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On the boundary limits of harmonic functions

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1. Introduction

This paper deals with the boundary behavior of harmonic functions u on a bounded open set $G \subset R^n$ satisfying

$$\int_G |\text{grad}\, u(x)|^p \omega(x) dx < \infty,$$

where $p > 1$ and ω is a nonnegative measurable function on G. The function $\omega(x)$ is mainly of the form $\varphi(d(x))$, where $d(x)$ denotes the distance of x from the boundary ∂G and φ is a monotone function on the interval $(0, \infty)$. Moreover, G is assumed to satisfy certain smoothness conditions mentioned later.

Our first aim in this paper is to find a positive function $A(x)$ on G for which $A(x)u(x)$ tends to zero as x tends to the boundary ∂G. We shall next give conditions which assure the boundedness of u on G or near a boundary point of G. In special cases, u will be shown to have a finite limit at a boundary point; our discussion below will include the proof of the existence of nontangential limits.

We here remark that the case $p = 1$ can be treated similarly with a small modification.

2. Boundary limits of harmonic functions on general bounded domains

Throughout this paper, let G be a bounded domain in R^n satisfying the following condition: There exist a compact set K and a positive number c such that any point x in G is joined to K by a piecewise smooth curve $x(t)$ in G having the following properties:

(C_1) $x(1) \in K$. (C_2) $x(0) = x$.

(C_3) $|x(t_2) - x(t_1)| \leqq c(t_2 - t_1)|x(0) - x(1)|$ whenever $0 \leqq t_1 \leqq t_2 \leqq 1$.

(C_4) $|x(t_2) - x(t_1)| \geqq c^{-1}(t_2 - t_1)|x(0) - x(1)|$ whenever $0 \leqq t_1 \leqq t_2 \leqq 1$.

(C_5) If $y \in B(x(t), 2^{-1}d(x(t)))$, then $d(x) + |x - y| < cd(y)$.

REMARK. Condition (C_4) implies the following:

(C_6) For any $y \in G$, the linear measure of the set of all t such that $y \in B(x(t), 2^{-1}d(x(t)))$ is dominated by $M|x(0) - x(1)|^{-1}d(y)$, where M is a positive constant independent of y and $x(t)$. In fact, if $y \in B(x(t_i), 2^{-1}d(x(t_i)))$, $i=1, 2$, then $d(y) \geqq 2^{-1}d(x(t_i))$, so that $|x(t_1)-x(t_2)| \leqq 2^{-1}[d(x(t_1))+d(x(t_2))] \leqq 2d(y)$. Hence ($C_4$) implies ($C_6$).

EXAMPLE 1. Let φ be a Lipschitz continuous function on R^{n-1}, and define $G(r_1, r_2)=\{(x', x_n) \in R^{n-1} \times R^1; |x'|<r_1, \varphi(x')<x_n<r_2\}$. If $r_2 > \sup_{|x'|<r_1} \varphi(x')$, then $G(r_1, r_2)$ satisfies the above condition on G.

EXAMPLE 2. Let φ be a nondecreasing continuous function on the interval $[0, \infty)$, and define $G(r_1, r_2)=\{(x', x_n) \in R^{n-1} \times R^1; |x'|<r_1, \varphi(|x'|)<x_n<r_2\}$. If $r_2 > \varphi(r_1)$, then $G(r_1, r_2)$ satisfies the above condition on G.

In fact, let $e=(0, r_3) \in G(r_1, r_2)$, where $\varphi(r_1)<r_3<r_2$, and note that for $x \in G(r_1, r_2)$, $x(t)=(1-t)x+te$ satisfies conditions (C_2)$\sim$(C_5).

For any positive numbers a and γ, we set $T_\gamma(a)=\{x=(x', x_n) \in R^{n-1} \times R^1; |x'|^\gamma < ax_n\}$. Then $T_\gamma(a) \cap B(0, 1)$ is a typical example of G, where $B(x, r)$ denotes the open ball with center at x and radius r.

Our first aim is to establish the following result.

