HIROSHIMA MATH. J.
18 (1988), 31–46

On oscillatory properties of the first order differential equations with one or two retarded arguments

YU. I. DOMSHLAK and A. I. ALIEV

(Received November 20, 1986)

1. Introduction

Oscillatory properties of first order differential equations with retarded argument have been studied for a long time, beginning with the fundamental investigations of A. D. Myshkis. The following result is due to him:

THEOREM 1 (A. D. Myshkis [1]). *If $\tau(t)\leqq t$ and there exist positive constants a_0 and τ_0 such that*

$$a(t) \geqq a_0, \quad t - \tau(t) \geqq \tau_0, \quad a_0\tau_0 > 1/e,$$

then all solutions of the equation

$$x'(t) + a(t)x(\tau(t)) = 0 \tag{1}$$

are oscillatory, that is, every solution has a sequence of zeros tending to infinity.

Soon it has become clear that the absence of nonoscillatory solutions of equation (1) shold be defined not so much by the behavior of the coefficient $a(t)$ itself, but by its averaging $\int_{\tau(t)}^{t} a(s)ds$. That is why the following step has been made in [2].

THEOREM 2 (G. Ladde [2]). *Let the following conditions hold:*

$$a(t) \geqq 0, \quad t \geqq t_1, \tag{2}$$

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} a(s)ds > 1/e. \tag{3}$$

Then all solutions of equation (1) *are oscillatory.*

The conditions of both theorems are best possible in the sense that the autonomous equation $x'(t)+ax(t-\tau)=0$ has nonoscillatory solutions if $a\tau\leqq 1/e$.

An important question arises: Is the nonnegativity of the coefficient $a(t)$ in Theorem 2 essential, or does this requirement appear only in connection with the method of proof?

The nonnegativity condition on $a(t)$ in theorems on oscillatory properties of all solutions of (1) was partially discarded apparently for the first time in [3, 4]. However, it was still necessary to demand that $a(t)$ be nonnegative on a sequence of intervals of length of order $t-\tau(t)$. Meanwhile it is desirable to get conditions ensuring oscillation of all solutions of (1) in such a way that would admit rapid oscillation of the coefficient $a(t)$. This problem is solved at the end of this paper.

We note that the desired result regarding the equation (1) with one retarded argument can be obtained only after investigating oscillatory properties of the equation with two retarded arguments, which is the basic contents of the present paper.

Consider the equation

$$x'(t) - a_1(t)x(\tau_1(t)) + a_2(t)x(\tau_2(t)) = 0, \tag{4}$$

where $a_2(t) \geqq 0$, and $-a_1(t)$ and $a_2(t)$ may have different signs.

In what follows we suppose that $\tau_i'(t) > 0$, $i=1, 2$, and $q_i(t)$, $i=1, 2$, are the inverse functions of $\tau_i(t)$: $\tau_i(q_i(t)) \equiv t$, $i=1, 2$, and also $\tau_2(t) \leqq \tau_1(t) \leqq t$.

Remark 1. The condition $\tau_2(t) \leqq \tau_1(t)$ is imposed without loss of generality. Indeed, if it does not hold, then it suffices to introduce

$$\bar{\tau}_1(t) = \max\{\tau_1(t), \tau_2(t)\}, \quad \bar{\tau}_2(t) = \min\{\tau_1(t), \tau_2(t)\}$$

and change the coefficients in the following manner:

$$\bar{a}_i(t) = \begin{cases} a_i(t) & \text{for} \quad \tau_1(t) \leqq \tau_2(t), \\ a_j(t) & \text{for} \quad \tau_1(t) > \tau_2(t), \end{cases} \qquad i, j = 1, 2, \quad i \neq j.$$

The problem of oscillation of solutions of such equations was first considered in [7] and [8] (for the case of constant delays $\tau_i(t) \equiv t - \tau_i$ and for the case of almost constant coefficients $a_i(t)$).

Two types of conditions for oscillation of solutions of equation (4) are given in the present paper. They become necessary and sufficient for the autonomous equation

$$x'(t) - a_1 x(t-\tau_1) + a_2 x(t-\tau_2) = 0, \quad \tau_2 \geqq \tau_1 > 0. \tag{5}$$

These conditions are derived from the bounds on sign preserving intervals of the solution of equation (4), obtained below. Such bounds for solutions of equation (4) are apparently given for the first time here.

We emphasize the important fact that it is possible to formulate conditions for the coefficients in the integral form as well. It follows therefore that Theorems 5 and 7 of the present paper are the development of Theorem 2, while Lemma 2.1 of [8] and our Theorems 4 and 6 can be considered as the development of Theorem 1.

The authors wish to underline the importance of the remarks dispersed throughout the paper, and the reader is requested to pay special attention to them.

2. Sturmian comparison theorem

One of the authors gave in [4] Sturmian comparison theorems for first and second order equations with several deviating arguments. In [9] these theorems are given in the most general form, where the functions $t-\tau_i(t)$ can change their sign. However, the nonnegativity of all those coefficients was stipulated at least on certain intervals. That is why none of these theorems is applicable to equation (4), if we wish to cover the case when not only $a_2(t)\geqq 0$ but also $a_1(t)\geqq 0$. A variant of the Sturmian comparison theorem free of this defect is provided below.

Sometimes for the sake of simplicity we use the notation $f_1f_2(t)$ for the superposition $f_1(f_2(t))$ of the functions $f_1(t)$ and $f_2(t)$.

