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Dirichlet integral and energy of potentials on harmonic spaces with adjoint structure

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Introduction

On a self-adjoint harmonic space, we can establish Green's formulae, which give relations between Dirichlet integral and energy of potentials (see [6; Part II]). On the other hand, for the potential theory with respect to the heat equation, relations between Dirichlet integral and energy have not been completely clarified; on a cylindrical domain, such relations have been discussed by M. Pierre [7], [8] in the framework of "parabolic Dirichlet space", and also some results can be found in the investigations of parabolic capacities (see, e.g., [4]).

The purpose of the present paper is to obtain such relations on a P-harmonic space having an adjoint structure. We shall establish a sort of Green's formula for continuous potentials with finite energy and show that the Dirichlet integral is majorized by the energy for such potentials.

In the last section, we shall investigate the case of the heat equation on a cylindrical domain $X=\Omega\times(0, T)$ ($\Omega\subset \boldsymbol{R}^d$, $d\geq 1$) and show that continuous heat potentials with finite energy on X belong to the class $L^2(0, T; H_0^1(\Omega))\cap L^\infty(0, T; L^2(\Omega))$, which is a space considered in [7], [8] (also, cf. [4]).

§1. Mutually adjoint harmonic structures

Let X be a connected locally compact space with a countable base, and suppose two harmonic sheaves $\mathscr{H}$ and $\mathscr{H}^*$ (or hyperharmonic sheaves $\mathscr{U}$ and $\mathscr{U}^*$) are given so that $(X, \mathscr{H})$ and $(X, \mathscr{H}^*)$ (or, $(X, \mathscr{U})$ and $(X, \mathscr{U}^*)$) are both P-harmonic spaces in the sense of Constantinescu-Cornea [2]. The set of all continuous potentials with respect to $\mathscr{H}$ (resp. $\mathscr{H}^*$) is denoted by $\mathscr{P}_C$ (resp. $\mathscr{P}_C^*$). We say that $\mathscr{H}$ and $\mathscr{H}^*$ are *mutually adjoint* if there exists a function (called an associated Green function) $G(x, y)$: $X\times X\to[0, +\infty]$ satisfying the following conditions:

(G.0) $G(x, y)$ is lower semicontinuous on $X\times X$ and continuous off the diagonal set;

(G.1) For each $y\in X$, $G(\cdot, y)$ is a potential for $\mathscr{H}$ and is harmonic for $\mathscr{H}$ on $X\setminus\{y\}$;

(G*.1) For each $x \in X$, $G(x, \cdot)$ is a potential for $\mathscr{H}^*$ and is harmonic for $\mathscr{H}^*$ on $X \setminus \{x\}$;

(G.2) For any $p \in \mathscr{P}_C$, there is a unique non-negative measure μ on X such that $p = G\mu$ $(G\mu(x) = \int G(x, y) d\mu(y))$;

(G*.2) For any $p^* \in \mathscr{P}_C^*$, there is a unique non-negative measure ν on X such that $p^* = G^*\nu$ $(G^*\nu(y) = \int G(x, y) d\nu(x))$.

REMARK 1.1. Given $(\mathscr{H}, \mathscr{H}^*)$, the Green function is unique up to a multiplicative constant.

REMARK 1.2. Existence of a Green function implies the proportionality axiom (cf. [1]) and Doob's convergence property (cf. [9]) for both $\mathscr{H}$ and $\mathscr{H}^*$.

Hereafter, we assume that a pair $(\mathscr{H}, \mathscr{H}^*)$ of mutually adjoint harmonic structures is given and let $G(x, y)$ be the associated Green function. For simplicity, we further assume that the constant function 1 is superharmonic for both $\mathscr{H}$ and $\mathscr{H}^*$. Since our assumptions are symmetric with respect to $\mathscr{H}$ and $\mathscr{H}^*$, every assertion relative to $\mathscr{H}$ has its counterpart (the dual statement) relative to $\mathscr{H}^*$. Thus in many cases, we shall state results only for $\mathscr{H}$, leaving the formulation of their dual statements to the reader.

