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Locally inner derivations of ideally finite Lie algebras

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Introduction

Let d be a linear endomorphism of a Lie algebra L. We call d a locally inner derivation of L if, for any finite-dimensional subspace F of L, there is an element $x \in L$ such that $yd=[y, x]$ for any $y \in F$. Evidently the set of locally inner derivations of L is an ideal of the derivation algebra $\mathrm{Der}(L)$. It will be denoted by $\mathrm{Lin}(L)$.

C. A. Christodoulou introduced the notion of cofinite Lie algebras by analogy with cofinite groups and investigated their structure in [2]. In group theory locally inner automorphisms and local conjugacy classes of FC-groups have been studied by many authors from various points of view (see for example [3, 4, 6, 9, 10]). In this paper, following their works we study locally inner derivations of ideally finite Lie algebras by making use of the notion of cofinite Lie algebras. In Section 1 we shall show that for a cofinite and ideally finite Lie algebra, its locally inner derivations are precisely those induced by elements of its idealizer in its profinite completion (Theorem 1). In Section 2 we shall show that for an ideally finite Lie algebra L, $\mathrm{Lin}(L)$ is a profinite completion of $\mathrm{Inn}(L)$ for some cofinite topology (Theorem 2), and by using it we shall determine the dimension of $\mathrm{Lin}(L)$ and when $\mathrm{Lin}(L)$ and $\mathrm{Inn}(L)$ coincide over some fields (Theorems 3 and 4, Corollary 2).

1.

We shall be concerned with Lie algebras which are not necessarily finite-dimensional over an arbitrary field $\mathfrak{k}$ of characteristic zero. A Lie algebra L is called a cofinite Lie algebra if it has a topology satisfying the following C1–C4, where $\mathscr{K}(L)$ will denote the set of closed ideals of L of finite codimension, and $\mathscr{T}(L)$ will denote the set of closed vector subspaces of L of finite codimension:

C1. $\cap\{K: K \in \mathscr{K}(L)\}=0$.

C2. For any $H \in \mathscr{T}(L)$, there exists $K \in \mathscr{K}(L)$ such that $K \subset H$.

C3. If H, K are vector subspaces of L such that $H \subset K$ and $H \in \mathscr{T}(L)$, then K is closed.

C4. The set $\{x+U: x \in L, U \in \mathscr{T}(L)\}$ is a subbase of closed sets of L.

This topology is called a cofinite topology, which was suggested by the definition of coset topology in Hochshild and Mostow [5]. A cofinite Lie algebra cannot be Hausdorff unlike cofinite groups. A compact cofinite Lie algebra is called a profinite Lie algebra. It is known that any cofinite Lie algebra L has a profinite completion P, that is, L can be embedded as a dense subalgebra in a profinite Lie algebra P (see [2, Proposition 3.2 and Theorem 3.3]).

Let L be a cofinite Lie algebra. A derivation d of L is called residually inner if, for all $K \in \mathscr{K}(L)$, we have $Kd \subset K$ and the derivation consequently induced on L/K is inner. It is easy to see that residually inner derivations are continuous.

Now we consider the relationship between locally inner derivations and residually inner derivations of a cofinite Lie algebra.

LEMMA 1. *Let L be a residually finite Lie algebra. Then a locally inner derivation of L is residually inner for any cofinite topology on L.*

PROOF. Let d be a locally inner derivation of L and let K be an ideal of L of finite codimension. Then there exists a finite-dimensional subspace F of L such that $L=K+F$. It is easy to see that $Kd \subset K$. Since d is locally inner, there is an element $x \in L$ such that $d|_F = \mathrm{ad}_L(x)|_F$. Thus the derivation induced by d on L/K coincides with $\mathrm{ad}_{L/K}(x+K)$.

We require a lemma describing some closure properties of confinite Lie algebras. A bar over a set will denote closure.

LEMMA 2. *Let L be a dense subalgebra of a cofinite Lie algebra P and let U be a vector subspace of P. Then*

(a) $\overline{U} = \cap \{U+M : M \in \mathscr{K}(P)\}$. *In particular $P=L+M$ for every $M \in \mathscr{K}(P)$.*

(b) $\mathscr{K}(P) = \{\overline{K} : K \in \mathscr{K}(L)\}$.