LEMMA 1. *Let*

$$\tau_2(t) \leqq \tau_1(t) \leqq t, \quad \tau_2(t_2) > t_1, \tag{6}$$

and suppose that the function $x(t)$ *is defined on* $(\tau_2 q_1(t_1),\, q_1(t_2))$ *and absolutely continuous on* $(q_1(t_1),\, q_2(t_2))$, *the function* $y(t)$ *is defined on* $(\tau_1(t_1),\, q_1(t_2))$ *and absolutely continuous on* $(t_1,\, t_2)$, *the functions* $a_1(t)$, $\tilde{a}_1(t)$ *and* $a_2(t)$ *are piecewise continuous on* $(q_1(t_1),\, q_1(t_2))$, *and* $\tilde{a}_2(t)$ *is piecewise continuous on* $(q_2(t_1),\, q_2(t_2))$.

Then the following identity holds:

$$\begin{aligned}
&\int_{q_1(t_1)}^{q_1(t_2)} \{x'(t)-a_1(t)x(\tau_1(t))+a_2(t)x(\tau_2(t))\}y(\tau_1(t))dt \\
&= \int_{t_1}^{t_2} \{-\tau_1'(t)y'(\tau_1(t))-q_1'(t)\tilde{a}_1(q_1(t))y(t)+q_2'(t)\tilde{a}_2(q_2(t))y(\tau_1 q_2(t))\}x(t)\,dt \\
&\quad + x(q_1(t_2))y(t_2) - x(q_1(t_1))y(t_1) + \int_{\tau_1(t_2)}^{t} x(q_1(s))y'(s)ds \\
&\quad - \int_{\tau_1(t_2)}^{t_2} x(q_1(s))y'(s)ds + \int_{t_1}^{t_2} q_1'(s)\{\tilde{a}_1(q_1(s))-a_1(q_1(s))\}x(s)y(s)ds \\
&\quad + \int_{\tau_1 q_2(t_1)}^{t_2} q_1'(s)\{a_2(q_1(s))-\tilde{a}_2(q_1(s))\}x(\tau_2 q_1(s))y(s)ds \\
&\quad + \int_{t_1}^{\tau_1 q_2(t_1)} q_1'(s)a_2(q_1(s))x(\tau_2 q_1(s))y(s)ds \\
&\quad - \int_{t_2}^{\tau_1 q_2(t_2)} q_1'(s)\tilde{a}_2(q_1(s))x(\tau_2 q_1(s))y(s)ds.
\end{aligned} \tag{7}$$

The proof is carried out by means of the following equalities:

$$\int_{q_1(t_1)}^{q_1(t_2)} x'(t)y(\tau_1(t))dt = x(q_1(t_2))y(t_2) - x(q_1(t_1))y(t_1) + \int_{\tau_1(t_1)}^{t_1} x(q_1(s))y'(s)ds$$
$$- \int_{\tau_1(t_2)}^{t_2} x(q_1(s))y'(s)ds - \int_{t_1}^{t_2} \tau_1'(t)y'(\tau_1(t))x(t)dt;$$
$$- \int_{q_1(t_1)}^{q_1(t_2)} a_1(t)x(\tau_1(t))y(\tau_1(t))dt = \int_{t_1}^{t_2} \{\tilde{a}_1(q_1(t)) - a_1(q_1(t))\}x(t)y(t)q_1'(t)dt$$
$$- \int_{t_1}^{t_2} \tilde{a}_1(q_1(t))x(t)y(t)q_1'(t)dt;$$
$$\int_{q_1(t_1)}^{q_1(t_2)} a_2(t)x(\tau_2(t))y(\tau_1(t))dt = \int_{t_1}^{t_2} q_1'(s)\tilde{a}_2(q_2(s))y(\tau_1 q_2(s))x(s)ds$$
$$+ \int_{t_1}^{\tau_1 q_2(t_1)} q_1'(s)a_2(q_1(s))x(\tau_2 q_1(s))y(s)ds$$
$$- \int_{t_2}^{\tau_1 q_2(t_2)} q_1'(s)\tilde{a}_2(q_1(s))x(\tau_2 q_1(s))y(s)ds$$
$$+ \int_{\tau_1 q_2(t_1)}^{t_2} q_1'(s)\{a_2(q_1(s)) - \tilde{a}_2(q_1(s))\}x(\tau_2 q_1(s))y(s)ds.$$

We now consider the differential inequality

$$(8) \quad -\tau_1'(t)y'(\tau_1(t)) - q_1'(t)\tilde{a}_1(q_1(t))y(t) + q_2'(t)\tilde{a}_2(q_2(t))y(\tau_1 q_2(t)) \geqq 0$$

and formulate the Sturmian comparison theorem.

THEOREM 3. *Let the following conditions hold*:

1) (6) *holds*;
2) *there exists* $y(t)$ *with the properties*

$$(9) \quad \begin{aligned} &y(t_1) = y(t_2) = 0; \quad y(t) > 0, \quad t \in (t_1, t_2); \\ &y(t) \leqq 0, \quad t \in (t_2, \tau_1 q_2(t_2)); \\ &y'(t) \geqq 0, \quad t \in (\tau_1(t_1), t_1); \quad y'(t) \leqq 0, \quad t \in (\tau_1(t_2), t_2) \end{aligned}$$

and satisfying (8) *on* (t_1, t_2) *with*

$$(10) \quad \tilde{a}_2(t) \geqq 0, \quad t \in (q_1(t_2), q_2(t_2));$$

3) *the following inequalities hold*:

$$(11) \quad a_1(t) \leqq \tilde{a}_1(t), \quad t \in (q_1(t_1), q_1(t_2)); \quad a_2(t) \geqq \begin{cases} 0, & t \in (q_1(t_1), q_2(t_1)), \\ \tilde{a}_2(t), & t \in (q_2(t_1), q_1(t_2)); \end{cases}$$

4) *at least one of the inequalities* (11) *becomes strict on some subinterval.*

Then any solution $x(t)$ *of equation* (4) *has at least one zero on* $(\tau_2 q_1(t_1), q_1(t_2))$.