THEOREM 1. *Let u be a function harmonic in G and satisfying*

$$\int_G |\text{grad}\, u(x)|^p d(x)^\alpha dx < \infty \tag{1}$$

with a real number α. Then

$$\lim_{x \to \partial G} d(x)^{(n-p+\alpha)/p} u(x) = 0 \qquad \text{in case } n-p+\alpha > 0,$$

$$\lim_{x \to \partial G} [\log(1/d(x))]^{1/p-1} u(x) = 0 \quad \text{in case } n-p+\alpha = 0$$

and

$$u(x) \text{ is bounded on } G \qquad \text{in case } n-p+\alpha < 0.$$

For a proof of Theorem 1, we need the following lemma.

LEMMA 1. *For a piecewise smooth curve $x(t)$, $t \in [0, 1]$, in an open set $G \subset R^n$, set $G(x(t))=\bigcup_{0 \leqq s \leqq t} B(x(s), 2^{-1}d(x(s)))$. If u is harmonic in G, then for any $x \in G$ and any piecewise smooth curve $x(t)$ satisfying conditions (C_2), (C_3) and (C_6),*

$$|u(x)-u(x(t))| \leqq M' \int_{G(x(t))} |\text{grad}\, u(y)| d(y)^{1-n} dy$$

for all $t \in [0, 1]$, where M' is a positive constant which depends only on c and M in conditions (C_3) and (C_6).

PROOF. By conditions (C_2), (C_3), (C_6) and the mean value property of harmonic functions, we have

$$|u(x)-u(X_t)| = \left|\int_0^t (d/ds)u(X_s)\,ds\right|$$
$$\leqq c|x-X_1|\int_0^t \left(M_1[2^{-1}d(X_s)]^{-n}\int_{B(X_s,2^{-1}d(X_s))}|\mathrm{grad}\,u(y)|dy\right)ds$$
$$\leqq M_2|x-X_1|\left(\int_{G(X_t)}|\mathrm{grad}\,u(y)|\left(\int_{\{s;y\in B(X_s,2^{-1}d(X_s))\}}d(X_s)^{-n}ds\right)dy\right.$$
$$\leqq M_3\int_{G(X_t)}|\mathrm{grad}\,u(y)|d(y)^{1-n}dy,$$

where $X_t=x(t)$ and M_1, M_2, M_3 are positive constants which depend only on c and M. Thus the lemma is proved.

PROOF OF THEOREM 1. Let u be as in the theorem. For $\varepsilon>0$, set $G_\varepsilon=\{x\in G;\ d(x)>\varepsilon\}$. We assume that $K\subset G_{2\varepsilon}$ and $x\in G-G_\varepsilon$. Take a piecewise smooth curve $x(t)$ with conditions $(C_1)\sim(C_5)$, and let $t_0=\inf\{t;\ x(t)\in G_\varepsilon\}$. Then, in view of Lemma 1, we find $M_1>0$ such that

$$|u(x)-u(x(t_0))| \leqq M_1\int_{G(x(t_0))}|\mathrm{grad}\,u(y)|d(y)^{1-n}dy.$$

Since $d(y)\leqq d(x)+|x-y|<cd(y)$ whenever $y\in G(x(t))$, Hölder's inequality gives

$$|u(x)-u(x(t_0))|$$
$$\leqq M_1\left(\int_{G(x(t_0))}d(y)^{p'(1-n)-\alpha p'/p}\,dy\right)^{1/p'}F(t_0)$$
$$\leqq M_2\left(\int_0^d (d(x)+r)^{-p'(n-p+\alpha)/p-1}\,dr\right)^{1/p'}F(t_0)$$
$$\leqq M_3F(t_0)\times\begin{cases} d(x)^{-(n-p+\alpha)/p} & \text{if } n-p+\alpha>0,\\ [\log(1/d(x))]^{1/p'} & \text{if } n-p+\alpha=0,\\ d^{-(n-p+\alpha)/p} & \text{if } n-p+\alpha<0,\end{cases}$$

where $F(t_0)=\left(\int_{G(x(t_0))}|\mathrm{grad}\,u(y)|^p d(y)^\alpha dy\right)^{1/p}$, $1/p+1/p'=1$, $d=\sup\{|x-y|;\ x,y\in G\}$ and M_2, M_3 are positive constants independent of x. Consequently, in case $n-p+\alpha>0$, we obtain

$$\limsup_{x\to\partial G} d(x)^{(n-p+\alpha)/p}|u(x)|$$
$$\leqq M_3\left(\int_{G-G_{2\varepsilon}}|\mathrm{grad}\,u(y)|^p d(y)^\alpha dy\right)^{1/p},$$

which implies that the left hand side is equal to zero. The remaining cases can be treated similarly, and thus Theorem 1 is established.

