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The first eigenvalue of the Laplacian on a certain generalized flag manifold

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§ 1. Introduction

Let (S^2, g) be a 2-dimensional sphere with a Riemannian metric g. Let Δ be the Laplacian with respect to g, acting on smooth functions on S^2. Let $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be eigenvalues of Δ, each of which is repeated as many times as its multiplicity, and let $\lambda_1 = \lambda_1(g)$ be the first positive eigenvalue in particular. J. Hersh [5] showed that

$$(1) \qquad 1/\lambda_1 + 1/\lambda_2 + 1/\lambda_3 \geq (3/8\pi)\,\mathrm{vol}\,(S^2, g)$$

holds and consequently,

$$(2) \qquad \lambda_1(g)\,\mathrm{vol}\,(S^2, g) \leq 8\pi,$$

where $\mathrm{vol}\,(M, g)$ denotes the volume of a Riemannian manifold (M, g). The equality in (1) or (2) holds if and only if (S^2, g) is the canonical sphere.

In various studies of spectra of Riemannian manifolds, one direction indicated by M. Berger is the generalization of the inequalities of (1) and (2) to other manifolds X. He has shown that the inequality generalizing (1) is false for $X = S^n$ $(n \geq 3)$ and for $X = T^2$ as follows ([1], $n^0 4$): For $X = S^n$, an n-dimensional sphere $(n \geq 3)$, there exists a Riemannian metric g on S^n such that

$$\textstyle\sum_{i=1}^{n} 1/\lambda_i < ((n+1)/n)(\mathrm{vol}\,(S^n, g)/V_0)^{2/n},$$

where V_0 = the volume of the canonical sphere S^{n-1}. And for $X = T^2$, a 2-dimensional torus, there exists a flat torus satisfying a similar inequality.

With respect to the generalization of (2), we know several results as follows.

M. Berger in [1], [2] has shown that for $X = T^n$, an n-dimensional torus, there exists a positive constant $k(T^n)$ such that

$$\lambda_1(g)\,\mathrm{vol}\,(T^n, g) \leq k(T^n)$$

for every left invariat metric g.

P. C. Yang and S. T. Yau in [12] have shown that if X is a Riemann surface of genus h, then for every metric g on X,

$$\lambda_1(g)\,\mathrm{vol}\,(X, g) \leq 8\pi(h+1).$$

H. Urakawa in [10], H. Muto and H. Urakawa in [8] have shown that if X is a certain homogeneous space stisfying their condition (C'), then there exists a family of invariant metrics $(g_t)_{0<t<\infty}$ on X such that

(3) $\lambda_1(g_t) \longrightarrow \infty$ when $t \longrightarrow \infty$, and $\mathrm{vol}(X, g_t)$ is constant in t.

Compact connected semisimple group manifolds, and real, complex, quaternionic Stiefel manifolds satisfy the condition (C') for example.

S. Tanno in [9], H. Muto in [7] have shown that for $X=S^n$, there exists a family of metrics satisfying (3).

The purpose of this article is to prove that for $X=G/K$, where G is a compact connected Lie group and K is the centralizer of a toral subgroup (such X is called a generalized flag manifold), if X has the reducible isotropy action, then there exists a family of invariant metrics satisfying (3) (Theorem in §3). A generalized flag manifold does not satisfy the condition (C').

Throughout this paper, for a real vector space V, its complexification is denoted by V_c and for a real or complex vector space V, its dual vector space is denoted by V^*.

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§ 2. The Laplacian for the invariant metric

Let (M, g) be an n-dimensional Riemannian manifold and let $C^\infty(M)$ be the space of complex valued smooth functions on M. Let $(x_1,\dots, x_n)$ denote a local coodinate system on an open set of M. The Laplacian Δ with respect to g is now defined by

$$\Delta f = -\sum_{i,j} g^{ij}(\partial_i\partial_j f - \sum_k \Gamma^k_{ij}\partial_k f), \qquad \text{for} \quad f \in C^\infty(M).$$

Here ∂_i stands for the vector field $\partial/\partial x_i$, (g^{ij}) is the inverse matrix of (g_{ij}) with $g_{ij}=g(\partial_i, \partial_j)$ and Γ^k_{ij} is the Christoffel symbol of the Riemannian connection for g as customary.

