langle x\rangle\lhd R$ using [6, Lemma 6.3]. Therefore we have $\langle x\rangle\lhd L$.

$(5)\Rightarrow(4)$: Let $H\lhd L$ and put $S_1=\{s\in S\colon \text{there is } r\in R \text{ such that } r+s\in H\}$. It is easy to show that S_1 is an ideal of S containing $S\cap H$. Also S_1 is semisimple by [2, Theorem 13.4.2 and Lemma 13.4.3]. Since $S_1\leq H+R$, $S_1=S_1^2\leq(H+R)^2$ $\leq H+R^2$ and so $S_1\leq(H+R^2)^2\leq H$. Hence we have $S_1=S\cap H$. Furthermore since $H\leq R+S_1$, we obtain $H=(H\cap R)\oplus(H\cap S)$, where $H\cap R$ is either abelian or almost-abelian. Now assume that $H \operatorname{si} L$. Then by repeating this argument we have $H=(H\cap R)\oplus(H\cap S)$. It follows that $H\cap R$ and $H\cap S$ are ideals of R and S respectively. Hence we see that $H\lhd L$.

REMARK. Over any field there exists an $\acute{\mathrm{E}}\mathfrak{A}\cap \mathrm{L}\mathfrak{N}$-algebra which belongs to $\mathfrak{C}$ but not to $\mathfrak{C}(\mathrm{asc})\cup\mathfrak{T}\cup\acute{\mathrm{E}}(\mathrm{si})\mathfrak{A}$ (Example 4.1). Therefore in Proposition 2.1 we can not extend the classes $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$ and $\acute{\mathrm{E}}(\mathrm{si})\mathfrak{A}$ (over a field of characteristic 0) to the class $\acute{E}\mathfrak{A}$. Furthermore by [11, Theorem 4] we can not remove the hypothesis "$\sigma(L) \in \acute{E}(\mathrm{si})\mathfrak{A}$" in Theorem 2.3.

3.

We denote by $\mathfrak{C}^*$ the class of C-algebras introduced in [14], that is, Lie algebras L in which each subalgebra of a nilpotent subalgebra H of L is an ideal in the idealizer of H in L. In this section we shall investigate not necessarily finite-dimensional $\mathfrak{C}^*$-algebras and related Lie algebras. We begin with the following

LEMMA 3.1. *Let L be a $\mathfrak{C}^*$-algebra.*

(a) *If N is a locally nilpotent subalgebra of L, then N is abelian.*

(b) *If N is a locally nilpotent ideal of L, then* ad x *is a scalar transformation on N for any $x \in L$ and* $\dim L/C_L(N) \leq 1$.

PROOF. Let N be a locally nilpotent subalgebra of L and let x, y be any elements of N. Then there exists a nilpotent subalgebra K of N which contains x and y. Since L is a $\mathfrak{C}^*$-algebra, any 1-dimensional subalgebra of K is an ideal of K. It follows from Lemma 1.1 that K is abelian. Therefore $[x, y]=0$ and N is abelian. Moreover if N is an ideal of L, then any 1-dimensional subalgebra of N is an ideal of L. By using Lemma 1.1 we have the assertion of (b).

We shall give several characterizations of $\mathfrak{C}^*$-algebras in the following proposition, which generalizes [14, Theorem 2.4], and in Theorem 3.5.

PROPOSITION 3.2. *Let L be a Lie algebra over an arbitrary field. Then the following are equivalent*:

(1) *L is a $\mathfrak{C}^*$-algebra.*

(2) *Every soluble subalgebra of L is either abelian or almost-abelian.*

(3) *Every $\acute{E}(\lhd)\mathrm{L}\mathfrak{N}$-subalgebra of L is either abelian or almost-abelian.*

PROOF. (1) $\Rightarrow$ (3): Let H be an $\acute{E}(\lhd)\mathrm{L}\mathfrak{N}$-subalgebra of L and set $N=\rho(H)$. Since H is a $\mathfrak{C}^*$-algebra, it follows from Lemma 3.1 that N is abelian and $\dim H/C_H(N) \leq 1$. Furthermore by Lemma 1.2 we have $C_H(N) \leq N$ and so $\dim H/N \leq 1$. If $\dim H/N=0$, then H is abelian. If $\dim H/N=1$, then we can pick up an element x of $H \setminus N$ and write $H=N \dotplus \langle x \rangle$, where ad x is a non-zero scalar transformation on N. Therefore H is almost-abelian.

