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Universal Wu classes

Dedicated to Professor Masahiro Sugawara on his 60th birthday

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§ 1. Introduction

Let BO be the space which classifies stable (real) vector bundles, and consider its mod 2 cohomology $H^*(BO; Z_2)$ (the coefficient Z_2 will be omitted often). Then, $H^*(BO)$ is the polynomial algebra over Z_2 on the universal Stiefel-Whitney classes $w_i \in H^i(BO)$ for $i \geqq 1$ [2, Th. 7.1]. Let $v_i \in H^i(BO)$ be the universal Wu classes (cf. [1, p. 225], [4, p. 315]) defined inductively by

(1) $\quad v_0 = 1 = w_0$ and $w_i = \sum_{j=0}^{i} Sq^j v_{i-j}$; i.e., $w = Sqv$ or $v = Sq^{-1}w$

(Wu's formula, cf. [2, Th. 11.14]) for $w = \sum_i w_i$, $v = \sum_i v_i$ and the Steenrod squaring operator $Sq = \sum_i Sq^i$ with Sq^{-1} given by $Sq^{-1}Sq = 1 = Sq\, Sq^{-1}$.

In this note, we prove a formula representing v_i by w_j's modulo

(2) the ideal $I^{(2)} = (w_1^2, w_2^2, \ldots)$ of $H^*(BO)$ generated by the squares w_i^2 for $i \geqq 1$:

THEOREM. (i) (Stong) $v_a \equiv v_{a_1} \cdots v_{a_l}$ mod $I^{(2)}$ *for any* $a \geqq 1$, *where* $a = a_1 + \cdots + a_l$ *is the dyadic expansion of* a.

(ii) $v_{2a} \equiv v_a^{(2)} + \sum_{i=0}^{a-1} w_i w_{2a-i}$ mod $I^{(2)}$ *for any power* a *of* 2, *and* $v_1 = w_1$.
Here, the notation $x^{(2)}$ *for* $x \in H^a(BO)$ *is used in the following sence*:

(3) *If* $x \equiv \sum_{i=1}^{k} x_i \in H^a(BO)$ mod $I^{(2)}$ *with monomials* x_i *on* w_j's, *we have uniquely* $x^{(2)} \in H^{2a}(BO)$ mod $I^{(2)}$ *given by*

$$x^{(2)} (= (x^2 - \textstyle\sum_{i=1}^{k} x_i^2)/2) = \sum_{1 \leqq i < j \leqq k} x_i x_j \quad \textit{and} \quad x^{(2)} = 0 \quad \textit{if} \quad k \leqq 1;$$
$(x+y)^{(2)} \equiv x^{(2)} + y^{(2)} + xy$ *and* $(xy)^{(2)} \equiv 0$ mod $I^{(2)}$ *for any* x, y.

COROLLARY. $v \equiv 1 + \sum w_{i_1} \cdots w_{i_l}$ mod $I^{(2)}$,
where $\sum$ *is taken over all sequences* $1 \leqq i_1 < \cdots < i_l$ $(l \geqq 1)$ *satisfying*

(4) $\{i_1, \ldots, i_l\} = \{\alpha_1, \beta_1, \ldots, \alpha_m, \beta_m, \gamma_1, \ldots, \gamma_n\}$ $(l = 2m + n,\ m \geqq 0,\ n \geqq 0)$ *such that* $\alpha_j + \beta_j$ *and* γ_j *are all powers of* 2.

A formula modulo the ideal generated by w_i^2 and $\prod_{j=1}^{4} w_{i_j}$ is previously known to the author. Theorem (i) is due to Professor Robert E. Stong, and the author is most grateful to his valuable advices during this work.

§ 2. Proof of Theorem (i)

Let RP^k be the k-dimensional real projective space, and consider the m-fold product space $X_{n,m}=(RP^1\times RP^n)^m$ with the projections p_i: $X_{n,m}\to RP^1\times RP^n$ to the ith factor, q_1: $RP^1\times RP^n\to RP^1$ and q_n: $RP^1\times RP^n\to RP^n$ $(n\geqq 2)$. Moreover, let ξ_k be the canonical line bundle over RP^k, and consider the vector bundle

$$\eta_{n,m} = \oplus_{i=1}^m (p_i^*q_n^*\xi_n\oplus\zeta_i^\perp),\ \zeta_i = p_i^*q_1^*\xi_1 \otimes p_i^*q_n^*\xi_n, \text{ over } X_{n,m},$$

where $\zeta^\perp$ is a bundle such that $\zeta\oplus\zeta^\perp$ is the trivial bundle.