In this section we review some fundamental facts on the Laplacian and its eigenvalues for an invariant metric on a compact homogeneous space along the same lines as [8]. Let $M=G/K$ be a compact homogeneous space where G is a compact connected Lie group and K is a closed connected subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively and let $\mathfrak{m}$ be a vector subspace of $\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ (direct sum) and $(\mathrm{Ad}\,K)\mathfrak{m}=\mathfrak{m}$. Then $\mathfrak{m}$ can be identified with the tangent space $T_o(G/K)$ at the origin $o=\{K\}$ in G/K. Every invariant metric on G/K determines an inner product on $\mathfrak{m}$ which is $(\mathrm{Ad}\,K)$-invariant and conversely every $(\mathrm{Ad}\,K)$-invariant inner product on $\mathfrak{m}$ can be extended to an invariant metric on G/K. Let $U(\mathfrak{g}_c)$ denote the universal enveloping algebra of $\mathfrak{g}_c$. Then $U(\mathfrak{g}_c)$ is naturally isomorphic to the algebra of left invariant differential operators on G. Let $U(\mathfrak{g}_c)^K$ denote the subspace of $(\mathrm{Ad}\,K)$-invariants in $U(\mathfrak{g}_c)$. Let $C^\infty(G)^K$ denote the space of all the elements $f \in C^\infty(G)$ such that $f(gk)=f(g)$ for $g \in G$, $k \in K$. Then $C^\infty(G)^K$ can be naturally identified with $C^\infty(G/K)$. Under this identification the algebra of invariant differential operators on G/K is isomorphic to the algebra of restrictions $\{D \mid C^\infty(G)^K \mid D \in U(\mathfrak{g}_c)^K\}$ (see [4], p. 390).

LEMMA 1. *Let g be an invariant metric on G/K and let Δ be the Laplacian with respect to g. Let $(X_i)_{i=1}^n$ be a basis of $\mathfrak{m}$. Put $g_{ij}=g(X_i, X_j)$ and let (g^{ij}) be the inverse matrix of (g_{ij}). Then*

$$\Delta = -\sum_{i,j} g^{ij} X_i X_j \quad \text{in} \quad U(\mathfrak{g}_c)^K.$$

For the proof, notice that the expression $\sum_{i,j} g^{ij} X_i X_j$ is independent of the choice of a basis and see Theorem 1 and its Corollary in [8].

Let $L^2(G/K)$ denote the L^2-completion of $C^\infty(G/K)$ with respect to the invariant Riemannian measure. The Laplacian Δ can be extended to a self-adjoint operator on $L^2(G/K)$. For a finite dimensional G-module V, V^* denotes the dual G-module and V^K denotes the subspace of K-fixed vectors in V. The Peter-Weyl theorem for a compact homogeneous space states that

$$L^2(G/K) = \sum_\lambda \oplus (V_\lambda \otimes (V_\lambda^*)^K),$$

where λ in the summation runs over the representative set of all equivalence classes of irreducible unitary G-modules (see [11], p. 118, 5.3.6). An element $v \otimes f$ in $V_\lambda \otimes (V_\lambda^*)^K$ is identified with the smooth function $(v \otimes f)(g)=f(g^{-1}v)$ on G/K. Hence we have $\Delta(v \otimes f)=v \otimes \Delta f$; in the right hand side Δ acts on $(V_\lambda^*)^K$ as an element in $U(\mathfrak{g}_c)^K$. From these facts we have the following lemma on the eigenvalues of Δ.

