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Convergence of sum product of a martingale difference sequence

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1. Introduction

Let $\{x_k(t)\}_{k\in \boldsymbol{N}}$ ($\boldsymbol{N}$ is the collection of all natural numbers) be a sequence of complex functions on [0, 1] such that $x_k(t)+1\neq 0$ for every $t\in[0, 1]$. Then the convergence of the product $\prod_k(1+x_k(t))$ has been investigated by many authors in connection with the convergence of the sum $\sum_k x_k(t)$. For example G. H. Hardy [1] proved that if $\{a_k\}$ is a sequence of positive numbers which converges monotonically to zero and $\sum_k a_k^n$ diverges for every $n\in \boldsymbol{N}$, then $\prod_k(1+a_k e^{2\pi ikt})$ diverges for every rational number t. J. E. Littlewood [2] proved that if $\{a_k\}$ is a sequence of positive numbers converges monotonically to zero, then $\prod_k(1+a_k e^{2\pi ikt})$ converges for every irrational number t with possible exception of the Liouville numbers. In the measure theoretical point of view, L. Carleson's theorem implies that if $\sum_k |a_k|^2<+\infty$, then $\prod_k(1+a_k e^{2\pi ikt})$ converges almost surely. All of these discussions concerned the convergence or the divergence of $\sum_k a_k e^{2\pi ikt}$.

The author investigated this problem from the probabilistic point of view and proved in [4] that if $\{X_k\}$ is a sequence of independent random variables with mean zero such that $1+X_k>0$, a.s., for every k, then the almost sure convergence of $\prod_k(1+X_k)$ is equivalent to that of $\sum_k X_k$. In this paper we shall extend this result to a martingale difference sequence and prove the following theorem.

THEOREM 1. *Let $\{X_k, \mathscr{B}_k\}$ be a martingale difference sequence such that $X_k+1>0$, a.s., for every k. Then $\prod_k(1+X_k)$ converges almost surely if and only if $\sum_k X_k$ converges almost surely.*

As an application we shall give a new criterion for the absolute continuity of locally equivalent measures.

2. Proof of Theorem 1

A sequence of random variables $\{X_k\}$ is a *submartingale difference sequence* iff X_k is $\mathscr{B}_k$-measurable and

$$E[X_{k+1}\,|\,\mathscr{B}_k]\geq 0,\quad \text{a.s.},$$

and *martingale difference sequence* iff

$$E[X_{k+1} \mid \mathscr{B}_k] = 0, \quad \text{a.s.},$$

for every k. Before proving Theorem 1, we remark several lemmas.

LEMMA 1 (A. N. Shiryaev [5], VII–5–Theorem 5). *Let $\{X_k\}$ be a submartingale difference sequence such that*

$$|X_k| < K, \quad \text{a.s.}, \quad k \in \boldsymbol{N},$$

for a positive constant K and assume that $\sum_k X_k$ converges almost surely. Then we have

$$\textstyle\sum_k E[X_{k+1}^2 \mid \mathscr{B}_k] < +\infty, \quad \text{a.s.}.$$

LEMMA 2. *Let $\{X_k\}$ be a submartingale difference sequence such that*

$$X_k < K, \quad \text{a.s.}, \quad k \in \boldsymbol{N},$$

for a positive constant K and assume that $\sum_k X_k$ converges almost surely. Then $\sum_k X_k^2$ converges almost surely.

PROOF. For any $k \in \boldsymbol{N}$ define

$$Y_k = \begin{cases} X_k, & \text{if} \quad |X_k| < K, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have for any $k \in \boldsymbol{N}$, $|Y_k| < K$ and

$$E[Y_{k+1} \mid \mathscr{B}_k] \geq E[X_{k+1} \mid \mathscr{B}_k] \geq 0, \quad \text{a.s.},$$

so that $\{Y_k\}$ is a submartingale difference sequence. Therefore by Lemma 1 we have

$$\textstyle\sum_k E[Y_{k+1}^2 \mid \mathscr{B}_k] < +\infty, \quad \text{a.s.}, \tag{1}$$

and consequently

$$\textstyle\sum_k E[Y_{k+1}^4 \mid \mathscr{B}_k] \leq K^2 \sum_k E[Y_{k+1}^2 \mid \mathscr{B}_k] < +\infty, \quad \text{a.s.}. \tag{2}$$

On the other hand, since $\sum_k X_k$ converges almost surely, we have

$$\textstyle\sum_k P(Y_{k+1} \neq X_{k+1} \mid \mathscr{B}_k) = \sum_k P(|X_{k+1}| > K \mid \mathscr{B}_k) < +\infty, \quad \text{a.s.}, \tag{3}$$

and by W. Stout [6], Theorem 2–8–8, (1), (2) and (3) imply the almost sure convergence of $\sum_k X_k^2$. Q. E. D.