Then, the total Stiefel-Whitney (resp. Wu) class $w(\eta_{n,m})=\sum_i w_i(\eta_{n,m})$ (resp. $v(\eta_{n,m})=\sum_i v_i(\eta_{n,m})=Sq^{-1}w(\eta_{n,m})$) of $\eta_{n,m}$ is given by the following

LEMMA 2.1. *Put* $\alpha_i=w_1(p_i^*q_n^*\xi_n)$ *and* $\sigma_i=w_1(p_i^*q_1^*\xi_1)$. *Then*:

(i) $H^*(X_{n,m}; Z_2)=Z_2[\sigma_1, \alpha_1,\dots, \sigma_m, \alpha_m]/(\sigma_1^2,\dots, \sigma_m^2, \alpha_1^{n+1},\dots, \alpha_m^{n+1})$.

(ii) $w(\eta_{n,m})=\prod_{i=1}^m \{1+\sigma_i(1+\alpha_i)^{-1}\}$; *i.e.*, $w_i(\eta_{n,m})=\sum (\prod_{k=1}^r \sigma_{i_k}\alpha_{i_k}^{s_k})$, *where the sum is taken over all* $1\leqq i_1<\cdots<i_r\leqq m$ *and* $s_k\geqq 0$ $(1\leqq k\leqq r)$ *with* $r+\sum_{k=1}^r s_k=i$.

(iii) $v(\eta_{n,m}) = \prod_{i=1}^m (1+\sum_{r\geqq 0}\sigma_i\alpha_i^{-1+2^r})$; *i.e.*, $v_i(\eta_{n,m}) = \sum(\prod_{k=1}^r \sigma_{i_k}\alpha_{i_k}^{-1+t_k})$, *where the sum is taken over all* $1\leqq i_1<\cdots<i_r\leqq m$ *and powers* t_k *of* $2 (1\leqq k\leqq r)$ *with* $\sum_{k=1}^r t_k=i$.

PROOF. (i) holds by the definition of ξ_k. $p_i^*q_k^*\xi_k$'s are line bundles, and the basic properties of the Stiefel-Whitney classes for line bundles imply that $w(p_i^*q_n^*\xi_n)=1+\alpha_i$, $w(\zeta_i^\perp)=(1+\sigma_i+\alpha_i)^{-1}$ and (ii), because $\sigma_i^2=0$ and so $(1+\alpha_i)(1+\sigma_i+\alpha_i)^{-1}=1+\sigma_i(1+\sigma_i+\alpha_i)^{-1}=1+\sigma_i(1+\alpha_i)^{-1}$. (ii) implies (iii), because the basic properties of Sq (cf. [3]) show that $Sq\,\sigma_i=\sigma_i$, $Sq(\alpha_i^t)=\alpha_i^t+\alpha_i^{2t}$ for $t=2^r$, and

$$Sq(1+\textstyle\sum_{r\geqq 0}\sigma_i\alpha_i^{-1+2^r})=Sq\{1+\sigma_i(1+\sum_{r\geqq 0}\alpha_i^{2^r})^{-1}\}=1+\sigma_i(1+\alpha_i)^{-1}. \quad \square$$

LEMMA 2.2. *Put* $w_i=w_i(\eta_{n,m})\in H^i(X_{n,m}; Z_2)$. *Then*:

(i) $w_i^2=0$ *for any* $i\geqq 1$, *and* $w_{i_1}\cdots w_{i_l}=0$ *for any* $i_k\geqq 1$ *and* $l>m$.

(ii) *In* $H^i(X_{n,m}; Z_2)$ *with* $i\leqq n+1$, *the monomials* $w_{i_1}\cdots w_{i_l}$, *for* $1\leqq l\leqq m$, $1\leqq i_1<\cdots<i_l$ *and* $\sum_{k=1}^l i_k=i$, *are linearly independent.*

PROOF. Lemma 2.1 (i) and (ii) show the lemma, because
$w_{i_1}\cdots w_{i_l}=\sum_{1\leqq j_1,\dots,j_l\leqq m}(\prod_{k=1}^l \sigma_{j_k}\alpha_{j_k}^{-1+i_k})+\sum_{l'>l}(\prod_{k=1}^{l'} \sigma_{j_k}\alpha_{j_k}^{s_k})$. $\square$

LEMMA 2.3. *Let* $(t_1,\dots, t_r)$ *be a sequence of powers* t_k *of* 2 *with* $\sum_{k=1}^r t_k=a$.