The proof is based on identity (7). Indeed, suppose that there exists a solution $x(t)$ of equation (4), preserving the (positive) sign on $(\tau_2 q_1(t_1), q_1(t_2))$. Then, in view of the conditions of Theorem 3, all the terms of the right-hand side of (7) are nonnegative, and at least one is strictly positive. At the same time, the left-hand side of (7) vanishes.

It should not present any difficulties in extending this theorem to the case of several retarded arguments.

The next lemma describes a collection of inequalities (8), the coefficients of which contain two arbitrary functions. All these inequalities admit solutions $y(t)$ with properties (9) and (10) of Theorem 3.

LEMMA 2. *Let* (6) *hold. Let* $\varphi(t)$ *and* $k(t)$ *be continuous functions on* $(\tau_1(t_1), \tau_1 q_2 q_1(t_1))$ *such that*

$$(12) \qquad 0 < \int_{t_1}^{t} \varphi(s)ds < \pi, \quad t \in (t_1, t_2); \quad \int_{t_1}^{t_2} \varphi(s)ds = \pi;$$

$$(13) \qquad 0 < \int_{\tau_2(t)}^{t} \varphi(s)ds < \pi, \quad t \in (q_2\tau_1(t_1), \tau_1 q_2 q_1(t_2))\,;$$

$$(14) \qquad k(t) \leqq \varphi(t) \operatorname{cosec} \int_{\tau_2 q_1(t)}^{q_1(t)} \varphi(s)ds, \quad t \in (\tau_1\tau_2 q_1(t_2), \tau_1(t_2)).$$

Further let in (8)

$$(15) \qquad (-1)^i q_i'(t)\tilde{a}_i(q_i(t))$$

$$\equiv \tau'(t)\, \frac{\varphi(\tau_1(t)) - (-1)^i k(\tau_1(t)) \sin \int_{\tau_1(t)}^{\tau_i q_2(t)} \varphi(s)ds}{\sin \int_{\tau_1(t)}^{t} \varphi(s)ds + \sin \int_{\tau_1(t)}^{\tau_1 q_2(t)} \varphi(s)ds} \exp\Big(-\int_{\tau_1(t)}^{\tau_1 q_i(t)} m(s)ds\Big)$$

$$i = 1, 2,$$

where

$$m(t) \equiv$$

$$\frac{\varphi(t)\cos\Big(\frac{1}{2}\int_t^{q_1(t)} \varphi(s)ds + \frac{1}{2}\int_t^{\tau_1 q_2 q_1(t)} \varphi(s)ds\Big) + k(t) \sin \frac{1}{2}\int_{q_1(t)}^{\tau_1 q_2 q_1(t)} \varphi(s)ds}{\sin\Big(\frac{1}{2}\int_t^{q_1(t)} \varphi(s(ds + \frac{1}{2}\int_t^{\tau_1 q_2 q_1(t)} \varphi(s)ds\Big)},$$

and also

$$(16) \qquad m(t)\sin\int_{t_1}^{t} \varphi(s)ds + \varphi(t)\cos\int_{t_1}^{t} \varphi(s)ds \begin{cases} \geqq 0, & t \in (\tau_1(t_1), t_1), \\ \leqq 0, & t \in (\tau_1(t_2), t_2). \end{cases}$$

Then inequality (8) *has a solution* $y(t)$ *with properties* (9), *where* (10) *holds.*

The proof proceeds by directly checking that

$$y(t) = \exp\left(\int_{t_1}^{t} m(s)ds\right)\cdot \sin \int_{t_1}^{t} \varphi(s)ds$$

is a solution of inequality (8). In view of (13) all the expressions above have meaning, in view of (14), (10) is fulfilled, and in view of (12) and (6), (9) is true.

3. Bounds for intervals between the consecutive zeros of the solutions

The following theorem, arising from Lemma 2 and Theorem 3, gives upper bounds for the intervals of consecutive zeros of solutions of equation (4) if restrictions of type (11) are imposed on the coefficients $a_i(t)$.

THEOREM 4. *Suppose that:*
1) *$\varphi(t)$ and $k(t)$ satisfy the conditions of Lemma* 2;
2) (11) *holds, where $\tilde{a}_i(t)$, $i=1, 2$, are defined in* (15);
3) *condition* 4 *of Theorem* 3 *holds.*
Then the statement of Theorem 3 *is valid.*

We now consider a very important special case of Theorem 4 in which

$$\varphi(t) \equiv 2\nu, \quad k(t) \equiv k, \quad t_2 = t_1 + \frac{\pi}{2\nu}.$$

COROLLARY 4.1. *If*

$$\tau_2 > \tau_1 > 0, \quad 0 < \nu < \frac{\pi}{2\tau_2}, \ k \leqq \frac{2\nu}{\sin 2\nu\tau_1}, \tag{17}$$

and

$$(-1)^i a_i(t) \geqq \psi_i(\nu), \quad i, j = 1, 2, \quad i \neq j, \tag{18}$$

where

$$\psi_i(\nu) \equiv \frac{2\nu-(-1)^i k \sin 2\nu\tau_j}{\sin 2\nu\tau_1+\sin 2\nu\tau_2} \exp\left(-\tau_i \frac{2\nu \cos \nu(\tau_1+\tau_2)+k \sin \nu(\tau_2-\tau_1)}{\sin \nu(\tau_1+\tau_2)}\right),$$

then any solution of the equation

$$x'(t) - a_1(t)x(t-\tau_1) + a_2(t)x(t-\tau_2) = 0 \tag{19}$$

has at least one zero on any interval (t_1, t_2) with $|t_2-t_1|>\tau_2+\pi/(2\nu)$.