We denote by $\mathscr{M}_{PC}^+$ the set of all non-negative measures μ on X such that $G\mu \in \mathscr{P}_C$. By (G.0) and (G.1), we easily see that if $\mu \in \mathscr{M}_{PC}^+$ and μ' is a non-negative measure on X such that $\mu' \leq \mu$, then $\mu' \in \mathscr{M}_{PC}^+$. The following lemma is a consequence of (G.2):

LEMMA 1.1. *Let $\mu_1, \mu_2 \in \mathscr{M}_{PC}^+$. If $(G\mu_1 - G\mu_2)|_U \in \mathscr{H}(U)$ for an open set U in X, then $\mu_1|_U = \mu_2|_U$.*

For a signed measure ν such that $|\nu| \in \mathscr{M}_{PC}^+$, we write $G\nu$ for $G\nu^+ - G\nu^-$, which is a continuous function on X.

Let $\mathscr{R}$ (resp. $\mathscr{R}^*$) be the sheaf of functions which are locally expressible as differences of two continuous $\mathscr{H}$- (resp. $\mathscr{H}^*$-) superharmonic functions. By Lemma 1.1 and [2; Theorem 2.3.2], given $f \in \mathscr{R}(U)$ (U: open) there exists a uniquely determined signed measure $\sigma(f)$ on U such that $|\sigma(f)||_V \in \mathscr{M}_{PC}^+$ and $(f - G[\sigma(f)|_V])|_V \in \mathscr{H}(V)$ for any relatively compact open set V with $\bar{V} \subset U$. It is easy to see that σ defines a sheaf morphism $\mathscr{R} \to \mathscr{M}$ (=the sheaf of signed measures on X) such that $\sigma(f) \geq 0$ on U if and only if f is $\mathscr{H}$-superharmonic on U; namely σ is a measure representation for $(X, \mathscr{H})$ as defined in [6]. If $\mu \in \mathscr{M}_{PC}^+$ and $h \in \mathscr{H}(X)$, then $\sigma(G\mu + h) = \mu$. Note that $\sigma(1) \geq 0$ by assumption. Likewise, we obtain a measure representation σ^*: $\mathscr{R}^* \to \mathscr{M}$ for $(X, \mathscr{H}^*)$.

LEMMA 1.2. *Let s be a continuous $\mathscr{H}$-superharmonic function on X such that $\sigma(s)(X) < +\infty$. Then $\sigma(s) \in \mathscr{M}_{PC}^+$ and $s - G(\sigma(s))$ is the greatest harmonic minorant of s.*

PROOF. Let $\mu=\sigma(s)$, let $\{X_n\}$ be an exhaustion of X and let $\mu_n=\mu|_{X_n}$. Then $\mu_n\in\mathcal{M}_{PC}^+$ and $(s-G\mu_n)|_{X_n}\in\mathcal{H}(X_n)$. For any compact set K in X_n, $\alpha=\sup_{x\in K, y\in\partial X_n} G(x, y)<+\infty$ by (G.0). By (G*.1) and the assumption that 1 is $\mathcal{H}^*$-superharmonic, we see that $G(x, y)\leq\alpha$ for all $x\in K$ and $y\in X\setminus X_n$ (cf. [6; Proposition 2.5]). Hence $G(\mu-\mu_n)$ is bounded on K, which implies that $G(\mu-\mu_n)|_{X_n}\in\mathcal{H}(X_n)$. Since these are true for all n, it follows that $\mu\in\mathcal{M}_{PC}^+$ and $s-G\mu\in\mathcal{H}(X)$. Since $G\mu$ is an $\mathcal{H}$-potential, $s-G\mu$ is the greatest harmonic minorant of s.

Let $C_0(X)$ be the set of all continuous functions with compact support in X, and write $\mathcal{R}_0(X)=\mathcal{R}(X)\cap C_0(X)$, $\mathcal{R}_0^*(X)=\mathcal{R}^*(X)\cap C_0(X)$. Also, we write $\mathcal{Q}_C=\mathcal{P}_C-\mathcal{P}_C$ and $\mathcal{Q}_C^*=\mathcal{P}_C^*-\mathcal{P}_C^*$. For $\varphi\in\mathcal{R}_0(X)$, $\sigma(\varphi)$ has compact support, so that $\varphi=G(\sigma(\varphi))\in\mathcal{Q}_C$. Hence, $\mathcal{R}_0(X)\subset\mathcal{Q}_C$. Similarly, $\mathcal{R}_0^*(X)\subset\mathcal{Q}_C^*$. Thus, if $f\in\mathcal{R}_0(X)$ and $f^*\in\mathcal{R}_0^*(X)$, then