Here we deal with the best possibility of Theorem 1 as to the order of convergence, when we restrict ourselves to the case G is a cone $\Gamma(a)=T_1(a)$.

PROPOSITION 1. *Let h be a nonincreasing positive function on the interval $(0, \infty)$ such that $\lim_{r\downarrow 0} h(r)=\infty$. Then there exists a nonnegative measurable function f such that*

$$\int_{\hat{\Gamma}(a)} f(y)^p |y_n|^\alpha dy < \infty$$

and

$$\limsup_{x\to 0, x\in\Gamma(a)} h(x_n)A(x_n)u(x) = \infty,$$

where $\hat{\Gamma}(a)=\{-y;\ y\in\Gamma(a)\}$, $A(x_n)=x_n^{(n-p+\alpha)/p}$ *if* $n-p+\alpha>0$, $A(x_n)=(\log(1/x_n))^{-1/p'}$ *if* $n-p+\alpha=0$, $A(x_n)=1$ *if* $n-p+\alpha<0$ *and* $u(x)=\int_{\hat{\Gamma}(a)}(x_n-y_n)|x-y|^{-n}f(y)dy$.

REMARK. If $-1<\alpha<p-1$, then, in view of [5; Lemma 1],

$$\int |\mathrm{grad}\, u(x)|^p |x_n|^\alpha dx \leqq M\int f(y)^p |y_n|^\alpha dy < \infty$$

with a positive constant M independent of f.

PROOF OF PROPOSITION 1. First we consider the case $n-p+\alpha=0$. Let $\{a_j\}$ be a sequence of positive integers such that $2a_j<a_{j+1}$, and take a sequence $\{b_j\}$ of positive numbers such that $\lim_{j\to\infty} b_j h(2^{-2a_j+1})=\infty$ and $\sum_{j=1}^\infty b_j^p<\infty$. We now define $f(y)=b_j|y|^{-1}(\log|y|^{-1})^{-1/p}$ if $y\in\hat{\Gamma}(a)\cap B(0, 2^{-a_j})-B(0, 2^{-2a_j})$ and $f=0$ otherwise. Then we see easily that the function u defined as in the proposition satisfies

$$\lim_{x\to 0, x\in A} h(x_n)A(x_n)u(x) = \infty$$

with $A=\cup_{j=1}^\infty\{x\in\Gamma(a);\ 2^{-2a_j}<|x|<2^{-2a_j+1}\}$. On the other hand we have $\int_{R^n} f(y)^p|y_n|^\alpha dy \leqq M\sum_{j=1}^\infty b_j^p<\infty$ with a positive constant M (cf. [5; Proof of Proposition 8]).

The case $n-p+\alpha\neq 0$ can be treated similarly, by suitably modifying the definition of f.

The boundedness of u is obtained under a weaker condition as stated below.

THEOREM 2. *Let g be a nonincreasing positive function on the interval $(0, \infty)$ such that $\int_0^1 g(r)^{1/(1-p)}r^{-1}dr<\infty$. Let u be a function which is harmonic in G and satisfies*

$$\int_G |\mathrm{grad}\, u(x)|^p g(d(x)) d(x)^{p-n} dx < \infty. \tag{2}$$

Then $u(x)$ is bounded on G.

In case $g(r)=r^{-\delta}$ with $\delta>0$, Theorem 2 is an immediate consequence of Theorem 1.