We omit the proof. Note that inequalities (16) and (14) reduce to restriction (17) in the special case under consideration.

REMARK 2. In the next section it will be shown that restrictions (18) are asymptotically sharp, that is, the right-hand side of (18) cannot be replaced by their limiting values as $\nu \to 0$.

It is possible to consider Theorem 4 as the development of Theorem 1, because restrictions of type (11) are imposed on the coefficients. At the same time an extension of Theorem 2 to equation (4) is also very important. This means that one should get bounds for the distances between successive zeros of solutions by way of imposing conditions not on $a_i(t)$ themselves, but on their averages $\int_{\tau_j(t)}^{t} a_i(s)ds$, $i, j=1, 2$.

THEOREM 5. *Suppose that $\tau_2(t) \to \infty$ as $t \to \infty$ and*

$$a_1(t) \geqq 0, \quad a_2(t) \geqq 0, \tag{20}$$

and that there exist functions $b_i(t)$, $i=1, 2$, such that the following conditions hold:

1)

$$b_1(t) \geqq a_1(t) \geqq 0, \quad t \geqq t_0\,; \tag{21}$$

$$0 \leqq b_2(t) \leqq a_1(t), \quad t \geqq t_0\,; \tag{22}$$

2) *there exist the finite limits*

$$\lim_{t\to\infty} \int_{\tau_i(t)}^{t} b_j(s)ds = c_{ij}, \quad i, j = 1, 2; \tag{23}$$

3) *the system*

$$\begin{cases} (c_{11}c_{22}-c_{12}c_{21})x_1x_2 - c_{11}x_1 + c_{22}x_2 - 1 = 0 \\ -c_{11}x_1 + c_{12}x_2 - \log x_1 < 0 \\ -c_{21}x_1 + c_{22}x_2 - \log x_2 > 0 \end{cases} \tag{24}$$

has a solution $\{x_1, x_2\}$ with $x_1>0$, $x_2>0$;

4)

$$\alpha_2 x_2 \int_{q_2q_1(t_1)}^{q_2q_2(t_2)} b_2(s)ds - \alpha_1 x_1 \int_{q_1q_1(t_1)}^{q_1q_1(t_2)} b_1(s)ds \geqq \frac{\pi}{2\nu}. \tag{25}$$

Here $0<\nu<\nu_0$ satisfies the inequalities

$$\varphi_1(\nu) > 1, \quad \varphi_2(\nu) < 1, \tag{26}$$

where

$$\varphi_i(\nu) \equiv \frac{\nu x_i}{\sin \nu - \cos \nu(\alpha_2-\alpha_1)} \exp\left(- \frac{2\nu\alpha_i}{\tan \nu} + \frac{\sin \nu(\alpha_2-\alpha_1)}{\sin \nu} (x_i c_{i1} + x_2 c_{i2})\right),$$

and $\{\alpha_1, \alpha_2\}$ *is a solution of the system*

$$(27)\qquad \begin{cases} (1+x_1c_{11})\alpha_1 - x_2c_{12}\alpha_2 = 0 \\ x_1c_{21}\alpha_1 + (1-x_2c_{22})\alpha_2 = 0 \\ \alpha_1 + \alpha_2 = 1 \end{cases}$$

which is solvable in view of (24_1).

Then any solution of equation (4) *has a zero on the interval* $(\tau_2(t_1), t_2)$ *provided* t_1 *is sufficiently large.*

PROOF. Put in (15)

$$(28)\qquad \begin{aligned} \varphi(t) &\equiv 2\nu\alpha_2x_2q_1'(t)q_1'(q_1(t))b_2(q_2q_1(t)) \\ &\quad - 2\nu\alpha_1x_1q_1'(t)q_1'(q_1(t))b_1(q_1q_1(t)), \\ k(t) &\equiv -x_1q_1'(t)q_1'(q_1(t))b_1(q_1q_1(t)) \\ &\quad - x_2q_2'(t)q_2'(q_2(t))b_2(q_2q_1(t)). \end{aligned}$$

This is admissible because of the imposed condition 4, since the restrictions on $\varphi(t)$ and $k(t)$ of Lemma 2 are satisfied. Then, taking into account the fact that

$$\int_{\tau_i(t)}^t q_1'(s)q_j'(q_1(s))b_j(q_jq_1(s))ds = \int_{q_jq_1\tau_1(t)}^{q_jq_1(t)} b_j(\xi)d\xi,$$

we have in view of (27)

$$(29)\qquad \lim_{t\to\infty}\int_{\tau_i(t)}^t \varphi(s)ds = 2\nu\alpha_2x_2c_{12} - 2\nu\alpha_1x_1c_{i1} = 2\nu\alpha_i,$$

$$(30)\qquad \lim_{t\to\infty}\int_{\tau_i(t)}^t k(s)ds = -x_1c_{i1} - x_2c_{i2}, \quad i = 1, 2.$$

From (28) we obtain

$$(31)\qquad \begin{aligned} \tau_1'(t)\{\varphi(\tau_1(t))+2\nu\alpha_2k(\tau_1(t))\} &= -2\nu x_1q_1'(t)b_1(q_1(t)), \\ \tau_1'(t)\{\varphi(\tau_1(t))-2\nu\alpha_1k(\tau_1(t))\} &= 2\nu x_2q_2'(t)b_2(q_2(t)). \end{aligned}$$

Inequalities (24_2) and (24_3) are equivalent, respectively, to the inequalities

$$\begin{aligned} x_1\exp(c_{11}x_1-c_{12}x_2) &> 1, \\ x_2\exp(c_{21}x_1-c_{22}x_2) &< 1. \end{aligned}$$

It is not difficult to calculate that

$$\lim_{\nu\to 0}\varphi_i(\nu) = x_i\exp(c_{i1}x_1-c_{i2}x_2), \quad i = 1, 2.$$

Therefore, we see that (26) holds for $0<\nu<\nu_0$, provided ν_0 is sufficiently small, which shows in particular the consistency of the conditions of the theorem.