$$\int f\,d\sigma^*(f^*)=\int f^*d\sigma(f). \tag{1.1}$$

PROPOSITION 1.1. *Equality* (1.1) *holds for $f\in\mathcal{R}(X)$ and $f^*\in\mathcal{R}_0^*(X)$, or for $f\in\mathcal{R}_0(X)$ and $f^*\in\mathcal{R}^*(X)$.*

PROOF. Let $f\in\mathcal{R}(X)$ and $f^*\in\mathcal{R}_0^*(X)$. By [6; Proposition 2.17], there exists $\varphi\in\mathcal{R}_0(X)$ such that $\varphi=1$ on a neighborhood V of $\mathrm{Supp}\,f^*$. Then $f\varphi\in\mathcal{R}_0(X)$ ([6; Corollary 2.7]), $f\varphi=f$ on $\mathrm{Supp}\,f^*$ and $\sigma(f\varphi)|_V=\sigma(f)|_V$. Hence

$$\int f\,d\sigma^*(f^*)=\int f\varphi\,d\sigma^*(f^*)=\int f^*d\sigma(f\varphi)=\int f^*d\sigma(f).$$

The case $f\in\mathcal{R}_0(X)$ and $f^*\in\mathcal{R}^*(X)$ is similar.

COROLLARY 1.1. *For $u\in\mathcal{R}(X)$, it is harmonic on X if and only if $\int u\,d\sigma^*(\psi)=0$ for all $\psi\in\mathcal{R}_0^*(X)$.*

PROOF. Since $\mathcal{R}_0^*(X)$ is dense in $C_0(X)$ (cf. [2; Theorem 2.3.1]), the corollary immediately follows from the proposition.

As in [2; pp. 39–40], we denote by R (resp. R^*) the reduction operator with respect to $\mathcal{H}$ (resp. $\mathcal{H}^*$), namely, for a non-negative function f on X,

$$Rf=\inf\{u\in\mathcal{U}\mid u\geq f \text{ on } X\}\quad(\text{resp. } R^*f=\inf\{v\in\mathcal{U}^*\mid v\geq f \text{ on } X\}).$$

By [2; Proposition 2.2.3] (also see [6; Proposition 2.6]), $Rf\in\mathcal{P}_C$ if f (≥ 0) is continuous and dominated by an $\mathcal{H}$-potential; in particular, if $f\in C_0(X)$. Furthermore, $\mathrm{Supp}\,\sigma(Rf)\subset\mathrm{Supp}\,f$.

LEMMA 1.3. *If $\mu_1, \mu_2\in\mathcal{M}_{PC}^+$ and $G\mu_1\leq G\mu_2$, then $\mu_1(X)\leq\mu_2(X)$.*

PROOF. If $\{\varphi_n\}$ is a monotone increasing sequence of functions in $C_0^+(X)$ such that $\varphi_n(x)\to 1$ for any $x \in X$, then $\{R^*\varphi_n\}$ is a monotone increasing sequence in $\mathscr{P}_C^*$ such that $R^*\varphi_n(x)\to 1$ for any $x\in X$, since we assumed that 1 is $\mathscr{H}^*$-superharmonic. Hence, if we put $\mu_n^*=\sigma^*(R^*\varphi_n)$, $n=1, 2,\dots$, then

$$\mu_1(X) = \lim_{n\to\infty}\int R^*\varphi_n\, d\mu_1 = \lim_{n\to\infty}\int G\mu_1\, d\mu_n^*$$
$$\leq \lim_{n\to\infty}\int G\mu_2\, d\mu_n^* = \lim_{n\to\infty}\int R^*\varphi_n\, d\mu_2 = \mu_2(X).$$

COROLLARY 1.2. *For $f\in \mathscr{Q}_C$ with $|\sigma(f)|(X)<+\infty$ (in partiuclar, for $f\in \mathscr{R}_0(X)$), $f\geq 0$ implies $\sigma(f)(X)\geq 0$.*