(3) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1): Let N be a nilpotent subalgebra of L and let K be a subalgebra of N. Then $N+\langle x \rangle$ is either abelian or almost-abelian for any element x of $I_L(N)$. If $N+\langle x \rangle$ is abelian, then $[K, x] \subseteq K$ is clear. We may assume that $N+\langle x \rangle$ is almost-abelian. Then $x \notin N$ and $(N+\langle x \rangle)^2=N$. Since ad x is a scalar transformation on N, we have $[K, x] \subseteq K$. This shows that K is an ideal of $I_L(N)$.

Varea showed the following results: Let L be an n-dimensional Lie algebra over a field $\mathfrak{k}$ of at least $n-1$ elements.

(V1) L is a $\mathfrak{C}^*$-algebra if and only if every subalgebra of L is a $\mathfrak{C}$-algebra ([14, Proposition 2.6]).

(V2) L is a $\mathfrak{C}^*$-algebra if and only if every subalgebra of L is a $\mathfrak{T}$-algebra ([14, Theorem 2.8]).

Now in these results we can remove the restriction on the cardinality of $\mathfrak{k}$ and the finite-dimensionality of L.

Before generalizing (V1) and giving other characterizations of $\mathfrak{C}^*$-algebras we shall consider a stronger form:

Lemma 3.3. *Let L be a Lie algebra over an arbitrary field. Then the following are equivalent*:

(1) *For any subalgebra H of L, every ascendant subalgebra of H is an ideal of H.*

(2) *For any subalgebra H of L, every subideal of H is an ideal of H.*

(3) *For $K \leq L$ and $x \in L$, if $[K, {}_n x, K] \subseteq K$ for any $n \geq 1$ then K is an ideal of $\langle K, x \rangle$.*

Proof. (1)$\Rightarrow$(2) is clear.

(2)$\Rightarrow$(3): Let $K \leq L$ and $x \in L$ such that $[K, {}_n x, K] \subseteq K$ for all $n \geq 1$. Since $K \lhd K^x \lhd \langle K, x \rangle$, we obtain $K \lhd \langle K, x \rangle$.

(3)$\Rightarrow$(1): Let $K \operatorname{asc} H$ and let $(A_\alpha)_{\alpha \leq \sigma}$ be an ascending series from K to H. We show by transfinite induction on α that K is an ideal of A_α. Let $\alpha > 1$ and assume that $K \lhd A_\beta$ for all $\beta < \alpha$. If α is a limit ordinal, then $K \lhd \bigcup_{\beta < \alpha} A_\beta = A_\alpha$. Otherwise by induction hypothesis $K \lhd A_{\alpha-1} \lhd A_\alpha$. Let $x \in A_\alpha$. Since $[K, {}_n x, K] \subseteq K$ for any $n \geq 1$, it follows that $K \lhd \langle K, x \rangle$. Hence we have $K \lhd A_\alpha$.

Let $\mathfrak{X}$ be a class of Lie algebras. Then we denote by $\mathfrak{X}^S$ the largest s-closed subclass of $\mathfrak{X}$. That is, L belongs to $\mathfrak{X}^S$ if and only if every subalgebra of L belongs to $\mathfrak{X}$.

Now as a direct consequence of Lemma 3.3 we have

Corollary 3.4. $\mathfrak{T}(\mathrm{asc})^S = \mathfrak{T}^S$.

As mentioned above we shall provide a generalization for (V1) and characterizations of $\mathfrak{C}^*$-algebras, which is the first main theorem in this section.

Theorem 3.5. *Let L be a Lie algebra over an arbitrary field. Then the following are equivalent*:

(1) *L is a $\mathfrak{C}^*$-algebra.*

(2) *For any subalgebra H of L, every 1-dimensional ascendant subalgebra of H is an ideal of H.*

(3) *For any subalgebra H of L, every 1-dimensional subideal of H is an*

ideal of H.