LEMMA 2. *The set of all eigenvalues of Δ on $L^2(G/K)$ coincides with the set of all eigenvalues of Δ on V_λ^K's where λ runs over the representative set of all finite dimensional irreducible unitary G-modules.*

§ 3. The Laplacian on a generalized flag manifold

Let G be a compact connected semisimple Lie group and $\mathfrak{g}$ its Lie algebra. Let T_1 be a toral subgroup of G and let K be the centralizer of T_1 in G. Then the homogeneous space G/K is called a generalized flag manifold. We recall another construction of a generalized flag manifold (cf. [11], p. 149, 6.2.10). Let T be a maximal torus of G and $\mathfrak{t}$ the corresponding subalgebra of $\mathfrak{g}$. Denote by R the roots system of the pair $(\mathfrak{g}_c, \mathfrak{t}_c)$. Let R_+ denote a positive system of R

and $\sum$ the set of simple roots contained in R_+. Let (,) denote the Killing form of $\mathfrak{g}_c$. Also (,) stands for the bilinear form on $\mathfrak{t}_c$ or the dual $\mathfrak{t}_c^*$ induced by the Killing form. One can choose a root vector E_α with $\alpha \in R$ as follows: $(E_\alpha, E_{-\alpha})=1$ and conj $E_\alpha = -E_{-\alpha}$ where conj denotes the conjugation of $\mathfrak{g}_c$ relative to $\mathfrak{g}$. Let S be a proper subset of $\sum$ and let R_S denote the set of roots which are linear combinations of elements in S. Put $\mathfrak{g}_S = \mathfrak{t}_c + \sum CE_\alpha$ where the summation is over $\alpha \in R_S$ and put $\mathfrak{k}_S = \mathfrak{g}_S \cap \mathfrak{g}$. Put $\mathfrak{t}_S = \{H \in \mathfrak{t} \mid \alpha(H)=0 \text{ for all } \alpha \in S\}$. Let K_S and T_S be the analytic subgroups of G corresponding to $\mathfrak{k}_S$ and $\mathfrak{t}_S$ respectively. Then K_S is the centralizer of a torus T_S in G and G/K_S is a generalized flag manifold.

Let $\mathfrak{n}^S$ denote the subalgebra of $\mathfrak{g}_c$ spanned by E_α's with $\alpha \in R_+ - R_S$ and put $\mathfrak{n}^{-S} = \text{conj}\, \mathfrak{n}^S$. Notice that $\mathfrak{g}_c = \mathfrak{g}_S + \mathfrak{n}^S + \mathfrak{n}^{-S}$ (direct sum), $[\mathfrak{g}_S, \mathfrak{n}^S] \subset \mathfrak{n}^S$ and $[\mathfrak{g}_S, \mathfrak{n}^{-S}] \subset \mathfrak{n}^{-S}$. Put $\mathfrak{m} = (\mathfrak{n}^S + \mathfrak{n}^{-S}) \cap \mathfrak{g}$. Then $\mathfrak{m}$ makes an (Ad K_S)-invariant complement to $\mathfrak{k}_S$ in $\mathfrak{g}$, and $\mathfrak{n}^S$ is isomorphic to $\mathfrak{m}$ as a $\mathfrak{k}_S$-module by the map $X \mapsto (X + \text{conj}\, X)/2$ $(X \in \mathfrak{n}^S)$. The remainder of this section is devoted to the proof of the following theorem.

THEOREM. *Let G be a compact connected simple Lie group and let G/K_S be a generalized flag manifold. Assume that the linear isotropy representation of K_S on G/K_S is reducible. Then there exists a family of invariant metrics $(g_t)_{t>0}$ on G/K_S which has the following properties.*

(1) *The Riemannian volume* vol$(G/K_S, g_t)$ *is constant in t.*
(2) *The first eigenvalue $\lambda_1(g_t)$ is not bounded.*

REMARK. If G is semisimple, then the generalized flag manifold G/K is decomposed as $G/K = G_1/K_1 \times \cdots \times G_n/K_n$, where each G_i is simple and each G_i/K_i is a generalized flag manifold. If every linear isotropy representation of G_i/K_i is reducible, then the above conclusion holds for G/K.