§ 2. Gradient measure and energy

For f, $g\in\mathscr{R}(X)$, their mutual gradient measure with respect to σ is defined by

$$\delta_{[f,g]} = \frac{1}{2}\{f\sigma(g)+g\sigma(f)-\sigma(fg)-fg\sigma(1)\}$$

and the gradient measure of $f\in\mathscr{R}(X)$ by

$$\delta_f = \delta_{[f,f]} = \frac{1}{2}\{2f\sigma(f)-\sigma(f^2)-f^2\sigma(1)\}$$

(see [6]). Similarly, $\delta^*_{[f,g]}$ and δ^*_f for f, $g\in\mathscr{R}^*(X)$ are defined in terms of σ^*.

THEOREM 2.1. *If f, $g\in\mathscr{R}(X)\cap\mathscr{R}^*(X)$, then $\delta_{[f,g]}=\delta^*_{[f,g]}$.*

PROOF. Let $\varphi\in\mathscr{R}_0(X)$. Then φf, φg, $\varphi fg\in\mathscr{R}_0(X)$ ([6; Corollary 2.7]). By [6; Theorem 3.2],

$$\begin{aligned}2\varphi\delta_{[f,g]} &= 2\delta_{[\varphi f,g]} - 2f\delta_{[\varphi,g]}\\ &= \{\varphi f\sigma(g)+g\sigma(\varphi f)-\sigma(\varphi fg)-\varphi fg\sigma(1)\}\\ &\quad - \{f\varphi\sigma(g)+fg\sigma(\varphi)-f\sigma(\varphi g)-\varphi fg\sigma(1)\}\\ &= g\sigma(\varphi f)+f\sigma(\varphi g)-fg\sigma(\varphi)-\sigma(\varphi fg).\end{aligned}$$

Hence, using Proposition 1.1, we have

$$\begin{aligned}2\int\varphi\, d\delta_{[f,g]} &= \int g\, d\sigma(\varphi f)+\int f\, d\sigma(\varphi g)-\int fg\, d\sigma(\varphi)-\int d\sigma(\varphi fg)\\ &= \int\varphi f\, d\sigma^*(g)+\int\varphi g\, d\sigma^*(f)-\int\varphi\, d\sigma^*(fg)-\int\varphi fg\, d\sigma^*(1)\\ &= 2\int\varphi\, d\delta^*_{[f,g]}.\end{aligned}$$

Since $\mathscr{R}_0(X)$ is dense in $C_0(X)$, it follows that $\delta_{[f,g]}=\delta^*_{[f,g]}$.

PROPOSITION 2.1. *If $f\in\mathscr{R}(X)$ and $g\in\mathscr{R}_0(X)$, then*

$$2\delta_{[f,g]}(X)+\int fg\,d\sigma(1)+\int fg\,d\sigma^*(1)=\int f\,d\sigma(g)+\int g\,d\sigma(f).$$

PROOF. Since $\sigma(fg)(X)=\int fg\,d\sigma^*(1)$ by Proposition 1.1, the equality of the proposition follows from the definition of $\delta_{[f,g]}$.

COROLLARY 2.1. *If $\sigma(1)=\sigma^*(1)=0$, then*

$$\delta_{[f,g]}(X)=\frac{1}{2}\left\{\int f\,d\sigma(g)+\int g\,d\sigma(f)\right\}$$

for $f\in\mathscr{R}(X)$ and $g\in\mathscr{R}_0(X)$.

We consider the following classes:

$$\mathscr{P}_{BC}=\{p\in\mathscr{P}_C\,|\,p\text{: bounded}\},\quad \mathscr{Q}_{BC}=\mathscr{P}_{BC}-\mathscr{P}_{BC},$$
$$\mathscr{P}_{IC}=\{p\in\mathscr{P}_C\,|\,\textstyle\int p\,d\sigma(p)<+\infty\},\quad \mathscr{Q}_{IC}=\mathscr{P}_{IC}-\mathscr{P}_{IC},$$
$$\mathscr{P}_{BIC}=\mathscr{P}_{BC}\cap\mathscr{P}_{IC},\quad \mathscr{Q}_{BIC}=\mathscr{P}_{BIC}-\mathscr{P}_{BIC}.$$