PROOF OF THEOREM 2. Let $x \in G$ and take a piecewise smooth curve $x(t)$ satisfying conditions $(C_1)\sim(C_5)$. In view of Lemma 1, we have

$$\begin{aligned}|u(x)-u(x(1))| &\leqq M_1 \int_{G(x(1))} |\mathrm{grad}\, u(y)| d(y)^{1-n} dy \\ &\leqq M_2 \Big(\int_{G(x(1))} |\mathrm{grad}\, u(y)|^p g(d(y)) d(y)^{p-n} dy\Big)^{1/p} \\ &\quad \times \Big(\int_{G(x(1))} g(d(y))^{-p'/p} d(y)^{-n} dy\Big)^{1/p'}\end{aligned}$$

with positive constants M_1 and M_2. Since $d(y)<d(x)+|x-y|<cd(y)$ for $y \in G(x(1))$,

$$\begin{aligned}&\int_{G(x(1))} g(d(y))^{-p'/p} d(y)^{-n} dy \\ &\leqq c^n \int_G g(d(x)+|x-y|)^{-p'/p} (d(x)+|x-y|)^{-n} dy \\ &\leqq M_3 \int_0^d g(d(x)+r)^{-p'/p} (d(x)+r)^{-1} dr \leqq M_3 \int_0^{2d} g(r)^{-p'/p} r^{-1} dr\end{aligned}$$

with a positive constant M_3. Thus the theorem is obtained.

PROPOSITION 2. *Let $\xi \in \partial G$, and assume that there exists a sequence $\{B(x_j, \delta_j)\}$ of balls such that $x_j \in G$, $\xi \in B(x_j, \delta_j)$ for each j, $\lim_{j\to\infty} \delta_j = 0$ and any $x \in G \cap B(x_j, \delta_j)$ is joined to x_j by a curve $x(t)$ in $G \cap B(x_j, \delta_j)$ satisfying conditions (C_3), (C_4) and (C_5) for some $c>0$. If u is a function harmonic in $G \cap B(\xi, r)$ and satisfying*

$$\int_{G\cap B(\xi,r)} |\mathrm{grad}\, u(x)|^p g(d(x)) d(x)^{p-n} dx < \infty \tag{2'}$$

for some $r>0$, where $d(x)$ denotes the distance of x from the boundary ∂G as before, then $u(x)$ has a finite limit as $x \in G$ tends to ξ.

PROOF. We may assume, without loss of generality, that $B(x_1, \delta_1) \subset B(\xi, r/2)$. Then, by Lemma 1 and the above proof we see that $\sup_{x\in G\cap B(x_j,\delta_j)} |u(x)-u(x_j)|$ tends to zero as $j\to\infty$. Hence, it follows that $\{u(x_j)\}$ is bounded. If

$\lim_{k\to\infty} u(x_{j_k}) = \ell$, then $u(x)$ tends to ℓ as $x \to \xi$, $x \in G$. Therefore the required assertion follows.

The following two results are easy consequences of Proposition 2.

COROLLARY 1. *Suppose G is a bounded Lipschitz domain in R^n. Let u be a function which is harmonic in G and satisfies* (2). *Then u is extended to a continuous function on $G \cup \partial G$.*

COROLLARY 2. *Let $G = T_\gamma(a) \cap B(0, 1)$. If u is a function harmonic in $G \cap B(0, r)$ and satisfying* (2)$'$ *with $\xi = 0$ for some $r > 0$, then $u(x)$ has a finite limit as $x \in G$ tends to the origin.*

PROPOSITION 3. *Let G be as in the above Corollary* 2. *If u is a function harmonic in G and satisfying $\int_G |\mathrm{grad}\, u(x)|^p g(x_n) x_n^{p-n} dx < \infty$ (which is a condition weaker than* (2)), *then $u(x)$ has a finite limit as x tends to the origin along $T_\gamma(b)$ for any b, $0 < b < a$.*

PROOF. For simplicity, write $G(a, r) = T_\gamma(a) \cap B(0, r)$. Let $0 < b < a$ and $x \in G(b, 1/2)$. For ε with $0 < \varepsilon < 1/8$, let $x_\varepsilon = (0, \varepsilon) \in T_\gamma(a)$. If $x(t) = (1-t)x + tx_\varepsilon$, $t \in [0, 1]$, then we can find b' such that $b < b' < a$ and $G(x(1)) \subset T_\gamma(b')$. Consequently, we have by Lemma 1