PROOF. We employ the above notation and consider $\mathfrak{n}^S$ as a $\mathfrak{g}_S$-module by the adjoint representation. Then the assumption in the theorem means that $\mathfrak{n}^S$ is reducible. Since $\mathfrak{g}_c$ is simple, R_+ has a unique maximal root γ. Let S' be the subset of S defined by the following condition: $\{-\gamma\} \cup S'$ is a connected component of $\{-\gamma\} \cup S$ in the extended Dynkin diagram of R. Let R' be the set of roots which are linear combinations of elements of $\{-\gamma\} \cup S'$. Put $R'_+ = R' \cap R_+$. Then R'_+ becomes a positive system of R'. Let $\mathfrak{g}'_c$ stand for the subalgebra of $\mathfrak{g}_c$ generated by E_α with $\alpha \in R'$. Now $\mathfrak{g}'_c$ is a simple Lie algebra because its Dynkin diagram is connected. If $\alpha \in \{-\gamma\} \cup S'$ and $\beta \in S - S'$, then $\alpha \pm \beta$ are not roots. Hence $[\mathfrak{g}_S, \mathfrak{g}'_c] \subset \mathfrak{g}'_c$. One can see that $\mathfrak{g}'_c \cap \mathfrak{n}^S$ is an irreducible $\mathfrak{g}_S$-submodule of $\mathfrak{n}^S$. In fact it is generated by a root vector E_γ as a $\mathfrak{g}_S$-module. We extend an inner product on $\mathfrak{m}$ to a Hermitian inner product on $\mathfrak{m}_c = \mathfrak{n}^S + \mathfrak{n}^{-S}$. By Lemma 1 and the root space decomposition of $\mathfrak{m}_c$, one knows that there exists a family of invariant metrics $(g_t)_{t>0}$ on G/K_S which has the following properties (cf. [10], p. 219, (4.1)):

(1) $\det g_t$ is constant in t, where $\det g$ means $\det(g_{ij})$ for a metric g.
(2) The Laplacian Δ_t of g_t has the form

$$\Delta_t = t^{-r} \sum_\alpha (E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha) + t \sum_\beta (E_\beta E_{-\beta} + E_{-\beta} E_\beta),$$

where α runs over all the roots in $R_+ - R_S$ whose root vectors belong to $\mathfrak{g}'_c \cap \mathfrak{n}^S$ and β runs over the rest roots in $R_+ - R_S$, and r is a positive constant determined by (1).

Let $(\ , \ )'$ denote the Killing form of $\mathfrak{g}'_c$. Since $\mathfrak{g}'_c$ is simple, one can put $(\ , \ )' = k(\ , \ )$, where k is a positive constant. Let C be the universal Casimir element of $\mathfrak{g}_c$ and C' that of $\mathfrak{g}'_c$. Let V_λ be a finite dimensional irreducible G-module with the (R_+-extreme) highest weight λ (cf. [11], p. 90, 4.4.2). Let V_λ^K denote the subspace consisting of K-fixed vectors of V_λ. Elements of $\mathfrak{g}_S$ act trivially on $V_\lambda^{K_S}$, so that we see $C = \sum_\beta (E_\beta E_{-\beta} + E_{-\beta} E_\beta)$, $kC' = \sum_\alpha (E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha)$ as operators acting on $V_\lambda^{K_S}$, where β runs over all roots in $R_+ - R_S$ and α runs over the roots in $R_+ - R_S$ whose root vectors belong to $\mathfrak{g}'_c \cap \mathfrak{n}^S$. Hence we can rewrite Δ_t in the form $\Delta_t = (t^{-r} - t)kC' + tC$. We know that the Casimir element C acts by a scalar $(\lambda, \lambda + 2\rho)$ on V_λ where ρ is half the sum of R_+. As for C' we need decompose V_λ into $\mathfrak{g}'_c$-primary components. Note that $(\mu, \mu)' = k^{-1}(\mu, \mu)$ for a linear form μ on $\mathfrak{t}_c \cap \mathfrak{g}'_c$. Therefore one knows that each eigenvalue of Δ_t on $V_\lambda^{K_S}$ is of the form