(c) *If $K \in \mathscr{K}(L)$ then $\overline{K} \cap L = K$.*

PROOF. (a) follows from [2, Proposition 1.6].

(b) Let $\mathscr{K}(P) = \{M_i : i \in I\}$. Suppose that $M \in \mathscr{K}(P)$. Then $M \cap L \in \mathscr{K}(L)$. From (a), we have

$$\begin{aligned}\overline{M \cap L} &= \cap_{i \in I}(M_i + (M \cap L)) \\ &= \cap \{M_i + (M \cap L) : M_i \subset M\} \\ &= \cap \{M \cap (M_i + L) : M_i \subset M\} = M.\end{aligned}$$

Conversely let $K \in \mathscr{K}(L)$ and F be a finite-dimensional subspace of L such that $L=K+F$. Then there exists $M \in \mathscr{K}(P)$ such that $L \cap M \leqq K$, since $\{L \cap M_i : i \in I\}$ is cofinal in $\mathscr{K}(L)$. From the aobve $M = \overline{L \cap M} \leqq \overline{K}$. Hence $\overline{K}$ has finite codimension in P and so $\overline{K}+F$ is closed in P by C3. Since L is a dense subalgebra of P and L is contained in $\bar{K}+F$, we have $P=\bar{K}+F$. Now $[\bar{K}, F]\subset\bigcap_{i\in I}[K+M_i, F]\subset\bigcap_{i\in I}(K+M_i)=\bar{K}$. Thus $\bar{K}$ is an ideal of P and so $\bar{K}\in\mathscr{K}(P)$.

(c) There is a closed subset C of P such that $K=L\cap C$. Hence $\bar{K}\subset C$ and so $\bar{K}\cap L\subset C\cap L=K$. Thus $\bar{K}\cap L=K$.

LEMMA 3. *Let L be a cofinite Lie algebra with a profinite completion P, and let $x\in I_P(L)$. Then $\mathrm{ad}_L(x)$ is a residually inner derivation of L, and conversely any residually inner derivation of L is induced by such an element.*

PROOF. Let $K\in\mathscr{K}(L)$. Then $\bar{K}\in\mathscr{K}(P)$ and $\bar{K}\cap L=K$ by Lemma 2. Hence $[K, x]\subset\bar{K}\cap L=K$. Since $P=\bar{L}=L+\bar{K}$ we can write $x=l+k$ with $l\in L$, $k\in\bar{K}$. For any $y\in L$, we have $[y, k]=[y, x]-[y, l]\in L\cap\bar{K}=K$. Thus $\mathrm{ad}_{L/K}(x+K)=\mathrm{ad}_{L/K}(l+K)$ and therefore $\mathrm{ad}_L(x)$ is residually inner.

Conversely let d be a residually inner derivation of L. For each $K\in\mathscr{K}(L)$, let $S(K)=\{a\in P\colon yd-[y, a]\in\bar{K}$ for all $y\in L\}$. Since d is residually inner, $S(K)$ is non-empty. Let $a\in S(K)$ and $B=\{s-a\colon s\in S(K)\}$. It is not hard to see that B is a closed subspace of P. Since $S(K)=a+B$, $S(K)$ is closed. Furthermore the set $\{S(K)\colon K\in\mathscr{K}(L)\}$ has the finite intersection property, and so we can take an element x from their intersection since P is compact. From Lemma 2, $\cap\{\bar{K}\colon K\in\mathscr{K}(L)\}=0$. If $y\in L$, then we have $yd-[y, x]\in\bar{K}$ for any $K\in\mathscr{K}(L)$. Thus we have $d=\mathrm{ad}_L(x)$ and $x\in I_P(L)$.

From Lemmas 1 and 3, we have the following

PROPOSITION 1. *Let L be a cofinite Lie algebra with a profinite completion P. Then every locally inner derivation of L is induced by an element of $I_P(L)$.*

Now we search for conditions under which the converse statement of the proposition holds.

LEMMA 4. *Let L be a cofinte Lie algebra with a profinite completion P. Soppose that L has a local system $\mathscr{L}$ of finite-dimensional subalgebras satisfying:*

(a) *If $H\in\mathscr{L}$, $x\in P$ and $[H, x]\subset L$, then $x-y\in I_P(H)$ for some $y\in L$.*

(b) *If f is a continuous homomorphism of L onto a finite-dimensional Lie algebra, then $I_{f(L)}(f(H))=f(I_L(H))$ for each $H\in\mathscr{L}$.*

Then each element of $I_P(L)$ induces a locally inner derivation of L.