LEMMA 2.1. *If $f\in\mathscr{Q}_{BC}$, then $f^2\in\mathscr{Q}_{BC}$. If $f\in\mathscr{Q}_{BIC}$, then*

$$\delta_f(X)+\frac{1}{2}\int f^2d\sigma(1)\le\int f\,d\sigma(f),$$
$$|\sigma(f^2)|(X)<+\infty\quad\textit{and}\quad \sigma(f^2)(X)\ge 0.$$

PROOF. Let $f=p-q$ with $p,\ q\in\mathscr{P}_{BC}$ and let $M=\max(\sup_X p,\ \sup_X q)$. Then $|f|\le M$ and $|\sigma(f)|\le\sigma(p)+\sigma(q)$. Let $v=2M(p+q)-f^2$. Since

$$0\le 2\delta_f=2f\sigma(f)-\sigma(f^2)-f^2\sigma(1)\le 2M\sigma(p+q)-\sigma(f^2)=\sigma(v),$$

v is $\mathscr{H}$-superharmonic on X. Also, $0\le v\le 2M(p+q)$ implies $v\in\mathscr{P}_{BC}$. Hence $f^2=2M(p+q)-v\in\mathscr{Q}_{BC}$.

Since $|f\sigma(f)|\le M\sigma(p+q)$ and f is bounded, we see that $G(f\sigma(f))\in\mathscr{Q}_{BC}$ and $G(f^2\sigma(1))\in\mathscr{P}_{BC}$. Hence $G(\delta_f)\in\mathscr{Q}_{BC}$ and

$$G(\delta_f)=G(f\sigma(f))-\frac{1}{2}f^2-\frac{1}{2}G(f^2\sigma(1))\le G(f\sigma(f))-\frac{1}{2}G(f^2\sigma(1)).$$

Hence, by Lemma 1.3,

$$\delta_f(X)+\frac{1}{2}\int f^2d\sigma(1)\le\int f\,d\sigma(f),$$

provided $\int |f|\, d|\sigma(f)| < +\infty$, i.e., $f \in \mathscr{Q}_{BIC}$. Furthermore, since

$$|\sigma(f^2)| \leq 2|f|\,|\sigma(f)| + 2\delta_f + f^2\sigma(1),$$

$\int |f|\, d|\sigma(f)| < +\infty$ implies $|\sigma(f^2)|(X) < +\infty$, and hence $\sigma(f^2)(X) \geq 0$ by Corollary 1.2.

PROPOSITION 2.2. *If $G\mu$, $G\nu \in \mathscr{P}_C$, then*

$$\int G\mu\, d\nu + \int G\nu\, d\mu \leq \int G\mu\, d\mu + \int G\nu\, d\nu.$$

PROOF. Let $\{X_n\}$ be an exhaustion of X and put $\mu_n = \mu|_{X_n}$ and $\nu_n = \nu|_{X_n}$. Since $G\mu_n \leq \sup_{X_n} G\mu$ and $G\nu_n \leq \sup_{X_n} G\nu$ (cf. [6; Proposition 2.5]), $G\mu_n$, $G\nu_n \in \mathscr{P}_{BC}$. Let $f_n = G\mu_n - G\nu_n$. Then $f_n \in \mathscr{Q}_{BC}$ and

$$\int |f_n|\, d|\sigma(f_n)| \leq \int_{X_n} |f_n|\, d\mu + \int_{X_n} |f_n|\, d\nu < +\infty.$$

Hence, by the above lemma, $\int f_n\, d\sigma(f_n) \geq 0$, i.e.,

$$\int G\mu_n\, d\nu_n + \int G\nu_n\, d\mu_n \leq \int G\mu_n\, d\mu_n + \int G\nu_n\, d\nu_n.$$

Letting $n \to \infty$, we obtain the required inequality.