Then, for any $b\geqq 1$, *the number of all subsequences* $(t_{j_1},\dots,t_{j_s})$ $(1\leqq j_1<\cdots<j_s\leqq r)$ *with* $\sum_{k=1}^{s} t_{j_k}=b$ *is congruent to* $\binom{a}{b}$ mod 2.

PROOF. If $t_k=1$ for all k, then the lemma is trivial. Assume $t_k\geqq 2$ for some k; and consider $T=(t_1,\dots,t_{k-1},u,v,t_{k+1},\dots,t_r)$ with $u=v=t_k/2$, and its subsequences $S\subset T$. Then, $\#\{S\mid S\ni u, S\not\ni v\}=\#\{S\mid S\not\ni u, S\ni v\}$ and $\#\{S\mid S\ni u, v,$ or $S\not\ni u, v\}=\#\{$all subsequences of $(t_1,\dots,t_r)\}$, where $\#$ denotes the number of elements. Thus the lemma holds by induction. $\square$

PROPOSITION 2.4. $v_a(\eta_{n,m})=\prod_{i=1}^{l} v_{a_i}(\eta_{n,m})$, *where* $a=a_1+\cdots+a_l$ *is the dyadic expansion of* $a\geqq 1$ *(i.e.,* $a_1>\cdots>a_l$ *and they are powers of* 2).

PROOF. Compare the both sides by Lemma 2.1 (iii), by noticing that $\sigma_i^2=0$. Then the equality follows from Lemma 2.3, since $\binom{a}{a_i}\equiv 1$ mod 2. $\square$

PROOF OF THEOREM (i). Take n and m to satisfy $n+1\geqq a$ and $(m+1)(m+2)>2a$, and let $\tilde{\eta}_{n,m}\colon X_{n,m}\to BO$ be the classifying map of the bundle $\eta_{n,m}$ over $X_{n,m}$. Then, $\tilde{\eta}^*_{n,m}(v_a)=v_a(\eta_{n,m})=\prod_{i=1}^{l} v_{a_i}(\eta_{n,m})=\tilde{\eta}^*_{n,m}(\prod_{i=1}^{l} v_{a_i})$ by Proposition 2.4; hence $v_a-\prod_{i=1}^{l} v_{a_i}$ is in $I^{(2)}$ by Lemma 2.2. $\square$

§3. Proof of Theorem (ii)

LEMMA 3.1. $\sum_{i=0}^{a-1} Sq^i(xv_{a-i})=\sum_{i=0}^{a-1}(Sq^ix)w_{a-i}$ *for any* $x\in H^*(BO\colon Z_2)$.

PROOF. $\sum_{i=0}^{a} Sq^i(xv_{a-i})=\sum_{i=0}^{a}\sum_{j=0}^{i}\ [=\sum_{j=0}^{a}\sum_{i=j}^{a}]\,(Sq^jx)(Sq^{i-j}v_{a-i})=\sum_{j=0}^{a}(Sq^jx)w_{a-j}$ by (1), and the lemma holds since $v_0=1=w_0$. $\square$

LEMMA 3.2. *Let a be a power of* 2, *and* $0\leqq b<2a$. *Then,*

$$w_{2a+b}+Sq^bv_{2a}+\sum_{i=0}^{b-1} w_{b-i}Sq^iv_{2a}\equiv\sum_{i=0}^{a-1} w_{a+b-i}Sq^iv_a \quad \textit{if} \quad b<a,$$

$$\equiv\sum_{i=b-a+1}^{a-1}\Big(w_{a+b-i}+\sum_{j=0}^{b-a} w_{b-i-j}Sq^jv_a\Big)Sq^iv_a \quad \textit{if} \quad b\geqq a, \quad \mathrm{mod}\ I^{(2)}.$$