(4) *For two elements* a *and* x *of* L, *if* $[a, {}_nx, a]\in\langle a\rangle$ *for any* $n\geq 1$ *then* $\langle a\rangle$ *is an ideal of* $\langle a, x\rangle$.

PROOF. The equivalence of (2), (3) and (4) can be proved as in Lemma 3.3.

(1)$\Rightarrow$(3): It is sufficient to show that if $\langle a\rangle \lhd^2 H$ then $\langle a\rangle\lhd H$. Suppose that $\langle a\rangle \ntrianglelefteq B \ntrianglelefteq H$. Let x be any element of $H\setminus B$. We put $M_n=\sum_{i=0}^{n}\langle[a, {}_ix]\rangle$ for any $n\geq 0$ and $M=\bigcup_{n=0}^{\infty}M_n$. Then we have $[a, {}_nx, B]\subseteq M_n$ by induction on n. Thus M_n is an ideal of B for all $n\geq 0$. Since $M_n=M_{n-1}+\langle[a, {}_nx]\rangle$, we obtain $M_n^{(1)}\leq M_{n-1}$. Therefore $M_n^{(n+1)}=0$. By virtue of Proposition 3.2 we conclude that $M_n^{(2)}=0$ for all $n\geq 0$ and so $M^{(2)}=0$. Now we set $K=M+\langle x\rangle$. Then $K^{(3)}=0$. Again by Proposition 3.2 K is either abelian or almost-abelian. If K is abelian, then $[a, x]=0$. If K is almost-abelian, then $[a, x]\in\langle a\rangle$ since $K^2=M$. Hence we have $[a, H]\subseteq\langle a\rangle$.

(3)$\Rightarrow$(1): Let H be a nilpotent subalgebra of L and let a be any element of H. Since $\langle a\rangle$ is a subideal of $I_L(H)$, $\langle a\rangle$ is an ideal of $I_L(H)$. Thus L is a $\mathfrak{C}^*$-algebra.

COROLLARY 3.6. $\mathfrak{C}^*=\mathfrak{C}(\mathrm{asc})^S=\mathfrak{C}^S$.

Next as for (V2) we need the following

LEMMA 3.7. *Let* L *be a finite-dimensional* $\mathfrak{C}^*$*-algebra over any field. If* K *is an ideal of* L, *then* $K^2+\langle h\rangle$ *is an ideal of* L *for any* $h\in K$.

PROOF. We assume that $K^2<K\ntrianglelefteq L$. Let x be an element of $L\setminus K$. We may moreover assume that $L=K+\langle x\rangle$. Let h be an element of $K\setminus K^2$. It suffices to show that $[L, h]\subseteq\langle h\rangle+K^2$. If $I_K(\langle h\rangle)=\langle h\rangle$, then $\langle h\rangle$ is a Cartan subalgebra of K. Let $K=K_0\dotplus K_1$ be the Fitting decomposition of K relative to $\mathrm{ad}\,h$. Using [5, Proposition 3.1] we have $K_0=\langle h\rangle$ and so $K=K^2+\langle h\rangle$. In this case there is nothing to prove. We next consider the case $I_K(\langle h\rangle)\neq\langle h\rangle$. Let k be an element of $I_K(\langle h\rangle)\setminus\langle h\rangle$ and let H be a maximal soluble subalgebra of L containing h and k. By Proposition 3.2 H is either abelian or almost-abelian. If H is abelian, then we can consider the Fitting decomposition of L relative to $\mathrm{ad}\,H$, say $L=L_0\dotplus L_1$. It turns out that $L=H+K$, since H is a Cartan subalgebra of L and $L_1\subseteq L^2\subseteq K$. Hence $[L, h]\subseteq K^2$. Next suppose that H is almost-abelian. If $h\notin H^2$, then $H=H^2\dotplus\langle h\rangle$ and $\mathrm{ad}\,h$ is a non-zero scalar transformation on H^2. Now we can write $k=y+\alpha h$, where $0\neq y\in H^2$ and $\alpha\in\mathfrak{k}$. Accordingly we have $[k, h]=[y, h]\neq 0$. Since $[k, h]\in K^2\cap\langle h\rangle=0$, we have a contradiction. Therefore we obtain $h\in H^2$ and so $H\subsetneq K$. This leads to $L=H+K$. It follows that $[L, h]\subseteq[H, h]+[K, h]\subseteq\langle h\rangle+K^2$. This completes the proof.