PROOF. Let $x\in I_P(L)$ and $H\in\mathscr{L}$. By (a) there is an element $y\in L$ such that $x-y\in I_P(H)$. So to prove that x induces a locally inner derivation of L, we may replace x by $x-y$ and assume that $x\in I_P(H)$.

Now we put $I=I_P(H)$ and show that $I\cap L$ is dense in I. Let $M\in\mathscr{K}(P)$ and $J=I\cap L$. Then $P=L+M$ and it is easy to see that the canonical homomorphism of L onto P/M is continuous. Therefore we can apply (b) to obtain $I_{P/M}(H+M/M)=J+M/M$. Since $[I+M, H+M]\subset H+M$, we have $I\subset J+M$

and $I=J+(I \cap M)$. From Lemma 2, it follows that $\bar{J}=\cap\{J+(I \cap M): M \in \mathscr{K}(P)\}=I$.

Let $C=C_I(H)$. Then I/C is finite-dimensional. For each $M \in \mathscr{K}(P)$ we have $[C+I \cap M, H] \subset H \cap M$. Therefore $\bar{C}$ centralizes H, whence C is closed and $C \in \mathscr{K}(I)$. By Lemma 2, $I=J+C$. So we can write $x=z+c$ with $z \in J$, $c \in C$. Then $[h, x]=[h, z]$ for any $h \in H$, which completes the proof.

If L is cofinite and ideally finite, then each finite-dimensional ideal of L is a closed ideal of P (see [2, Proposition 1.18 and Corollary 2.26]). So it is easy to see that the local system consisting of all finite-dimensional ideals of L satisfies the conditions (a), (b) of Lemma 4. As a consequence of Lemmas 1, 3 and 4 we have the following

THEOREM 1. *Let L be a cofinite and ideally finite Lie algebra and let P be a profinite completion of L. Then the following subalgebras of* Der(L) *are coincident*:

(a) *The algebra of all locally inner derivations of L.*

(b) *The algebra of all residually inner derivations of L.*

(c) *The algebra of all derivations induced on L by elements of $I_P(L)$.*

We note that (b) and (c) are independent of the cofinite topology of L. In group theory, [3, Theorem 5.5] is the corresponding result on locally inner automorphisms.

We now construct a cofinite $\mathrm{L}(\lhd^2)\mathfrak{F}$-algebra in which the algebra of (c) is not contained in the algebra of (a). Let H_i be the three-dimensional Heisenberg algebra with basis $\{x_i, y_i, z_i\}$ $(i=1, 2,\dots)$, where $[x_i, y_i]=z_i$, $[H_i, z_i]=0$. Next, we put $H=\mathrm{Dr}_{i\in \boldsymbol{N}} H_i$ and $C=\mathrm{Cr}_{i\in \boldsymbol{N}} H_i$. Then there is a derivation d of H such that

$$x_i d = z_i - z_{i+1}$$

$$y_i d = z_i d = 0 \qquad (i \in \boldsymbol{N}).$$

d can be uniquely extended to a derivation of C, which we also denote by d. Now we can form the split extensions $L=H \dotplus \langle d\rangle$, $P=C \dotplus \langle d\rangle$, and we regard L as a subalgebra of P.

It is clear that $L^3=0$ and $L \in \mathrm{L}(\lhd^2)\mathfrak{F}$. For each $i \in \boldsymbol{N}$, let $K_i=\sum_{j>i} H_j$, $M_i=\mathrm{Cr}_{j>i} H_j$. Then we can give L and P the cofinite topologies by K_i's and M_i's. Now let R be the projective limit of $\{P/M_i; p_{ij}\}$, where p_{ij} is the canonical homomorphism of P/M_i onto P/M_j $(i \geqq j)$. It is well known that the homomorphism f: $P \to R$ such that $af=(a+M_i)$ is a topological and algebraic embedding (see [2, Corollary 2.18]). Moreover it is not hard to see that f is surjective. Thus P is topologically and algebraically isomorphic to R, and so P is a profinite Lie algebra. Since $L+M_i=P$ for each $i \in \boldsymbol{N}$, P is a profinite completion of L by Lemma 2.