COROLLARY 2.2. *If $G\mu$, $G\nu \in \mathscr{P}_C$ and $G\mu \leq G\nu$ on X, then*

$$\int G\mu\, d\mu \leq 4 \int G\nu\, d\nu.$$

PROOF.

$$\begin{aligned}\int G\mu\, d\mu &\leq \int G\nu\, d\mu = \int G(\sqrt{2}\,\nu)\, d(\mu/\sqrt{2})\\ &\leq \int G(\sqrt{2}\,\nu)\, d(\sqrt{2}\,\nu) + \int G(\mu/\sqrt{2})\, d(\mu/\sqrt{2})\\ &= 2\int G\nu\, d\nu + \frac{1}{2}\int G\mu\, d\mu.\end{aligned}$$

Hence, we obtain the required inequality if $\int G\mu\, d\mu < +\infty$. In case $\int G\mu\, d\mu = +\infty$, consider μ_n as in the proof of the above proposition. Then, since $G\mu_n \leq G\mu \leq G\nu$ and $\int G\mu_n\, d\mu_n < +\infty$, $\int G\mu_n\, d\mu_n \leq 4 \int G\nu\, d\nu$ by the above. Letting $n \to \infty$, we obtain $\int G\nu\, d\nu = +\infty$ in this case.

REMARK 2.1. The inequalities in Proposition 2.2 and Corollary 2.2 fail to hold if we omit the assumption that $G\mu$ and $G\nu$ are continuous. For example, let $X=\boldsymbol{R}^{n+1}$ $(n\geq 1)$ and $\mathscr{H}$ (resp. $\mathscr{H}^*$) be the harmonic sheaf defined by the solutions of the heat (resp. adjoint heat) equation. If $\mathrm{Supp}\,\mu\subset\boldsymbol{R}^n\times\{0\}$ and $\mathrm{Supp}\,\nu\subset\boldsymbol{R}^n\times\{1\}$, then $\int G\mu\,d\mu=\int G\nu\,d\nu=0$, while $\int G\mu\,d\nu>0$ provided $\mu\neq 0$ and $\nu\neq 0$. If $\mathrm{Supp}\,\nu\subset\boldsymbol{R}^n\times\{0\}$, $\nu\neq 0$ and $\mu=\varphi(x,t)dxdt$ with some $\varphi\in C_0^+(\boldsymbol{R}^{n+1})$ such that $\mathrm{Supp}\,\varphi\subset\boldsymbol{R}^n\times\{t>0\}$, then $G\mu\leq G\nu$ for sufficiently small non-zero φ, while $\int G\nu\,d\nu=0$ and $\int G\mu\,d\mu>0$. (See, e.g., [3] and [10] for properties of potentials for the heat equation.)

COROLLARY 2.3. *$\mathscr{Q}_{IC}$ (and hence $\mathscr{Q}_{BIC}$) is a linear space, and for any f, $g\in\mathscr{Q}_{IC}$, $\int|f|\,d|\sigma(g)|<+\infty$. If we write*

$$\langle f,g\rangle=\frac{1}{2}\left\{\int f\,d\sigma(g)+\int g\,d\sigma(f)\right\}\qquad \text{for}\quad f,g\in\mathscr{Q}_{IC},$$

then $\langle\cdot,\cdot\rangle$ is a non-negative definite symmetric bilinear form on $\mathscr{Q}_{IC}$, so that

$$|\langle f,g\rangle|^2\leq\langle f,f\rangle\cdot\langle g,g\rangle\qquad \text{for}\quad f,g\in\mathscr{Q}_{IC}.$$

PROOF. If f, $g\in\mathscr{Q}_{IC}$, then $f=G\mu_1-G\mu_2$, $g=G\nu_1-G\nu_2$ with $G\mu_j$, $G\nu_j\in\mathscr{P}_{IC}$ $(j=1,2)$. Then, by the above proposition,

$$\begin{aligned}\int|f|\,d|\sigma(g)|&\leq\int(G\mu_1+G\mu_2)\,d(\nu_1+\nu_2)\\&\leq 2\left\{\int G\mu_1d\mu_1+\int G\mu_2d\mu_2+\int G\nu_1d\nu_1+\int G\nu_2d\nu_2\right\}<+\infty.\end{aligned}$$

Hence, $\langle f,g\rangle$ is well-defined for f, $g\in\mathscr{Q}_{IC}$. Also, by the above proposition again, $\langle f,f\rangle\geq 0$ for any $f\in\mathscr{Q}_{IC}$. The rest of the corollary is shown by standard arguments.