From Lemma 3.7 we can deduce the following result, which generalizes [14, Proposition 2.7].

PROPOSITION 3.8. *Let L be a locally finite $\mathfrak{C}^*$-algebra over an arbitrary field. If N is an ideal of L, then L/N is a $\mathfrak{C}^*$-algebra.*

PROOF. Let K/N be a nilpotent subalgebra of L/N and let H/N be a subalgebra of K/N. Suppose that $x \in I_L(K)$ and $h \in H$. We set $K_1 = K \cap \langle h, x \rangle$ and recursively define $K_{n+1} = K_n^2 + \langle h \rangle$ for any $n \geq 1$. We first show that

$$K_n^2 \geq K_1^n \text{ and } K_n \lhd K_n + \langle x \rangle \text{ for any positive integer } n. \qquad (*)$$

When $n=1$ it is clear that $(*)$ is true. Suppose that $(*)$ is true for $n \geq 1$. Then we have

$$K_{n+1}^2 = (K_n^2 + \langle h \rangle)^2 \subseteq (K_1^n + \langle h \rangle)^2 \subseteq K_1^{n+1}.$$

Since K_n is finite-dimensional, we deduce from Lemma 3.7 that $K_{n+1} = K_n^2 + \langle h \rangle \lhd K_n + \langle x \rangle$. Hence $K_{n+1} \lhd K_{n+1} + \langle x \rangle$. It follows that $(*)$ is true for any $n \geq 1$. Since K/N is nilpotent, we have $K^m \leq N$ for some m. Then

$$[h, x] \in K_m^2 + \langle h \rangle \subseteq K_1^m + \langle h \rangle \subseteq N + \langle h \rangle \subseteq H.$$

This shows that $H \lhd I_L(K)$ and so $H/N \lhd I_{L/N}(K/N)$. Thus L/N is a $\mathfrak{C}^*$-algebra.

Now we can show the second main theorem in this section, which is a generalization for (V2).

THEOREM 3.9. *Let L be a locally finite Lie algebra over an arbitrary field. Then the following are equivalent:*

(1) *For any subalgebra H of L, every serial subalgebra of H is an ideal of H.*

(2) *For any subalgebra H of L, every 1-dimensional serial subalgebra of H is an ideal of H.*

(3) *For any subalgebra H of L, every ascendant subalgebra of H is an ideal of H.*

(4) *For any subalgebra H of L, every 1-dimensional ascendant subalgebra of H is an ideal of H.*

(5) *For any subalgebra H of L, every subideal of H is an ideal of H.*

(6) *For any subalgebra H of L, every 1-dimensional subideal of H is an ideal of H.*

PROOF. Implications $(1) \Rightarrow (3) \Rightarrow (5) \Rightarrow (6)$ and $(1) \Rightarrow (2) \Rightarrow (4) \Rightarrow (6)$ are trivial.

$(6) \Rightarrow (5)$: Suppose that $K \text{ si } H \leq L$. Then $K^\omega \lhd H$ by [2, Lemma 1.3.2] and $K/K^\omega \text{ si } H/K^\omega$. By using Theorem 3.5 and Proposition 3.8 we see that H/K^ω is a $\mathfrak{C}^*$-algebra. Since K/K^ω is locally nilpotent, K/K^ω is abelian by Lemma 3.1. Hence $(\langle x \rangle + K^\omega)/K^\omega \lhd K/K^\omega \text{ si } H/K^\omega$ and so $(\langle x \rangle + K^\omega)/K^\omega \lhd H/K^\omega$ for all $x \in K$ by Theorem 3.5. Thus we obtain $K \lhd H$.

$(5) \Rightarrow (1)$: Suppose that $K \operatorname{ser} H \leq L$. Let k and x be elements of K and H respectively. Then by [2, Proposition 13.2.4] $K \cap \langle k, x\rangle$ si $\langle k, x\rangle$ and so $K \cap \langle k, x\rangle \lhd \langle k, x\rangle$. Hence we have $[k, x] \in K$ for any $k \in K$ and $x \in H$.