$$\begin{aligned} E_{\lambda,\mu} &= (t^{-r} - t)k(\mu, \mu + 2\rho')' + t(\lambda, \lambda + 2\rho) \\ &= (t^{-r} - t)(\mu, \mu + 2\rho') + t(\lambda, \lambda + 2\rho), \end{aligned}$$

where μ is a $\mathfrak{t}_c$-weight of an R'_+-extreme highest weight vector in a $\mathfrak{g}'_c$-primary component in V_λ generated by $V_\lambda^{K_S}$ and ρ' is half the sum of R'_+. Since $(\lambda, \lambda) - (\mu, \mu) \geq 0$ and $(\lambda - \mu, 2\rho) \geq 0$, we obtain that

$$\begin{aligned} E_{\lambda,\mu} &= t^{-r}(\mu, \mu + 2\rho') + t\{(\lambda, \lambda) - (\mu, \mu) + (\lambda, 2\rho) - (\mu, 2\rho')\} \\ &\geq t\{(\lambda, 2\rho) - (\mu, 2\rho')\} = t\{(\lambda - \mu, 2\rho) + (\mu, 2\rho - 2\rho')\} \\ &\geq t(\mu, 2\rho - 2\rho'). \end{aligned}$$

Note that since $V_\lambda^{K_S}$ is contained in V_λ^T, elements of $V_\lambda^{K_S}$ are of $\mathfrak{t}_c$-weight zero. Hence μ is a nonnegative integral linear combination of roots in R'_+. Since μ is R'_+-dominant, one can see by inspection of the table of fundamental weights (cf. [3] or [6]) that μ is a positive integral linear combination of elements of $\sum'$ if $\mu \neq 0$. Here we denote by $\sum'$ the set of simple roots in R'_+. We will show that $2\rho - 2\rho'$ is R'_+-dominant and non-zero. Then one sees that $E_{\lambda,\mu} \geq$ (a positive constant)$\cdot t$ if $\mu \neq 0$. This shows in particular that the first eigenvalue $\lambda_1(g_t)$

is not bounded.

Let β be the lowest weight (root) of a $\mathfrak{g}_S$-module $\mathfrak{g}'_c \cap \mathfrak{n}^S$ relative to $R_+ \cap R_S$. Then it is easy to see that $\sum' = \{\beta\} \cup S'$. For a root α, let $\alpha^\vee$ denote the coroot $2\alpha/(\alpha, \alpha)$. Notice that $(2\rho - 2\rho', \alpha^\vee) = (2\rho, \alpha^\vee) - (2\rho', 2\alpha')'/(\alpha, \alpha)' = 2 - 2 = 0$ for $\alpha \in S'$ and $(2\rho - 2\rho', \beta^\vee) = (2\rho, \beta^\vee) - 2$. Therefore it is sufficient to prove that $(2\rho, \beta^\vee) \geq 3$. Let $\sum = \{\alpha_1, \ldots, \alpha_l\}$. For a root $\alpha = \sum_i n_i \alpha_i$, define $\mathrm{ht}\,\alpha = \sum_i n_i$. We first prove the following Lemma.

Lemma 3. *$\mathfrak{n}^S$ is an irreducible $\mathfrak{g}_S$-module if and only if $\sum - S$ consists of one simple root whose coefficient in the maximal root γ is equal to one.*