With consideration of (21), from (26) we have for $i=1, 2$

$$(32) \qquad (-1)^i q_i'(t) a_i(q_i(t)) >$$
$$(-1)^i \frac{\nu x_i q_i'(t) b_i(q_i(t))}{\sin \nu \cdot \cos \nu(\alpha_2-\alpha_1)} \exp\Big(-\frac{2\nu\alpha_i}{\tan \nu} + \frac{\sin \nu(\alpha_2-\alpha_1)}{\sin \nu}(x_1 c_{i1} + x_2 c_{i2})\Big).$$

On the other hand,

$$\lim_{t\to\infty} \frac{\exp\Big(-\int_{\tau_1(t)}^{\tau_1 q_i(t)} m(s)ds\Big)}{\sin \int_{\tau_1(t)}^{t} \varphi(s)ds + \sin \int_{\tau_1(t)}^{\tau_1 q_2(t)} \varphi(s)ds} = \varphi_i(\nu), \quad i=1, 2.$$

Therefore, it follows from (31), (32), (15) that (11) holds for t sufficiently large, and this completes the proof.

REMARK 3. In fact, a stronger statement holds. It is possible to impose conditions (21) and (22) not on the whole semi-axis, but on the set $G \equiv \bigcup_{k=1}^{\infty} (\tau_2(t_1^{(k)}), q_2(t_2^{(k)}))$, where $(t_1^{(k)}, t_2^{(k)})$ are intervals of type (25). In this case, $\lim_{t\to\infty}$ should be replaced by $\lim_{t\to\infty, t\in G}$ in (23); outside G no limitation is imposed on the coefficients $a_i(t)$. The details will be omitted.

4. Theorems on oscillation of all solutions

It turns out that theorems on oscillation of all solutions can be obtained from the bounds for intervals of consecutive zeros of solutions, which are derived in Theorems 4 and 5. They are obtained as the limiting one for $\nu \to 0$ from Theorems 4 and 5.

THEOREM 6. *Suppose that*

$$(33) \qquad \tau_2 > \tau_1 > 0, \quad m \leqq 1/\tau_1,$$

$$(34) \qquad a_1(t) \leqq a_1^0 < \frac{1-m\tau_2}{\tau_2-\tau_1} e^{-m\tau_1}, \quad t > t_0,$$

$$(35) \qquad a_2(t) \geqq a_2^0 > \frac{1-m\tau_1}{\tau_2-\tau_1} e^{-m\tau_2}, \quad t > t_0.$$

Then all solutions of equation (19) *are oscillatory.*

PROOF. The validity of this statement follows from Corollary 4.1. Indeed, letting $m \equiv [2+k(\tau_2-\tau_1)]/(\tau_1+\tau_2)$, we have

$$
\begin{aligned}
\lim_{\nu\to 0} \psi_1(\nu) &= \frac{1+k\tau_2}{\tau_1+\tau_2}\exp\left(-\frac{2+k(\tau_2-\tau_1)}{\tau_1+\tau_2}\tau_1\right)\\
&= -\frac{1-m\tau_2}{\tau_2-\tau_1}e^{m\tau_1},\\
\lim_{\nu\to 0} \psi_2(\nu) &= \frac{1-k\tau_2}{\tau_1+\tau_2}\exp\left(-\frac{2+k(\tau_2-\tau_1)}{\tau_1+\tau_2}\tau_2\right)\\
&= -\frac{1-m\tau_1}{\tau_2-\tau_1}e^{-m\tau_2}.
\end{aligned}
\tag{36}
$$

The restriction $k \leqq 1/\tau_1$ implies $m \leqq 1/\tau_1$. But if (34) and (35) are valid, then (18) holds for sufficiently small $\nu > 0$.

REMARK 4. It is possible to impose restrictions (34) and (35) not on the whole of the semi-axis, but on $G = \bigcup_{k=1}^{\infty}(t_k-\tau_2,\ t_k+\tau_2+(\pi/2\nu))$ if $a_1^0 < -\psi_1(\nu)$, $a_2^0 > \psi_2(\nu)$. Then there are no restrictions imposed on $a_i(t)$ outside G.

COROLLARY 6.1. *Let*

$$\tau_2 > \tau_1 > 0; \quad a_1(t) \leqq a_1^0, \quad a_2(t) \geqq a_2^0 \geqq 0, \quad t > t_0 \tag{37}$$

and suppose that the characteristic quasi-polynomial

$$F(\lambda) \equiv \lambda - a_1^0 e^{-\lambda\tau_1} + a_2^0 e^{-\lambda\tau_2} \tag{38}$$

has no real roots. Then all solutions of equation (10) *are oscillatory.*

PROOF. Consider the boundary of the "oscillation zone" on the plane $\{a_1^0, a_2^0\}$:

$$\bar{a}_1 = \frac{1-m\tau_2}{\tau_2-\tau_1}e^{-m\tau_1}, \quad \bar{a}_2 = \frac{1-m\tau_1}{\tau_2-\tau_1}e^{-m\tau_2}, \quad m \leqq 1/\tau_1. \tag{39}$$

It is not difficult to see that it coincides with the boundary of the zone $F(\lambda) > 0$, $F'(\lambda) = 0$, which is equivalent to the absence of real roots of $F(\lambda)$. Indeed, we have

$$\lambda - \bar{a}_1 e^{-\lambda\tau_1} + \bar{a}_2 e^{-\lambda\tau_2} = 0, \quad 1 + \bar{a}_1\tau_1 e^{-\lambda\tau_1} - \bar{a}_2\tau_2 e^{-\lambda\tau_2} = 0$$

if and only if

$$\bar{a}_1 = \frac{1+\lambda\tau_2}{\tau_2-\tau_1}e^{\lambda\tau_1}, \quad \bar{a}_2 = \frac{1+\lambda\tau_1}{\tau_2-\tau_1}e^{\lambda\tau_2},$$

and further we assume that $m = -\lambda$.