Now let $x=(x_i)\in C$. Then $[H, x]\subset H$ and $[x, d]=xd=z_1\in L$. Hence $x\in I_P(L)$. Finally we show that $\mathrm{ad}_L(x)$ is not a locally inner derivation of L. Suppose that $\mathrm{ad}_L(x)$ is locally inner. Then there is an element $y=h_1+\cdots+h_n+td\in L$ such that $[y, d]=z_1$ with $h_i\in H_i$ and $t\in\mathfrak{k}$. For each i, writing $h_i=a_ix_i+b_iy_i+c_iz_i$ with a_i, b_i, $c_i\in\mathfrak{k}$ $(i=1,\dots, n)$, we have $[y, d]=yd=a_1(z_1-z_2)+\cdots+a_n(z_n-z_{n+1})\neq z_1$, which is a contradiction. This establishes the claim.

2.

In this section we investigate locally inner derivations of ideally finite Lie algebras. In general an ideally finite Lie algebra H is not residually finite, but the algebra of its inner derivations $\mathrm{Inn}(H)$ is residually finite and can be a cofinite Lie algebra. Its profinite completion is given in the following

THEOREM 2. *Let H be an ideally finite Lie algebra. Then* $\mathrm{Lin}(H)$ *is a profinite completion of* $\mathrm{Inn}(H)$ *for some cofinite topology.*

PROOF. Let $\{F_j: j\in J\}$ be a collection of finite-dimensional ideals of H such that $\sum_{j\in J}F_j=H$ and J is directed, that is, for any $i, j\in J$ there exists $k\in J$ such that $F_i+F_j\subset F_k$. Put $L=\mathrm{Lin}(H)$, $I=\mathrm{Inn}(H)$ and let $L(F)=\{d\in L\colon Fd=0\}$, $I(F)=L(F)\cap I$ for each finite-dimensional ideal F of H. $L(F_j)$'s and $I(F_j)$'s form finite residual systems of L and I respectively, which give L and I the cofinite topologies.

We now let $P=\varprojlim\{L/L(F_i);\ p_{ij}\}$ where p_{ij} is the canonical homomorphism of $L/L(F_i)$ onto $L/L(F_j)$ for $F_i\geqq F_j$. We claim that the natural embedding $f\colon L\to P$ is surjective. For this let $(d_j+L(F_j))\in P$ with $d_j\in L$. If $F_i\geqq F_j$, then $d_i-d_j\in L(F_j)$ since $d_j+L(F_j)=(d_i+L(F_i))p_{ij}=d_i+L(F_j)$. Therefore we can define a locally inner derivation δ of H such that $x\delta=xd_j$ for $x\in F_j$. Then $\delta f=(d_j+L(F_j))$ and so f is a topological and algebraic isomorphism. Hence L is profinite.

Let $d\in L$ and $F=F_j$ for some $j\in J$. Then there exists $x\in H$ such that $d|_F=ad_F(x)$. It is clear that $d-\mathrm{ad}_H(x)\in L(F)$. Therefore $d\in L(F)+I$ and so $L=L(F)+I$. It follows that $\bar{I}=L$ from Lemma 2. This completes the proof.

It was shown in [2, Theorem 4.18] that a profinite Lie algebra cannot have countable-dimension. Taking account of this result, we can deduce the following

COROLLARY 1. *Let L be an ideally finite Lie algebra. Then* $\mathrm{Inn}(L)\in\mathfrak{F}$ *if and only if* $\mathrm{Lin}(L)\in\mathfrak{F}$. *Further if L is contable-dimensional, then* $\mathrm{Lin}(L)=\mathrm{Inn}(L)$ *if and only if* $\mathrm{Inn}(L)\in\mathfrak{F}$.

Let L be a semisimple ideally finite Lie algebra. It is well known that L is decomposed as a direct sum $\mathrm{Dr}_{i\in I}S_i$, and $\mathrm{Der}(L)$ is isomorphic to $\mathrm{Cr}_{i\in I}S_i$ where each S_i is a non-abelian simple $\mathfrak{F}$-algebra (see [1, Theorem 13.4.2 and Proposition 13.4.5]). It is not hard to see that if L is infinite-dimensional, then $\dim \mathrm{Lin}(L) = \dim \mathrm{Cr}_{i\in I} S_i = |\mathfrak{k}|^{|I|}$. We shall extend this result to general ideally finite Lie algebras.