REMARK. In general the class $\mathfrak{C}^*$ is not equal to the class $\mathfrak{T}^S$ (see Example 4.4).

Varea determined the structure of finite-dimensional $\mathfrak{C}^*$-algebras over an algebraically closed field ([14, Theorem 2.9]). By making use of this result we can determine the structure of locally finite $\mathfrak{C}^*$-algebras over an algebraically closed field.

PROPOSITION 3.10. *Let L be a locally finite $\mathfrak{C}^*$-algebra over an algebraically closed field. Then L is either abelian, almost-abelian or 3-dimensional simple.*

PROOF. Let F be any finite-dimensional subalgebra of L. Then by [14, Theorem 2.9] F is either abelian, almost-abelian or 3-dimensional simple. If every finite-dimensional subalgebra of L is either abelian or almost-abelian, then $L^{(2)}=0$. It follows from Proposition 3.2 that L is either abelian or almost-abelian. We now assume that there exists a 3-dimensional simple subalgebra F of L. Then for all $x \in L$, $\langle F, x\rangle$ is finite-dimensional. Hence $\langle F, x\rangle$ must be 3-dimensional simple. From this we can deduce that L is 3-dimensional simple.

Finally in the following result we see that every $\text{L\'E}\mathfrak{L}\mathfrak{N}$-subalgebras of a $\mathfrak{C}^*$-algebra is either abelian or almost-abelian, which generalizes Proposition 3.2.

PROPOSITION 3.11. *Let L be a Lie algebra over any field. If $L \in \text{L\'E}\mathfrak{L}\mathfrak{N}$, then the following are equivalent:*

(1) *For any subalgebra H of L, every serial subalgebra of H is an ideal of H.*

(2) *For any subalgebra H of L, every 1-dimensional serial subalgebra of H is an ideal of H.*

(3) *For any subalgebra H of L, every ascendant subalgebra of H is an ideal of H.*

(4) *For any subalgebra H of L, every 1-dimensional ascendant subalgebra of H is an ideal of H.*

(5) *For any subalgebra H of L, every subideal of H is an ideal of H.*

(6) *For any subalgebra H of L, every 1-dimensional subideal of H is an ideal of H.*

(7) *L is either abelian or almost-abelian.*

PROOF. Implications $(1) \Rightarrow (3) \Rightarrow (5) \Rightarrow (6)$ and $(1) \Rightarrow (2) \Rightarrow (4) \Rightarrow (6)$ are trivial.

$(6) \Rightarrow (7)$: Let a be any element of $L^{(2)}$. Then there is an $\text{\'E}\mathfrak{L}\mathfrak{N}$-subalgebra H

of L such that $a \in H^{(2)}$. Let $(H_\alpha)_{\alpha \leq \sigma}$ be an ascending $\mathrm{L}\mathfrak{N}$-series of H. We shall show by transfinite induction on α that $H_\alpha \lhd H$ for any ordinal $\alpha \leq \sigma$. Assume that $H_\beta \lhd H$ for all $\beta < \alpha$. If α is a limit ordinal, then $H_\alpha = \bigcup_{\beta<\alpha} H_\beta \lhd H$. Otherwise, since $H_{\alpha-1}$ belongs to $\acute{\mathrm{E}}(\lhd)\mathrm{L}\mathfrak{N}$ we have $H_{\alpha-1}^{(2)} = 0$ by using Proposition 3.2. If $M/H_{\alpha-1}$ is a soluble subalgebra of $H/H_{\alpha-1}$, then M is soluble. Hence by Proposition 3.2 we see that $H/H_{\alpha-1}$ is a $\mathfrak{C}^*$-algebra. Therefore $H_\alpha/H_{\alpha-1}$ is abelian by Lemma 3.1. It follows that $(\langle x \rangle + H_{\alpha-1})/H_{\alpha-1} \operatorname{asc} H/H_{\alpha-1}$ for all $x \in H_\alpha$. By Theorem 3.5 we have $\langle x \rangle + H_{\alpha-1} \lhd H$, whence $H_\alpha \lhd H$. Now Proposition 3.2 leads to $H^{(2)} = 0$. Therefore $L^{(2)} = 0$. Finally by using Proposition 3.2 we can conclude that L is either abelian or almost-abelian.