Proof of Lemma 3. We first recall the $\mathfrak{g}_S$-module structure of $\mathfrak{n}^S$. γ is the maximal root in R_+, so that we may write $\gamma = n_1\alpha_1 + \cdots + n_l\alpha_l$ where all n_i are positive and $\gamma \in R_+ - R_S$. If $\alpha \in R_+ - R_S$, $\beta \in R_+$ and $\alpha + \beta \in R_+$, then $\alpha + \beta \in R_+ - R_S$. For $\alpha \in R_+ - R_S$, put $R(\alpha) = \{\alpha + \beta \in R_+ \mid \beta \in R_S\}$, $\mathfrak{n}(\alpha) = \sum_{\beta \in R(\alpha)} CE_\beta$. Then $\mathfrak{n}(\alpha)$ is a $\mathfrak{g}_S$-submodule of $\mathfrak{n}^S$. Assume that $\mathfrak{n}^S$ is irreducible. Take α_1 in $\sum - S$. Then $\mathfrak{n}(\alpha_1) = \mathfrak{n}^S$, $R(\alpha_1) = R_+ - R_S$. Hence $\sum - S = \{\alpha_1\}$. Since $\gamma \in R(\alpha_1)$, γ is of the form $\gamma = \alpha_1 + n_2\alpha_2 + \cdots + n_l\alpha_l$. Assume conversely that $\sum - S = \{\alpha_1\}$ and $\gamma = \alpha_1 + n_2\alpha_2 + \cdots + n_l\alpha_l$. If $\alpha_i \in S$ then $\alpha_1 - \alpha_i$ is not a root. Hence α_1 is an extremal (the lowest) weight of $\mathfrak{n}^S$. $\mathfrak{n}^S$ is generated by E_{α_1} as a $\mathfrak{g}_S$-module, and thus $\mathfrak{n}^S$ is irreducible (see [6], 20.2 and 6.3).

Now assume that $\mathfrak{g}_c$ is of the type A_l, D_l, or E_l. Then all the roots are of the same length. Hence $(2\rho, \beta^\vee) = \sum n_i(2\rho, \alpha_i^\vee) = 2\,\mathrm{ht}\,\beta$ if $\beta = \sum n_i\alpha_i$. Because β is of the form $\beta = \gamma - \sum k_i\alpha_i$ $(\alpha_i \in S)$ and $\mathfrak{n}^S$ is a reducible $\mathfrak{g}_S$-module, the contraposition of Lemma 3 implies that $\mathrm{ht}\,\beta \geq 2$. Hence we have $(2\rho, \beta^\vee) \geq 4$.

Next we assume that $\mathfrak{g}_c$ is of the type B_l $(l \geq 3)$, C_l, F_4. Note that $(2\rho, \beta^\vee) \geq \mathrm{ht}\,\beta$ when $\mathrm{rank}\,\mathfrak{g}_c \geq 3$. So it is enoguh to show that $\mathrm{ht}\,\beta \geq 3$. In the case of B_l, we get the following pairs of S' and β with the help of the table in [3]:

$$S' = \{\alpha_1, \ldots, \alpha_p\}\,(p \leq l-1) \quad \text{and} \quad \beta = \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_l, \quad \text{or}$$

$$S' = \{\alpha_2, \ldots, \alpha_p\}\,(p \leq l-1) \quad \text{and} \quad \beta = \alpha_1 + \cdots + \alpha_p + 2\alpha_{p+1} + \cdots + 2\alpha_l.$$

Also the table shows that these are the only possible pairs of S' and β which can occur. Note that $l \geq 3$. Hence we obtain $(2\rho, \beta^\vee) \geq \mathrm{ht}\,\beta \geq 3$. The same checking process goes through in the case of C_l and F_4.

Finally we assume that $\mathfrak{g}_c$ is of the type B_2 or G_2. In the case of B_2 the contraposition of Lemma 3 implies that the only one pair $S' = \emptyset$ and $\beta = \gamma = \alpha_1 + 2\alpha_2$ occurs. Hence we obtain that $(2\rho, \beta^\vee) \geq \mathrm{ht}\,\beta = 3$. In the case of G_2 we know that the only two pairs occur:

$$S' = \emptyset, \quad \beta = \gamma = 3\alpha_1 + 2\alpha_2, \quad \text{hence} \quad (2\rho, \beta^\vee) = 6 \quad \text{or}$$

$$S' = \{\alpha_2\}, \quad \beta = 3\alpha_1 + \alpha_2, \quad \text{hence} \quad (2\rho, \beta^\vee) = 4.$$

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