LEMMA 5. *Let L be an infinite-dimensional ideally finite* $\mathrm{R}\mathfrak{F}$*-algebra and* $\{N_i: i\in I\}$ *be a set of ideals of L of finite codimension such that* $\bigcap_{i\in I} N_i = 0$. *Then* $\dim \mathrm{Inn}(L) \leqq |I|$.

PROOF. Adding all intersections of finitely many N_i we may assume that for any $i, j\in I$ there exists $k\in I$ such that $N_k \subset N_i \cap N_j$. For each $i\in I$, let $C_i = C_L(N_i)$. Then there exists a finite-dimensional ideal H of L such that $L = N_i + H$. It is clear that $C_L(H)\cap C_i \cap N_i$ has finite codimension in C_i, and is contained in $Z = \zeta(L)$. Hence $C_i/Z \in \mathfrak{F}$.

Now let B be any finite-dimensional ideal of L. Then there exists $i\in I$ such that $B\cap N_i = 0$. Hence $[B, N_i] = 0$ and so $B \subset C_i$. Thus $L/Z = \sum_{i\in I} C_i/Z$, and therefore $\dim \mathrm{Inn}(L) = \dim L/Z \leqq |I|\aleph_0 = |I|$.

LEMMA 6. *Let L be an ideally finite* $\mathrm{R}\mathfrak{F}$*-algebra, and let N be an ideal of L with infinite-dimension α. Then there exists an ideal M of L such that* $\dim L/M = \alpha$ *and* $M\cap N = 0$.

PROOF. Since L is ideally finite, there exist $\mathfrak{F}$-ideals F_j of L such that $N = \sum_{j\in J} F_j$, $|J| = \alpha$ and J is directed. Then for each $j\in J$ there is an ideal N_j of L of finite codimension such that $F_j \cap N_j = 0$. Let $K = \bigcap_{j\in J} N_j$ and $C/K = \zeta(L/K)$. Then by Lemma 5, $\dim L/C \leqq \alpha$. Replacing L by L/K we can assume $\dim L/Z \leqq \alpha$, where $Z = \zeta(L)$. Let M be a subspace of Z such that $Z = (N\cap Z) \dotplus M$. Clearly, M is an ideal of L and $\dim Z/M \leqq \dim N \leqq \alpha$. Hence $\dim L/M \leqq \alpha$. On the other hand $M\cap N = M\cap N\cap Z = 0$, and therefore $\dim L/M = \alpha$.

Next we state the following facts about extensions and liftings of locally inner derivations, which are analogous to [8, Corollaries 2.3 and 2.4].

LEMMA 7. *Let L be an ideally finite Lie algebra.*

(a) *If $H \leqq L$ and $d\in \mathrm{Lin}(H)$, then there exists $\delta \in \mathrm{Lin}(L)$ such that $\delta|_H = d$.*

(b) *If $K \lhd L$ and $d\in \mathrm{Lin}(L/K)$, then there exists $\delta\in \mathrm{Lin}(L)$ such that δ induces the derivation d on L/K.*

PROOF. (a) Let $\{F(i): i\in I\}$ be the set of all finite-dimensional ideals of L. For each $i\in I$ let $A_i = \mathrm{ad}(L)|_{F(i)}$ and $B_i = \{f\in A_i: f|_{H\cap F(i)} = d|_{H\cap F(i)}\}$. If we give A_i the affine topology, then A_i is compact and T_1 (see [2, Proposition 2.2]). Since there exists $h\in H$ such that $d = \mathrm{ad}(h)$ on $F(i)\cap H$, B_i is a non-empty closed subset of A_i and so compact.

For $F(i) \geqq F(j)$, the restriction $f_{ij}: A_i \to A_j$ is continuous and closed by [2, Lemma 2.4]. $\{B_i, f_{ij|B_i}\}$ forms a projective limit system and $\varprojlim \{B_i\}$ is non-empty by [7, Theorem 7.1]. Choosing $(d_i) \in \varprojlim \{B_i\}$, we can define $\delta \in \mathrm{Der}(L)$ by $\delta |_{F(i)} = d_i$. Then δ is locally inner and $\delta |_H = d$.