PROOF. Hereafter, 'mod $I^{(2)}$' is often omitted. We notice that

(5) $Sq^i(I^{(2)})\subset I^{(2)}$, and $Sq^iv_k\equiv 0$ if $i\geqq k\geqq 1$ (e.g., $k=2a+b-i\leqq a$),

by the definition of $I^{(2)}$ in (2) and the dimensional reason. Hence

$$\sum_{i=0}^{b} Sq^iv_{2a+b-i}\equiv\sum_{i=0}^{b} Sq^i(v_{2a}v_{b-i})=\sum_{i=0}^{b}(Sq^iv_{2a})w_{b-i}=A, \quad \text{and}$$

$$w_{2a+b}+A\equiv\sum_{i=b+1}^{c-1} Sq^iv_{a+c-i}\equiv\sum_{i=b+1}^{c-1} Sq^i(v_av_{c-i}) \quad (c=a+b)$$

$$=\sum_{i=b+1}^{c-1}\sum_{j=0}^{i}\Big[=\sum_{j=0}^{b}\sum_{i=b+1}^{c-1}+\sum_{j=b+1}^{c-1}\sum_{i=j}^{c-1}\Big](Sq^jv_a)(Sq^{i-j}v_{c-i})=B,$$

by (1), Theorem (i) and Lemma 3.1. Moreover, if $0\leqq b<a$, then

$$B \equiv \textstyle\sum_{j=0}^{b} (Sq^j v_a)(w_{c-j}+C) + \sum_{j=b+1}^{a-1} (Sq^j v_a) w_{c-j}, \quad \text{where}$$

$$C = \textstyle\sum_{i=0}^{b-j} Sq^i v_{c-j-i} \equiv \sum_{i=0}^{b-j} Sq^i(v_a v_{b-j-i}) = \sum_{i=0}^{b-j} (Sq^i v_a) w_{b-j-i};$$

hence $\sum_{j=0}^{b} (Sq^j v_a) C \equiv 0$. If $a \leqq b < 2a$, then

$$B \equiv \textstyle\sum_{j=b-a+1}^{a-1} (Sq^j v_a)(w_{c-j}+C) \quad \text{and} \quad C \equiv \sum_{i=0}^{b-a} (Sq^i v_a) w_{b-j-i}. \quad \square$$

Now, since $v_1 = w_1$, Theorem (ii) follows from the following

PROPOSITION 3.3. *Let a be a power of 2. Then,*

$$v_{2a} \equiv w_{2a} + \textstyle\sum_{i=0}^{a-1} w_{a-i} Sq^i v_a \quad \textit{and} \quad v_{4a} \equiv v_{2a}^{(2)} + \sum_{i=0}^{2a-1} w_i w_{4a-i} \bmod I^{(2)}.$$

PROOF. Lemma 3.2 implies the first congruence by taking $b=0$, and the second one by (3) as follows: $w_{4a}+v_{4a}+w_{2a}v_{2a} \equiv \sum_{i=1}^{2a-1} w_{2a-i} Sq^i v_{2a} \equiv \sum_{i=1}^{4} A_i$, where

$$A_1 = \textstyle\sum_{i=1}^{2a-1} w_{2a-i} w_{2a+i},\ A_2 = \sum_{i=1}^{2a-1} \sum_{j=0}^{i-1} [= \sum_{j=0}^{2a-1} \sum_{i=j+1}^{2a-1}] w_{2a-i} w_{i-j} Sq^j v_{2a} \equiv 0,$$

$$A_3 = (\textstyle\sum_{i=1}^{a-1} \sum_{j=0}^{a-1} + \sum_{i=a}^{2a-1} \sum_{j=i-a+1}^{a-1}) [= \sum_{j=0}^{a-1} \sum_{i=1}^{a+j-1}] w_{2a-i} w_{a+i-j} Sq^j v_a \equiv 0,$$

$$A_4 = \textstyle\sum_{i=a}^{2a-1} \sum_{j=i-a+1}^{a-1} \sum_{k=0}^{i-a} [= \sum_{0 \leqq k < j < a} \sum_{i=a+k}^{a+j-1}] w_{2a-i} w_{i-j-k} (Sq^j v_a)(Sq^k v_a) \equiv \sum_{0 \leqq k < j < a} w_{a-k} w_{a-j} (Sq^j v_a)(Sq^k v_a). \quad \square$$

§ 4. Proof of Corollary

For the set $\boldsymbol{N}$ of all positive integers, denote by $\boldsymbol{N}_2 \subset \boldsymbol{N}$ the subset of all powers of 2, and consider the collection $\mathfrak{S}$ of all finite subsets $S \subset \boldsymbol{N}$ satisfying

(6) $S = \{t_1 - r_1, r_1, \ldots, t_l - r_l, r_l, t_{l+1}, \ldots, t_m\}$, $\sharp S = m + l \geqq 1$ and $0 \leqq l \leqq m$, for $t_i \in \boldsymbol{N}_2$ $(1 \leqq i \leqq m)$ and $r_i \in \boldsymbol{N}$ with $r_i < t_i/2$ $(1 \leqq i \leqq l)$, (see (4)).