$$
\begin{aligned}
|u(x) - u(x_\varepsilon)| &\leqq M_1 \int_{G(b',4\varepsilon)} |\mathrm{grad}\, u(y)| y_n^{1-n} dy \\
&\leqq M_1 \left(\int_{G(b',4\varepsilon)} |\mathrm{grad}\, u(y)|^p g(y_n) y_n^{p-n}\, dy \right)^{1/p} \\
&\quad \times \left(\int_{G(b',4\varepsilon)} g(y_n)^{-p'/p} y_n^{-n} dy \right)^{1/p'} \\
&\leqq M_2 \left(\int_{G(b',4\varepsilon)} |\mathrm{grad}\, u(y)|^p g(y_n) y_n^{p-n} dy \right)^{1/p},
\end{aligned}
$$

since $M_3 y_n < d(y) < y_n$ for $y \in G(b', 1/2)$, where $b < b' < a$ and M_1 M_2, M_3 are positive constants. Hence it follows that u is bounded on $G(b, 1/2)$ and $\lim_{\varepsilon \downarrow 0} \sup_{x \in G(b,\varepsilon)} |u(x) - u(x_\varepsilon)| = 0$. If we take a sequence $\{\varepsilon_j\}$ of positive numbers such that $\varepsilon_j \to 0$ and $u(x_{\varepsilon_j}) \to \ell$ as $j \to \infty$, then $u(x)$ tends to ℓ as $x \to 0$ along $T_\gamma(b)$. Thus the required assertion follows.

This proposition gives the following result, which was already shown in [3; Theorem 6].

COROLLARY. *If u is a function harmonic in $\Gamma(a) \cap B(0, 1)$ and satisfying $\int_{\Gamma(a) \cap B(0,1)} |\mathrm{grad}\, u(x)|^p g(|x|) |x|^{p-n} dx < \infty$, then $u(x)$ has a finite limit as $x \to 0$ along $\Gamma(b)$, $0 < b < a$.*

REMARK 1. In the above Corollary we can not take $g(r)\equiv 1$. In fact, according to Remark 4 in [4], for given $\gamma>1$ we can find a function u on $D=\{(x_1,\dots,x_n)\in R^n;\ x_n>0\}$ satisfying the following conditions:

(i) u is harmonic in D.

(ii) $\int_{T_\gamma(a)} |\mathrm{grad}\, u(x)|^p x_n^{p-n} dx<\infty$.

(iii) u has a nontangential limit at 0.

(iv) $\limsup_{x\to 0, x\in T_\gamma(b)} u(x)=\infty$ for any b with $0<b<a$.

REMARK 2. In the Corollary to Proposition 3, u may fail to have a finite limit at 0 along $\Gamma(a)$. In fact, according to the proof of Theorem 8 in [3], we can find a nonnegative measurable function f such that $f=0$ on $\Gamma(a)$, $R_2 f(x)\equiv \int_{R^n} R_2(x-y)f(y)dy$ tends to ∞ as $x\to 0$ along $\Gamma(a)$ and $\int_{R^n} |\mathrm{grad}\ R_2\ f\ (x)|^p|x|^{p-n} dx<\infty$, where $R_2(x)=|x|^{2-n}$ in case $n\geqq 3$ and $R_2(x)=\log(1/|x|)$ in case $n=2$. Thus, if $\int_0^1 g(r)r^{p-1}dr<\infty$, then $u(x)=\sum_{j=1}^\infty(-1)^j R_2 f_j(x-x_j)$ is determined to satisfy the required conditions, where $\{x_j\}$ is a sequence of points on $\partial\Gamma(a)$ tending to 0 and $f_j=f$ on $B(0, r_j)$ and $f_j=0$ elsewhere.

3. Boundary limits of harmonic functions on $T_\gamma(a)$

In this section we are concerned with boundary limits at the origin for harmonic functions defined in $T_\gamma(a)$ and satisfying a condition weaker than (2).

THEOREM 3. *Let u be a function which is harmonic in $T_\gamma(a)\cap B(0, 1)$ and satisfies*

$$\int_{T_\gamma(a)\cap B(0,1)} |\mathrm{grad}\, u(x)|^p x_n^\alpha dx < \infty. \tag{3}$$

If $0<b<a$, then

$$\lim_{x\to 0, x\in T_\gamma(b)} A(x_n)u(x) = 0, \qquad \textit{in case } n-p+\alpha\geqq 0,$$

and

$$\lim_{x\to 0, x\in T_\gamma(b)} u(x) \textit{ exists and is finite}, \quad \textit{in case } n-p+\alpha<0,$$

where $A(x_n)$ is as in Proposition 1.