PROOF OF THEOREM 1. Let $\{X_k, \mathscr{B}_k\}$ be a martingale difference sequence such that $X_k+1>0$, a.s., for every k. Then $Z_k=\prod_{j=1}^{k}(1+X_j)$ is a positive martingale and we have $\sup_k \boldsymbol{E}[Z_k]=1<+\infty$. Therefore $Z_\infty=\lim_{k\to+\infty} Z_k\geq 0$ exists almost surely.

Assume that $\sum_k X_k$ converges almost surely. Then, since $\{-X_k\}$ is a sub-martingale difference sequence such that $-X_k<1$, a.s., for every k, by Lemma 2 we have $\sum_k X_k^2<+\infty$, a.s.. Therefore by the Cauchy's principle (see also H. Sato [3], Lemma 8), we have

$$Z_\infty = \prod_{k=1}^{+\infty}(1+X_k) > 0, \quad \text{a.s.}. \tag{4}$$

Conversely, assume that $Z_\infty>0$, a.s., and define $U_k=X_kZ_{k-1}$, and $v_k=(Z_{k-1})^{-1}$, $k\in \boldsymbol{N}$. Then (4) implies that

$$\sup_k v_k = (\inf_k Z_k)^{-1} < +\infty, \quad \text{a.s.},$$

and we have

$$\sup_k \boldsymbol{E}[|\textstyle\sum_{j=1}^{k} U_k|] = \sup_k \boldsymbol{E}[Z_k] = 1 < +\infty.$$

Therefore, since v_k is $\mathscr{B}_{k-1}$-measurable, by Burkholder's theorem (W. Stout [6], Theorem 2–9–4)

$$\textstyle\sum_k X_k = \sum_k v_k U_k$$

converges almost surely. Q. E. D.

3. Absolute continuity of locally equivalent masures

In this section, we shall prove the following theorem.

THEOREM 2. *Let $(\Omega, \mathscr{B})$ be a measurable space, $\{\mathscr{B}_k\}$ be an increasing sequence of σ-algebras which generates $\mathscr{B}$, P and Q be probability measures on $(\Omega, \mathscr{B})$, P_k and Q_k be the restrictions of P and Q to $\mathscr{B}_k$, $k\in \boldsymbol{N}$, respectively, and assume that for any $k\in \boldsymbol{N}$ P_k and Q_k are mutually absolutely continuous (denoted by $P_k\sim Q_k$). Then P is absolutely continuous with respect to Q if and only if*

$$\sum_k \left(\frac{dQ_{k+1}}{dP_{k+1}} - \frac{dQ_k}{dP_k}\right)\Big/\frac{dQ_k}{dP_k}$$

converges almost surely (P).

PROOF. Define

$$Z_k = \frac{dQ_k}{dP_k}, \quad k\in \boldsymbol{N},$$

$$X_1 = Z_1 - 1,$$

$$X_k = \frac{Z_k - Z_{k-1}}{Z_{k-1}}. \quad k = 2, 3, 4, \ldots.$$

In the discussions below, the mathematical expectation and the conditional expectation are always taken with respect to P.

Obviously, for any $k \in \boldsymbol{N}$ Z_k and X_k are $\mathscr{B}_k$-measurable and we have

$$Z_k = \prod_{j=1}^{k} (1+X_j), \quad \boldsymbol{E}[X_{k+1} \mid \mathscr{B}_k] = 0, \quad \text{a.s.}\,(P),$$

and

$$X_k + 1 > 0, \quad \text{a.s.}\,(P).$$

Therefore $\{Z_k\}$ is a martingale and $\{X_k\}$ is a martingale difference sequence. By Doob's theorem (W. Stout [6], Theorem 2–7–2) we have

$$P(0 \leq Z_\infty = \lim_k Z_k < +\infty) = 1.$$

and by A. N. Shiryaev [5], VII–6–Theorem 1,

$$Q(A) = \int_A Z_\infty dP + Q(A \cap \{Z_\infty = +\infty\}), \quad A \in \mathscr{B}. \tag{5}$$

Then it is easy to show that P is absolutely continuous with respect to Q if and only if $Z_\infty > 0$, a.s. (P), and the remaining part of the proof is obvious from Theorem 1. Q. E. D.

References

[1] G. H. Hardy, A note on the continuity or discontinuity of a function defined by an infinite product, Proc. London Math. Soc. (2), **7** (1909), 4–48.

[2] J. E. Littlewood, On a class of conditionally convergent infinite products, Proc. London Math. Soc. (2), **8** (1910), 195–199.

[3] H. Sato, Characteristic functional of a probability measure absolutely continuous with respect to a Gaussian Radon measure, J. Func. Anal., **61** (1985), 222–245.

[4] H. Sato, On the convergence of the product of independent random variables (to appear in J. Math. Kyoto Univ.).

[5] A. N. Shiryaev, Probability, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1984).

[6] W. F. Stout, Almost sure convergence, Academic Press, New York, San Francisco and London, 1974.

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