LEMMA 4.1. *In (6), m, l, t_i and r_i are unique for S, by ordering elements to satisfy $t_i > t_{i+1}$, or $t_i = t_{i+1}$ and $r_i < r_{i+1}$ for $i < l$, and $t_j > t_{j+1}$ for $j > l$.*

PROOF. We note that $t_i/2 < t_i - r_i \notin \boldsymbol{N}_2$ for $i \leqq l$ in (6). Hence, if $S \subset \boldsymbol{N}_2$, then $l = 0$ and so $m = \sharp S$ and the lemma holds. Let $S \not\subset \boldsymbol{N}_2$. Then $l \geqq 1$ and $s_1 = \max(S - \boldsymbol{N}_2) = t_1 - r_1$ by the above order. Here, $t_1/2 < t_1 - r_1 = s_1 < t_1$; hence $t_1 \in \boldsymbol{N}_2$ is unique, and so is r_1. Since $S - \{s_1, r_1\} \in \mathfrak{S}$, the lemma is proved by induction. $\square$

For any $a \in \boldsymbol{N}$, put $\mathfrak{S}(a) = \{S \in \mathfrak{S} \mid \sum_{s \in S} s = a\}$. Then, we have the following

LEMMA 4.2. *Assume that $a = a_1 + a_2$ for $a_1 \in \boldsymbol{N}_2$ and $a_2 \in \boldsymbol{N}$ with $a_1 \geqq a_2$. Then:*

(i) *$S_1 \cup S_2 \in \mathfrak{S}(a)$ for $S_k \in \mathfrak{S}(a_k)$ with $S_1 \cap S_2 = \phi$; and $\sharp(S_1 \cup S_2) \geqq 3$ if $a_1 = a_2$.*

(ii) *Conversely, for any $S \in \mathfrak{S}(a)$ with $\sharp S \geqq 3$ if $a_1 = a_2$, there are an odd number of unordered pairs $\{S_1, S_2\}$ of $S_k \in \mathfrak{S}(a_k)$ with $S_1 \cap S_2 = \phi$ and $S_1 \cup S_2 = S$.*

PROOF. (i) is clear by definition. For $S = \{t_1 - r_1, r_1, \dots, t_l - r_l, r_l, t_{l+1}, \dots, t_m\} \in \mathfrak{S}(a)$ and any $S_k \in \mathfrak{S}(a_k)$ in (ii), Lemma 4.1 means that if $t_i - r_i \in S_k$ $(i \leqq l)$, then $r_i \in S_k$. Thus, the number of all such $\{S_1, S_2\}$ is equal to that of all subsequences $(t_{i_1}, \dots, t_{i_n})$ of $(t_1, \dots, t_m)$ satisfying $\sum_{j=1}^{n} t_{i_j} = a_1$ (resp. $i_1 = 1$, in addition, if $a_1 = a_2$). Now, the latter is congruent to $\binom{a}{a_1}\left(\text{resp.} \binom{a-t_1}{a_1-t_1} \text{ if } a_1 = a_2\right)$ mod 2 by Lemma 2.3, which is odd by assumption. Thus (ii) is proved. □

Now, according to this lemma, the Theorem implies immediately that $v_a \equiv \sum_{S \in \mathfrak{S}(a)} \prod_{s \in S} w_s \bmod I^{(2)}$ by induction, which is the Corollary by definition.

References

[1] J. W. Milnor, On the Stiefel-Whitney numbers of complex manifolds and of spin manifolds, Topology **3** (1965), 223–230.

[2] J. W. Milnor and J. D. Stasheff, Characteristic classes, Ann. of Math. Studies **76**, Princeton Univ. Press, Princeton, 1974.

[3] N. E. Steenrod and D. B. A. Epstein, Cohomology operations, Ann. of Math. Studies **50**, Princeton Univ. Press, Princeton, 1962.

[4] R. E. Stong, Notes on cobordism theory, Princeton Math. Notes, Princeton Univ. Press, Princeton, 1968.

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