(7)$\Rightarrow$(1) is clear by [6, Lemma 6.3].

REMARKS. (1) The classes $\mathfrak{C}$, $\mathfrak{C}$(asc), $\mathfrak{C}$(ser), $\mathfrak{C}^*$, $\mathfrak{T}$, $\mathfrak{T}$(asc), $\mathfrak{T}$(ser) and $\mathfrak{T}^S$ are L-closed but not E-closed (see Example 4.5).

(2) The classes $\mathfrak{T}$, $\mathfrak{T}$(asc) and $\mathfrak{T}^S$ are Q-closed, but the classes $\mathfrak{C}$, $\mathfrak{C}$(asc) and $\mathfrak{C}^*$ are not Q-closed (see Example 4.4).

4.

In this section we shall give several examples concerning the classes in Sections 2 and 3.

EXAMPLE 4.1. Let U be the universal enveloping algebra of a Lie algebra L over any field and let V be the associative ideal in U which is generated by L. Suppose that L is nilpotent of class $n-1$ $(n \geq 2)$. The set of elements of V with weight $> n$ form an ideal in V, which is denoted by S. Then $N = N(L) = V/S$ is an associative nilpotent algebra of class n and can be considered as a right L-module in the usual way. Thus we form the split extension $E = E(L) = N \dotplus L$, which is nilpotent of class n.

Let L_1 be a 1-dimensional Lie algebra, say $L_1 = \langle e \rangle$. Then $N_1 = N(L_1) = \langle \bar{e}, \bar{e}^2 \rangle$ and so $L_2 = E(L_1) = N_1 \dotplus L_1 = \langle \bar{e}, \bar{e}^2 \rangle \dotplus \langle e \rangle$. We recursively define $N_n = N(L_n)$ and $L_{n+1} = E(L_n) = N_n \dotplus L_n$ for all $n \geq 2$. Set $L = \bigcup_{n=1}^\infty L_n$ (the direct limit of $\{L_n\}$). Then by [9, Theorem 4] we know that $L \in \mathrm{L}\mathfrak{N} \cap \acute{\mathrm{E}}\mathfrak{A}$ and that L has no non-zero bounded left Engel elements. Hence L is a $\mathfrak{C}$-algebra. Since by [1, Theorem 4.6] L is a Gruenberg algebra, L is not a $\mathfrak{C}$(asc)-algebra. Moreover $L \notin \acute{\mathrm{E}}(\mathrm{si})\mathfrak{A}$ (cf. [8, Example 6.7]). We put $I = \sum_{i=2}^\infty N_i$, which is an ideal of L. Then $L/I \cong L_2$. As we have seen above $L_2 = \langle e_1, e_2, e_3 \rangle$ with $[e_1, e_2] = e_3$ and $\zeta(L_2) = \langle e_3 \rangle$. From this we deduce that $\langle e_1 \rangle$ is a 2-step subideal but not an ideal of L_2. Thus L does not belong to $\mathfrak{T}$. That is to say, we have over an arbitrary field

$$\mathfrak{T} < \mathfrak{C} \quad \text{and} \quad \mathfrak{C}(\mathrm{asc}) < \mathfrak{C}.$$

EXAMPLE 4.2. Let S be the 3-dimensional simple Lie algebra with basis $\{e_{-1}, e_0, e_1\}$ over any field and with multiplication $[e_{-1}, e_0]=e_{-1}$, $[e_0, e_1]=e_1$ and $[e_{-1}, e_1]=e_0$. We set $L=\oplus_{i=1}^{\infty} S_i$, where each S_i is an isomorphic copy of S. Using [6, Theorem 4.4] we know that L belongs to $\mathfrak{T}$(ser). On the other hand, there is a soluble subalgebra of L which is neither abelian nor almost-abelian. Hence by Proposition 3.2 L is not a $\mathfrak{C}^*$-algebra. That is to say, we have over an arbitrary field

$$\mathfrak{C}^* < \mathfrak{C}(\mathrm{asc}), \quad \mathfrak{T}^S < \mathfrak{T}(\mathrm{asc}) \quad \text{and} \quad \mathfrak{T}(\mathrm{ser})^S < \mathfrak{T}(\mathrm{ser}).$$