COROLLARY 6.2. *For all solutions of the autonomous equation* (5) *to be oscillatory, it is necessary and sufficient that its characteristic quasi-polynomial* (38) *should have no real roots.*

PROOF. Sufficiency follows from Corollary 4.1. Conversely, if $F(\lambda_1)=0$, $\operatorname{Im}\lambda_1=0$, then $x(t)=\exp(\lambda_1 t)$ is a solution of (5).

REMARK 5. This statement solves completely the problem of oscillation of all solutions of equation (19) in terms of restrictions of type (11). It is not possible to improve the statement of Corollary 6.2 in these frames. The matter is that no restrictions are placed a priori on the sign of the coefficients a_1, a_2 in Corollary 6.2. From the conditions it follows immediately that $a_2 \geqq 0$. Indeed, if $a_2<0$, then $\lim_{\lambda\to-\infty} F(\lambda)=\infty$, $\lim_{\lambda\to\infty} F(\lambda)=-\infty$, and therefore $F(\lambda)$ has a real root. As for a_1, it may be either positive or negative.

THEOREM 7. *Suppose that*

$$\tau_2(t) \leqq \tau_1(t) \leqq t, \quad \tau_2(t) \to \infty, \quad a_i(t) \geqq 0, \quad i = 1, 2,$$

and conditions 1, 2, 3 *of Theorem* 5 *hold. Then all solutions of equation* (4) *are oscillatory.*

We will show that for equation (19) this theorem turns into the statement of Corollary 6.1 with the additional restriction $a_1^0>0$ (Note that the case $-a_1^0 \geqq 0$, $a_2^0 \geqq 0$ is simpler and has been considered in [10]). Indeed, let $\tau_i(t)=t-\tau_i$, $\tau_2>\tau_1>0$, and (37) hold with $a_1^0>0$. Then $c_{ij}=\tau_i a_j^0$, $i, j=1, 2$. Therefore the system (24) takes the form

$$\begin{cases} \tau_1 a_1^0 x_1 - \tau_2 a_2^0 x_2 + 1 = 0 \\ \tau_1(-a_1^0 x_1 + a_2^0 x_2) - \log x_1 < 0 \\ \tau_2(-a_1^0 x_1 + a_2^0 x_2) - \log x_2 > 0. \end{cases}$$

Setting $a_i^0 x_i = z_i$, we have an equivalent system

$$\text{(40)} \qquad \begin{cases} \tau_1 z_1 - \tau_2 z_2 + 1 = 0 \\ \tau_1(-z_1+z_2) - \log z_1 + \log a_1^0 < 0 \\ \tau_2(-z_1+z_2) - \log z_2 + \log a_2^0 > 0. \end{cases}$$

Evidently, if the system (40) is solved for some pair $\{a_1^0, a_2^0\}$, then it is also solved for the pair $\{a_1, a_2\}$ such that $0<a_1<a_1^0$, $a_2>a_2^0$. Therefore, the set M of pairs $\{a_1, a_2\}$ for which (40) is solvable is a part of the first quadrant of the plane $\{a_1, a_2\}$ and as its boundary has the set of such points $\{\bar{a}_1, \bar{a}_2\}$ for which the following system is solvable

$$\text{(41)} \qquad \begin{cases} \tau_1 z_1 - \tau_2 z_2 + 1 = 0 \\ \tau_1(-z_1+z_2) - \log z_1 + \log \bar{a}_1 = 0 \\ \tau_2(-z_1+z_2) - \log z_2 + \log \bar{a}_2 = 0. \end{cases}$$

It is not difficult to see that if λ_0 is a solution of the equation

$$F'(\lambda_0) \equiv 1 + a_1^0\tau_1 e^{-\lambda_0\tau_1} - a_2^0\tau_2 e^{-\lambda_0\tau_2} = 0,$$

then $z_i = a_i^0 e^{-\lambda_0\tau_i}$, $i=1, 2$, is a solution of system (40), with its second and third equations coinciding with $F(\lambda_0)=0$. Thus, the set M coincides with the set of those pairs $\{a_1, a_2\}$ for which the quasi-polynomial $F(\lambda)$ has no real roots. This completes the proof.

Example 1. Consider the equation

$$x'(t) - cx(t-\tau(t)) + dx(t-1) = 0, \tag{42}$$

where $c>0$, $d>0$, $\tau(t)\leqq 1$, $\tau(t)\geqq 0$ and $\tau(t)\to 0$ as $t\to\infty$.