(b) is similarly proved.

Now by making use of these lemmas we show the following

THEOREM 3. *Let L be an ideally finite* $\mathrm{R}\mathfrak{F}$*-algebra over a field* $\mathfrak{k}$ *of characteristic zero, and suppose that* $\mathrm{Inn}(L)$ *has infinite-dimension* α. *Then* $\dim \mathrm{Lin}(L) = |\mathfrak{k}|^{\alpha}$.

PROOF. Let $\{F_i: i \in I\}$ be the set of all finite-diemsnional ideals of L, ordering I by inclusion. We can choose a directed subset I' of I such that $L = \zeta(L) + \sum_{i \in I'} F_i$ and $|I'| = \alpha$. As in the proof of Theorem 2, we can see that $\mathrm{Lin}(L)$ is isomorphic to $\varprojlim \{\mathrm{Lin}(L)/L_i: i \in I'\}$, where $L_i = \{d \in \mathrm{Lin}(L): F_i d = 0\}$. Hence $\dim \mathrm{Lin}(L) \leqq \dim \mathrm{Cr}_{i \in I'}(\mathrm{Lin}(L)/L_i) \leqq |\mathfrak{k}|^{\alpha}$.

Conversely we show that $\dim \mathrm{Lin}(L) \geqq |\mathfrak{k}|^{\alpha}$. Let J be a maximal subset of I' such that $\sum_{j \in J} F_j = \oplus_{j \in J} F_j$ and each F_j is non-abelian. Let $N = \oplus_{j \in J} F_j$ and $|J| = \beta$. If β is finite, then there is an ideal K of L of finite codimension such that $K \cap N = 0$. For any two elements x, y of K, there exists an $\mathfrak{F}$-ideal F of L such that $\langle x, y \rangle \leqq F \leqq K$. Then $F \cap N = 0$. By the maximality of N, F is abelian and $[x, y] = 0$. Thus K is abelian. On the other hand there is an $\mathfrak{F}$-ideal E of L such that $L = E + K$. Then $C_K(E)$ is contained in $\zeta(L)$ and $\dim L/C_K(E)$ is finite. Then $C_K(E)$ is contained in $\zeta(L)$ and $\dim L/C_K(E)$ is finite. This contradicts the fact that $\mathrm{Inn}(L) \notin \mathfrak{F}$.

Therefore N is infinite-dimensional. By Lemma 6, there is an ideal M of L such that $\dim L/M = \beta$ and $M \cap N = 0$. As before, M is abelian. Let H be an ideal of L with dimension β such that $L = M + H$. By Lemma 6 again, there exists an ideal M_1 of L such that $\dim L/M_1 = \beta$ and $H \cap M_1 = 0$. Then $M_1 \cap M \leqq \zeta(L)$ and $\dim L/M_1 \cap M \leqq \beta$. Hence we have $\dim N = \alpha$.

It is not hard to see that $\mathrm{Lin}(N) \simeq \mathrm{Cr}_{j \in J} \mathrm{Inn}(F_j)$. Each $\mathrm{Inn}(F_j)$ is non-trivial because F_j is non-abelian. Therefore $\dim \mathrm{Lin}(N) = |\mathfrak{k}|^{\alpha}$. From Lemma 7, it follows that $\dim \mathrm{Lin}(L) \geqq |\mathfrak{k}|^{\alpha}$. This completes the proof.

For general ideally finite Lie algebras, we show the same fact as the above with some restriction.

THEOREM 4. *Let L be an ideally finite Lie algebra over a field* $\mathfrak{k}$ *of characteristic zero. Suppose that* $\mathrm{Inn}(L)$ *has infinite-dimension* α *and either* $\alpha > |\mathfrak{k}|$ *or* $\alpha = \aleph_0$. *Then* $\dim \mathrm{Lin}(L) = |\mathfrak{k}|^{\alpha}$.

PROOF. It is sufficient to show that $\dim \mathrm{Lin}(L) \geqq |\mathfrak{k}|^{\alpha}$ as in the case of

Theorem 3. Let $Z=\zeta(L)$ and $W=\zeta_2(L)$. If $\dim L/W=\alpha$, then we can apply Theorem 3 to conclude that $\dim \mathrm{Lin}(L/Z)=|\mathfrak{f}|^\alpha$ since L/Z is residually finite. It follows that $\dim \mathrm{Lin}(L)\geqq|\mathfrak{f}|^\alpha$ from Lemma 7.