PROOF. Let $0<b<a$ and $x_\varepsilon=(0,\dots, 0, \varepsilon)$ with $0<\varepsilon<1/8$. As in the proof of Proposition 3, we can find b' such that $b<b'<a$ and for any $x\in T_\gamma(b)\cap B(0, 1/2)=G(b, 1/2)$,

$$|u(x)-u(x_\varepsilon)| \leqq M_1 \int_{G(b',4\varepsilon)} |\mathrm{grad}\, u(y)|d(y)^{1-n}dy$$

with a positive constant M_1 which is independent of x and ε. Since there exists $M_2>0$ such that $d(y)>M_2 y_n$ whenever $y \in G(b', 1/2)$, applying the proof of Theorem 1, we obtain

$$|u(x)-u(x_\varepsilon)| \leqq M_3 F(\varepsilon) \times \begin{cases} x_n^{-(n-p+\alpha)/p} & \text{if } n-p+\alpha>0, \\ [\log(1/x_n)]^{1/p'} & \text{if } n-p+\alpha=0, \\ \varepsilon^{-(n-p+\alpha)/p} & \text{if } n-p+\alpha<0, \end{cases}$$

where $F(\varepsilon)=\left(\int_{T_\gamma(a)\cap B(0,4\varepsilon)} |\mathrm{grad}\, u(y)|^p y_n^\alpha dy\right)^{1/p}$ and M_3 is a positive constant independent of x and ε. Thus, the case $n-p+\alpha \geqq 0$ is proved. The case $n-p+\alpha<0$ follows from Proposition 3.

REMARK 1. In Theorem 3, $A(x_n)u(x)$ may not have a finite limit as $x \to 0$ along $T_\gamma(a)$.

We shall give an example of such u in case $\gamma=1$. First we consider the case $n-p+\alpha>0$ and $p<n$. We shall show that there is a nonnegative measurable function f on R^n such that $f=0$ on $\Gamma(a)$, $\int_{R^n} f(y)^p |y_n|^\alpha dy<\infty$ and

(4) $$\limsup_{x\to 0, x\in P_\gamma} A(x_n)u(x)=\infty,$$

where $u(x)=\int_{R^n}(x_n-y_n)|x-y|^{-n}f(y)dy$ and $P_\gamma=\{x=(x', x_n); |x'|+|x'|^\gamma<ax_n\}$, $\gamma>1$. For this purpose, take a sequence $\{x^{(j)}\}$ of points in $\partial\Gamma(a)$ such that $|x^{(j)}|=2^{-j}$, and find a sequence $\{a_j\}$ of positive numbers such that $\limsup_{j\to\infty} ja_j=\infty$ and $\sum_{j=1}^\infty a_j^p<\infty$. We now define

$$f(y)=a_j 2^{j(n-p+\alpha)/p}|x^{(j)}-y|^{-1}$$

for $y \in B_j \equiv B(x^{(j)}, 2^{-j-2})-\Gamma(a)$; we also define $f(y)=0$ outside $\cup_{j=1}^\infty B_j$. Then it is easy to see that

$$\int f(y)^p |y_n|^\alpha dy \leqq \sum_{j=1}^\infty a_j^p 2^{j(n-p+\alpha)} \int_{B_j} |x^{(j)}-y|^{-p}|y_n|^\alpha dy \leqq M_1 \sum_{j=1}^\infty a_j^p<\infty$$

with a positive constant M_1. Further we have for t such that $0<t<2^{-j-3}$

$$u(x^{(j)}+(0, t)) \geqq M_2 \int_{\Gamma_j}(t+|x^{(j)}-y|)^{1-n} f(y)dy$$
$$\geqq M_3 a_j 2^{j(n-p+\alpha)/p} \log(2^{-j}/t),$$