Setting $b_1(t)\equiv a_1(t)\equiv c$, $b_2(t)\equiv a_2(t)\equiv d$, we have

$$c_{11} = \lim_{t\to\infty}\int_{t-\tau(t)}^{t} cds = 0, \quad c_{12} = 0, \quad c_{21} = c, \quad c_{22} = d.$$

System (24) turns into

$$-dx_2 + 1 = 0, \quad -\log x_2 < 0, \quad -cx_1 + dx_2 - \log x_2 > 0,$$

which is solved if and only if

$$de^{-c} > 1/e. \tag{43}$$

Thus, as one should have expected, the condition of oscillation of all solutions of (42) is the same as for the equation

$$y'(t) - cy(t) + dy(t-1) = 0.$$

Example 2. Consider the case $\tau_2 = 2\tau_1 > 0$, for which the solvablity criterion of the system (24) could be written in a convenient form. Without loss of generality, we assume that $\tau_1 = 1$, and consider the equation

$$x'(t) - q(t)x(t-1) + p(t)x(t-2) = 0, \tag{44}$$

where $p(t)\geqq 0$, $q(t)\geqq 0$. Suppose that $q_0(t)\geqq q(t)$, $0\leqq p_0(t)\leqq p(t)$, and there exist the finite limits

$$\lim_{t\to\infty}\int_{t-1}^{t} q_0(s)ds = q, \quad \lim_{t\to\infty}\int_{t-1}^{t} p_0(s)ds = p.$$

Then, $c_{21} = 2c_{11} = 2q$, $c_{22} = 2c_{12} = 2p$. If $p=0$, then system (24) is incompatible. If $p>0$, then it will take the form

$$\begin{cases} qx_1 - 2px_2 + 1 = 0 \\ -qx_1 + px_2 - \log x_1 < 0 \\ -2qx_1 + 2px_2 - \log x_2 > 0, \end{cases}$$

which is equivalent to

$$(45) \qquad -qx_1 + 1 - 2\log x_1 < 0, \quad -qx_1 + 1 - 2\log \frac{qx_1+1}{2p} > 0.$$

It is possible to show (we omit the proof) that a necessary and sufficient condition for the system (45) to be solvable is

$$(46) \qquad \frac{4p - q^2 - q(q^2+8p)^{1/2}}{8p} - \log \frac{q+(q^2+8p)^{1/2}}{4p} > 0.$$

Therefore, if (46) holds, then all solutions of (44) are oscillatory.

This condition cannot be improved, because it is necessary and sufficient for the characteristic quasi-polynomial of the corresponding equation

$$x'(t) - qx(t-1) + px(t-2) = 0$$

to have no real roots.

5. Equations with one delay argument and quick-oscillating coefficients

Theorems 5 and 7 and their corollaries could be applied to the investigation of oscillatory properties of equation (1), when the condition (2) is not imposed a priori.

Consider equation (1), and represent $a(t)$ in the form

$$(47) \qquad a(t) = a_+(t) - a_-(t), \quad a_+(t) \geqq 0, \quad a_-(t) \geqq 0.$$

For example,

$$a_+(t) = \frac{1}{2}\{|a(t)| + a(t)\}, \quad a_-(t) = \frac{1}{2}\{|a(t)| - a(t)\}.$$

Then equation (1) can be rewritten in the form

$$(48) \qquad x'(t) - a_-(t)x(\tau(t)) + a_+(t)x(\tau(t)) = 0,$$

that is, in the form (4) with $\tau_1(t) \equiv \tau_2(t)$, and Theorems 5 and 7 can be applied to this special case.

THEOREM 8. *Suppose that $\tau(t) \to \infty$ as $t \to \infty$ and that for some representation of $a(t)$ in the form* (47), *there are continuous functions $a_-^0(t)$, $a_+^0(t)$ such that*

(49) $$a_-^0(t) \geqq a_-(t), \quad 0 \leqq a_+^0(t) \leqq a_+(t), \quad t \geqq t_0,$$

and the exist the finite limits

$$\lim_{t\to\infty} \int_{\tau(t)}^t a_-^0(s)ds = \beta_-, \quad \lim_{t\to\infty} \int_{\tau(t)}^t a_+^0(s)ds = \beta_+$$

with

(50) $$\beta_+ - \beta_- > 1/e.$$

Then any solution of equation (1) *has at least one zero on* $(\tau(t_1), t_2)$ *for sufficiently large* t_1, *provided*:

a) *in the case of* $\beta_-=0$

(51) $$\int_{qq(t_1)}^{qq(t_2)} \left\{\frac{1}{\beta_+} a_+^0(t) - (e+\delta)a_-^0(t)\right\} dt > \frac{\pi}{\nu}, \quad 0 < \nu < \nu_0,$$

where

(52) $$\beta_+ > \frac{\nu}{\sin \nu} \exp(-\nu \cot \nu) \equiv D_\nu, \quad \delta > 0;$$

b) *in the case of* $\beta_->0$

(53) $$\int_{qq(t_1)}^{qq(t_2)} \{x_2 a_+^0(t) - x_1 a_-^0(t)\}dt > \frac{\pi}{\nu},$$

where $x_1, x_2>0$ *are such that*

(54) $$\beta_+ x_2 - \beta_- x_1 = 1, \quad x_2 < 1/D_\nu.$$

PROOF. It is not difficult to see that

$$c_{11} = c_{21} = \beta_-, \quad c_{12} = c_{22} = \beta_+.$$

Therefore the system (24) turns into

$$\begin{cases} \beta_- x_1 - \beta_+ x_2 + 1 = 0 \\ -\beta_- x_1 + \beta_+ x_2 - \log x_1 < 0 \\ -\beta_- x_1 + \beta_+ x_2 - \log x_2 > 0, \end{cases}$$

which is equivalent to

(55) $$-\beta_- x_1 + \beta_+ x_2 = 1, \quad 0 < x_2 < e < x_1.$$

The latter is solvable if and only if (50) holds. This is evident for $\beta_-=0$. One can verify this for $\beta_->0$ by noticing that all solutions of the system (55) are written in the form

$$x_1 = e + (1-\mu)\,\frac{(\beta_+ - \beta_-)e - 1}{\beta_-}, \quad x_2 = e - \mu\,\frac{(\beta_+ - \beta_-)e-1}{\beta_+}, \quad 0 < \mu < 1.$$

Furthermore, $\varphi_i(v) \equiv x_i D_v$, $i=1, 2$, and system (27) has the only solution $\alpha_1 = \alpha_2 = 1/2$ both in the case of $\beta_- = 0$ and of $\beta_- > 0$. The requirement (26) is satisfied in view of (52) for $\beta_- = 0$ and in view of (54) for $\beta_- > 0$, and (25) turns into (51) and (53), respectively. This proves the theorem.