EXAMPLE 4.3. By [2, Theorem 6.5.5] over any field of characteristic zero there exists a non-zero locally nilpotent Lie algebra L with trivial Gruenberg radical. Evidently we see that L belongs to $\mathfrak{C}$(asc) but not to $\mathfrak{C}$(ser). Hence over any field of characteristic zero we have

$$\mathfrak{C}(\mathrm{ser}) < \mathfrak{C}(\mathrm{asc}).$$

EXAMPLE 4.4. Let W be a Witt algebra, that is, a Lie algebra over a field of characteristic zero with basis $\{w_1, w_2,\dots\}$ and multiplication $[w_i, w_j]=(i-j)w_{i+j}$. Since every non-zero soluble subalgebra of W is 1-dimensional ([7, Corollary to Theorem 1]), it follows from Proposition 3.2 that W is a $\mathfrak{C}^*$-algebra. Let N be the subalgebra of W generated by $w_4, w_5,\dots$. Then it is easy to see that W/N is not a $\mathfrak{C}$-algebra. Hence W is not a $\mathfrak{T}$-algebra. These tell us that over any field of characteristic zero the classes $\mathfrak{C}$, $\mathfrak{C}$(asc) and $\mathfrak{C}^*$ are not Q-closed and

$$\mathfrak{T}^S < \mathfrak{C}^*.$$

EXAMPLE 4.5. We consider a metabelian Lie algebra L not belonging to $\mathfrak{C}$. For example, let X be an abelian Lie algebra with basis $\{x_0, x_1, x_2,\cdots\}$ and σ be the upward shift on X, that is, $x_i\sigma=x_{i+1}$ for all $i\geq 0$. Then clearly the split extension $L=X \dotplus \langle\sigma\rangle$ is not a $\mathfrak{C}$-algebra. Thus from such a Lie algebra L we can deduce that any class $\mathfrak{X}$ satisfying $\mathfrak{A}\leq\mathfrak{X}\leq\mathfrak{C}$ is not E-closed.

References

[1] R. K. Amayo: Ascendant subalgebras of Lie algebras, preprint, Universität Bonn, 1975.
[2] R. K. Amayo and I. Stewart: Infinite-dimensional Lie Algebras, Noordhoff, Leyden, 1974.
[3] M. Honda: Joins of weak subideals of Lie algebras, Hiroshima Math. J. **12** (1982), 657–673.
[4] T. Ikeda: Hyperabelian Lie algebras, Hiroshima Math. J. **15** (1985), 601–617.
[5] N. Jacobson: Lie Algebras, Interscience, New York, 1962.
[6] Y. Kashiwagi: Lie algebras which have an ascending series with simple factors, Hiroshima Math. J. **11** (1981), 215–227.

[7] F. Kubo: A note on Witt algebras, Hiroshima Math. J. **7** (1977), 473–477.
[8] T. Sakamoto: Lie algebras satisfying the weak minimal condition on ideals, Hiroshima Math. J. **16** (1986), 51–75.
[9] L. A. Simonjan: Certain examples of Lie groups and algebras, Sibirsk. Mat. Ž. **12** (1971), 837–843, translated in Siberian Math. J. **12** (1971), 602–606.
[10] I. Stewart: Lie Algebras, Lecture Notes in Mathematics 127, Springer, Berlin, 1970.
[11] S. Tôgô: Serially finite Lie algebras, Hiroshima Math. J. **16** (1986), 443–448.
[12] S. Tôgô: Infinite-dimensional Lie Algebras (in Japanese), Maki, Tokyo (to appear).
[13] S. Tôgô and H. Miyamoto: Lie algebras in which every ascendant subalgebra is a subideal, Hiroshima Math. J. **8** (1978), 491–498.
[14] V. R. Varea: On Lie algebras in which the relation of being an ideal is transitive, Comm. Algebra **13** (1985), 1135–1150.

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