where $\Gamma_j=\{y \in B_j; |(x^{(j)}-y)'|<a(x^{(j)}-y)_n\}$ and M_2, M_3 are positive constants independent of j and t. Hence if $\gamma>1$ and $t=2^{-j\gamma}$, then $u(x(t)) \geqq M_3(r-1)a_j \cdot j2^{j(n-p+\alpha)/p}$, from which (4) follows. In case $n-p+\alpha=0$ and $p<n$, the above

function u satisfies $u(x^{(j)}+(0, t)) \geqq M_4 a_j j^{1+\varepsilon}$ for $t=2^{-j}\exp(-j^{1+\varepsilon})$ with $\varepsilon>0$, so that

$$\limsup_{x\to 0, x\in Q_\varepsilon} A(x_n)u(x) = \infty, \tag{5}$$

where $Q_\varepsilon=\{(x', x_n);\ |x'|[1+\exp(-(\log|x'|^{-1})^{1+\varepsilon})]<ax_n\}$, if $\{a_j\}$ is taken so that $\limsup_{j\to\infty} j^\varepsilon a_j=\infty$.

Next we consider the case $p=n$. In this case, let $B_j=B(x^{(j)}, 2^{-j-2})-B(x^{(j)}, 2^{-2j-2})-\Gamma(a)$ and take $\{a_j\}$ such that $\limsup_{j\to\infty} ja_j=\infty$ and $\sum_{j=1}^{\infty} ja_j^p<\infty$. Then the function u defined as above satisfies (4) or (5) with $\varepsilon=1$ according as $n-p+\alpha>0$ or $n-p+\alpha=0$.

Finally, in case $p>n$, let $B_j=B(x^{(j)}, 2^{-j-2})-B(x^{(j)}, 2^{-j-3})-\Gamma(a)$ and take a sequence $\{a_j\}$ such that $\limsup_{j\to\infty} ja_j=\infty$ and $\sum_{j=1}^{\infty} a_j^p<\infty$. Then the same conclusion as above holds.

REMARK 2. Let u be a function which is harmonic in $\Gamma(a)\cap B(0, 1)$ and satisfies $\int_{\Gamma(a)\cap B(0,1)} |\mathrm{grad}\, u(x)|^p x_n^{p-n} dx<\infty$. Then $u(x)$ has a finite limit as $x\to 0$ along $\Gamma(b)$, $0<b<a$, if there exists a sequence $\{x^{(j)}\}$ having the following properties:

(i) $\{x^{(j)}\}\subset \Gamma(a')$ for some a' such that $0<a'<a$.

(ii) $x^{(j)}\to 0$ as $j\to\infty$.

(iii) $|x^{(j)}|<M|x^{(j+1)}|$ for any j, where $M>1$ is a constant.

(iv) $\{u(x^{(j)})\}$ has a finite limit as $j\to\infty$.

To prove this fact, it suffices to note the following fact as was seen in the proof of Theorem 3: if $x\in\Gamma(b)$, $0<b<a$, and $M^{-1}|x^{(j)}|\leqq|x|\leqq M|x^{(j)}|$, then

$$\begin{aligned}|u(x)-u(x^{(j)})| &\leqq M_1[(x^{(j)})_n]^{-n}\int_{\Gamma_j}|\mathrm{grad}\, u(y)|dy\\ &\leqq M_2\left(\int_{\Gamma_j}|\mathrm{grad}\, u(y)|^p y_n^{p-n}dy\right)^{1/p},\end{aligned}$$

where $\Gamma_j=\{y\in\Gamma(a);\ (2M)^{-1}|x^{(j)}|<|x|<(2M)|x^{(j)}|\}$ and M_1, M_2 are positive constants.

REMARK 3. According to Remark 2, if u is a function which is harmonic in $\Gamma(a)$ and satisfies $\int_{\Gamma(a)} |\mathrm{grad}\, u(x)|^p x_n^{p-n} dx<\infty$, then we have (cf. Jackson [2])

$$C(u, \ell_0) = C(u, \Gamma(b)) \quad \text{for any } b \text{ with } 0<b<a,$$

where $\ell_0=\{(0, t);\ t>0\}$ and $C(u, F)=\bigcap_{r>0} \mathrm{cl}\,\{u(x);\ x\in F,\ x_n<r\}$. Here $\mathrm{cl}\, E$ denotes the closure of a set E in R^n.