Thus we may assume that $\dim L/W=\beta<\alpha$ and so $\dim W/Z=\alpha$. Let $\{x_i+Z: i\in I\}$ be a basis of W/Z and let $F_i=\langle x_i^L\rangle+Z$. Adding all the sums of finitely many F_i we can write $W/Z=\sum_{i\in I}F_i/Z$, where $F_i/Z\in\mathfrak{F}$ and I is a directed set of cardinal α. For each $i\in I$, $C_i=C_L(F_i)$ has finite codimension in L. There is an $\mathfrak{F}$-ideal F of L such that $L=C_i+F$. It is clear that $Z=C_W(C_i)\cap C_L(F)$ and $C_W(C_i)/Z\in\mathfrak{F}$. From the choise of F_i's, we can see that for each C_i there are only finitely many F_j which commute with C_i. Hence there are α subalgebras C_i.

Now let $C=\bigcap_{i\in I}C_i$, $A=\mathrm{ad}(L)|_W$ and $H=\mathrm{Lin}(L)|_W$. Then $A\simeq L/C$ and A is an infinite-dimensional abelian algebra. Let $A_i=C_A(F_i)$ and $H_i=C_H(F_i)$ for each $i\in I$. Then it is easily seen that the families of subalgebras A_i and H_i foım finite residual systems of A and H respectively, under which A and H are cofinite Lie algebras. Further since $A+H_i=H$ for each $i\in I$, A is dense in H by Lemma 2.

Let $A'=\{f\in A^*: A_if=0$ for some $i\in I\}$, where A^* is the dual space of A. For each $i\in I$ let $B_i=\{f\in A^*: A_if=0\}$. Then each B_i is a finite-dimensional subspace of A', and $A_i=\cap\{\mathrm{Ker}(f): f\in B_i\}$. Hence there is a one-to-one correspondence between the families A_i and B_i. Thus $\alpha\leqq|\mathfrak{f}|$ $(\dim A')$ since there are $|\mathfrak{f}|$ $(\dim A')$ finite-dimensional subspaces of A'. By the assumption on α, we have $\alpha\leqq\dim A'$. On the other hand $A'=\bigcup_{i\in I}B_i=\sum_{i\in I}B_i$ and so $\dim A'=\alpha$. By [2, Remark 4.17] we have $\dim H=|\mathfrak{f}|^\alpha\leqq\dim \mathrm{Lin}(L)$. The proof is completed.

COROLLARY 2. *Let L be an ideally finite Lie algebra over a countable field of characteristic zero. Then* $\mathrm{Lin}(L)=\mathrm{Inn}(L)$ *if and only if* $\mathrm{Inn}(L)\in\mathfrak{F}$.

COROLLARY 3. *Let L be an infinite-dimensional ideally finite Lie algebra over a countable field of characteristic zero. Then* $\mathrm{Der}(L)\neq\mathrm{Inn}(L)$. *In paticular*, $\dim \mathrm{Der}(L)=2^{(\dim L)}$.

PROOF. If $\dim \mathrm{Inn}(L)=\dim L$, then there is nothing to prove by Theorem 4. Thus we may assume that $\dim \mathrm{Inn}(L)<\dim L$. Then there is an ideal F of L such that $L=\zeta(L)+F$ and $\dim \mathrm{Inn}(L)=\dim F$. Let A be a subspace of $\zeta(L)$ such that $L=A\dotplus F$. Then each linear endomorphism f of A induces a derivation d of L as follows: $(a+x)d=af$ for $a\in A$, $x\in F$. Hence $\dim \mathrm{Der}(L)\geqq\dim \mathrm{End}(A)=2^{(\dim A)}$ and so $\dim \mathrm{Der}(L)=2^{(\dim L)}$.

Finally we note that for a semisimple serially finite Lie algebra L, $\mathrm{Der}(L)=\mathrm{Lin}(L)$. However, for semisimple locally finite Lie algebras we do not know whether $\mathrm{Der}(L)=\mathrm{Lin}(L)$ or not, even if they are simple.

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