Note that, for the special case of $a_-(t) \equiv a_-^0(t) \equiv 0$, the assertion of Theorem 8 can be found in [4].

COROLLARY 8.1. *Under the conditions of Theorem* 8 *all solutions of equation* (1) *are oscillatory.*

EXAMPLE 3. Consider the equation

$$x'(t) + \{2A \sin^2 n\pi t - Bt^{-\alpha} \sin^2 \omega\pi t\} x(t-1) = 0, \tag{56}$$

where A, B and α are positive constants and n is a positive integer.

Setting $a_+(t) \equiv a_+^0(t) \equiv 2A \sin^2 n\pi t$, $a_-(t) \equiv a_-^0(t) \equiv Bt^{-\alpha} \sin^2 \omega\pi t$, we have $\beta_+ = A$, $\beta_- = 0$; consequently, if $A > 1/e$, then all solutions of (56) are oscillatory on the basis of Corollary 8.1.

Note that if ω is irrational, then $a(t)$ is necessarily oscillating. Since n and ω may be arbitrarily large, oscillation rapidity may be arbitrarily high.

It is not difficult to give an example with $\liminf_{t\to\infty} a(t) < 0$ and even $\liminf_{t\to\infty} a(t) = -\infty$.

It would be appropriate to state here the hypotheses which the authors believe are valid.

Hypothesis 1. *The condition* (2) *of Theorem* 2 *could be neglected wihout any offset.*

The statement of Corollary 8.1 is a good approximation to this hypothesis.

It seems however that even a more general hypothesis is valid.

Hypothesis 2. *Suppose that* $\tau_2(t) \leqq \tau_1(t) \leqq t$, $\tau_2(t) \to \infty$ *as* $t \to \infty$, *and* $a_i(t) \geqq 0$, $i=1, 2$. *Denote* $A_{ij}(t) \equiv \int_{\tau_i(t)}^{t} a_j(s)ds$, $i, j = 1, 2$. *If, for any* $t \geqq t_0$ *and for some* $\delta > 0$, *the following system is solvable*

$$\begin{cases} (A_{11}A_{22} - A_{12}A_{21})x_1x_2 - A_{11}x_1 + A_{22}x_2 - 1 = 0 \\ -A_{11}x_1 + A_{12}x_2 - \log x_1 < -\delta \\ -A_{21}x_1 + A_{22}x_2 - \log x_2 > \delta, \end{cases}$$

then all solutions of equation (4) *are oscillatory.*

REMARK 6. It is obvious that the expression of equation (1) in the form of (48) (with a view to involve oscillating $a(t)$) is meaningful only when Theorems 5 and 7 are used. At the same time, Theorem 4, Theorem 6, Corollary 4.1 and, what is more, Lemma 2.1 from the paper by Kreith and Ladas [8] do not accomplish this end. Indeed it follows from (15) or from (18) that if $\tau_1(t)\equiv\tau_2(t)$, then $\tilde{a}_2(t)\geqq\tilde{a}_1(t)$, which, in turn, implies $a_+(t)\geqq a_-(t)$.

References

[1] A. D. Myshkis, Linear Differential Equations with Reatrded Argument, Nauka, Moscow, 1972 (Russian).

[2] G. S. Ladde, Oscillation caused by retarded perturbations of first order linear ordinary differential equations, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **63** (1977), 351–359.

[3] Y. I. Domshlak, Sturmian comparison method and its application to localization of zeros of solutions of differential equations with delay argument, **1**, Izv. Akad. Nauk Azerbaidzan SSR Ser. Fiz.-Tehn. Mat. Nauk, No. 5 (1979), 120–127 (Russian).

[4] Y. I. Domshlak, Sturmian comparison method and its application to localization of zeros of solutions of differential equations with delay argument, **2**, VINITI (USSR), 2578–79 (Russian), 18s.

[5] Y. I. Domshlak, Sturmian Comparison Method in Investigation of Behavior of Solutions of Differential-Operator Equations, Elm, Baku, 1986, 154s., (Russian).

[6] Y. I. Domshlak and A. I. Aliev, On theorems of Myshkis-Tramov type in investigation of oscillatory properties of functional-differential equations, Preprint: Institute of Physics, Akad. Sci. Azerbai. SSR, 1985 (Russian).

[7] G. Ladas and Y. Sficas, Oscillations of differential equations with positive and negative coefficients, Proc. Intern. Conf. Qualit. Theory Diff. Eq., Univ. Alberta, June, 1984.

[8] K. Kreith and G. Ladas, Allowable delays for positive diffusion processes, Hiroshima Math. J. **15** (1985), 437–443.

[9] Y. I. Domshlak, Sturmian comparison theorems for differential equations of first and second order with sign-variable deflections of an argument, Ukrain. Mat. Ž. **34** (1982), 158–163 (Russian).

[10] Y. I. Domshlak, Oscillatory properties of solutions of first order differential equations with two retarded arguments, Trans. "SAPR and ASPR in Land-Reclamation", Nalchik, 1983, pp. 101–111 (Russian).

Institute of Mathematics and Mechanics,
Azerbaidzan Academy of Sciences,
Baku, USSR