REMARK 4. The conclusions in Remarks 2 and 3 are not necessarily true if we replace $\Gamma(\cdot)$ by $T_\gamma(\cdot)$, $\gamma>1$, in view of Remark 1 given after the Corollary to Proposition 3.

Finally, in the two dimensional case, we give a result on the cluster sets for harmonic functions defined in the cone $\Gamma(a)$.

THEOREM 4. *Let $n=2$ and u be a function which is harmonic in $\Gamma(a)\cap B(0, 1)$ and satisfies* (3) *with $\gamma=1$ and $\alpha=p-2$. Then there exists a sequence $\{r_j\}$ having the following properties.*

(i) $2^{-j}<r_j<2^{-j+1}$.

(ii) *If $x^{(j)}\in\Gamma(a)\cap\partial B(0, r_j)$, then $C(u, \Gamma(b))=C(u, \{x^{(j)}\})$, for any b with $0<b<a$.*

In case $p=2$, Theorem 4 was proved by Bercovici, Foias and Pearcy [1].

PROOF OF THEOREM 4. Let $\tan\theta_0=a^{-1}$, $0<\theta_0<\pi/2$. By our assumption, we have

$$\infty > \iint_{\Gamma(a)} |\operatorname{grad} u(x_1, x_2)|^p x_2^{p-2} dx_1 dx_2$$
$$\geqq \int_0^1 \left(\int_{\theta_0}^{\pi-\theta_0} |(\partial/\partial\theta)u(r\cos\theta, r\sin\theta)|^p \sin^{p-2}\theta d\theta\right) r^{-1}dr.$$

Hence, setting $I_j=\inf\left\{\int_{\theta_0}^{\pi-\theta_0} |(\partial/\partial\theta)u(r\cos\theta, r\sin\theta)|d\theta;\ 2^{-j}<r<2^{-j+1}\right\}$, we see that $\sum_{j=1}^{\infty} I_j^p<\infty$. Let $\{r_j\}$ be a sequence such that $2^{-j}<r_j<2^{-j+1}$ and $\int_{\theta_0}^{\pi-\theta_0} |(\partial/\partial\theta)u(r_j\cos\theta, r_j\sin\theta)|d\theta<I_j+2^{-j}$. Let $e^{(j)}=(0, r_j)$ and $x^{(j)}\in\partial B(0, r_j)\cap\Gamma(a)$. Then we have

$$|u(x^{(j)})-u(e^{(j)})| \leqq I_j + 2^{-j} \quad \text{for any} \quad j.$$

Hence it follows that $C(u, \{x^{(j)}\})=C(u, \{e^{(j)}\})$. As in Remarks 2 and 3 after Theorem 3, we can prove that $C(u, \Gamma(b))=C(u, \{e^{(j)}\})$ for any b with $0<b<a$. Thus the theorem is proved.

REMARK. Let $n=2$ and u be a function which is harmonic in the half ball $D\cap B(0, 1)$ and satisfies $\int_{D\cap B(0,1)} |\operatorname{grad} u(x)|^p|x|^{p-2}dx<\infty$. Then, in view of the proof of Theorem 4, we can find a sequence $\{r_j\}$ satisfying (i) in Theorem 4 and

(ii)′ $C(u, \{x^{(j)}\})=C(u, \Gamma(a))$ for any $a>0$ and any $\{x^{(j)}\}$ such that $x^{(j)}\in D\cap\partial B(0, r_j)$.

References

[1] H. Bercovici, C. Foias and C. Pearcy, A spectral mapping theorem for functions with finite Dirichlet integral, J. Reine Angew. Math. **366** (1986), 1–17.

[2] H. L. Jackson, On the boundary behavior of BLD functions and some applications, Acad. Roy. Berg. Bull. Cl. Sci. (5) **66** (1980), 223–239.

[3] Y. Mizuta, On the radial limits of potentials and angular limits of harmonic functions, Hiroshima Math. J. **8** (1978), 415–437.

[4] Y. Mizuta, On the boundary limits of harmonic functions with gradient in L^p, Ann. Inst. Fourier **34** (1984), 99–109.

[5] Y. Mizuta, Boundary behavior of p-precise functions on a half space of R^n, this issue, 73–94.

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