§ 3. Green's formula

LEMMA 3.1. *Let U be an open set in X, $u_n\in\mathscr{H}(U)$, $n=1,2,\ldots$, and suppose $u_n\to 0$ locally uniformly on U. Then, for any compact set K in U,*

$$\lim_{n\to\infty}\delta_{u_n}(K)=0.$$

PROOF. Given a compact set K in U, choose $\varphi\in\mathscr{R}_0(U)$ such that $0\leq\varphi\leq 1$ on U and $\varphi=1$ on K (cf. [6; Proposition 2.17]). If $u\in\mathscr{H}(U)$, then

$$\begin{aligned}\varphi^2\delta_u&=\frac{1}{2}\{-\varphi^2\sigma(u^2)-u^2\varphi^2\sigma(1)\}\\&=\frac{1}{2}\{u^2\sigma(\varphi^2)-\sigma(u^2\varphi^2)-2u^2\varphi^2\sigma(1)\}-\delta_{[u^2,\varphi^2]}.\end{aligned}$$

By [6; Theorem 3.2],

$$\delta_{[u^2,\varphi^2]} = 4u\varphi\delta_{[u,\varphi]}.$$

Using [6; Proposition 3.3] and the continuity of u and φ, we obtain

$$-4u\varphi\delta_{[u,\varphi]} \leq \frac{1}{2}\varphi^2\delta_u + 8u^2\delta_\varphi.$$

Hence

$$\begin{aligned}\varphi^2\delta_u &\leq u^2\sigma(\varphi^2) - \sigma(u^2\varphi^2) - 2u^2\varphi^2\sigma(1) + 16u^2\delta_\varphi \\ &\leq u^2\sigma(\varphi^2)^+ + 16u^2\delta_\varphi - \sigma(u^2\varphi^2).\end{aligned}$$

By Corollary 1.2, we have $\sigma(u^2\varphi^2)(U)\geq 0$. Hence

$$\delta_u(K)\leq \int\varphi^2\, d\delta_u \leq \int u^2\, d\sigma(\varphi^2)^+ + 16\int u^2 d\delta_\varphi.$$

Since $\operatorname{Supp}\sigma(\varphi^2)^+$, $\operatorname{Supp}\delta_\varphi$ are both contained in $\operatorname{Supp}\varphi$, which is compact in U, we see that $\delta_{u_n}(K)\to 0$ if $u_n\to 0$ locally uniformly on U.

THEOREM 3.1. *If* $f\in\mathcal{Q}_{IC}$, *then* $\delta_f(X)<+\infty$, $|\sigma(f^2)|(X)<+\infty$, $\int f^2 d\sigma(1)<+\infty$, $\int|f|\, d|\sigma(f)|<+\infty$ *and the following hold*:

(i) (*Green's formula*)

$$\delta_f(X) + \frac{1}{2}\int f^2 d\sigma(1) + \frac{1}{2}\sigma(f^2)(X) = \int f\, d\sigma(f).$$

(ii) $(0\leq)\ \delta_f(X) + \frac{1}{2}\int f^2\, d\sigma(1) \leq \int f\, d\sigma(f)$, *i.e.*, $\sigma(f^2)(X)\geq 0$.

PROOF. In Corollary 2.3, we have shown that $\int |f|\, d|\sigma(f)|<+\infty$. Let $f=G\mu-G\nu$ with $G\mu,\ G\nu\in\mathcal{P}_{IC}$. Let $\{X_n\}$ be an exhaustion of X and let $\mu_n=\mu|_{X_n}$, $\nu_n=\nu|_{X_n}$, $p_n=G\mu_n$, $q_n=G\nu_n$ and $f_n=p_n-q_n$. Then, $f-f_n$ is $\mathcal{H}$-harmonic on X_n and $f-f_n\to 0$ locally uniformly on X. (Note that $p-p_n\downarrow 0$ and $q-q_n\downarrow 0$.) Hence, for any compact set K in X, $\delta_{f-f_n}(K)\to 0$ $(n\to\infty)$ by Lemma 3.1, so that $\delta_{f_n}(K)\to\delta_f(K)$ $(n\to\infty)$. Since $f_n\in\mathcal{Q}_{BIC}$ (cf. the proof of Proposition 2.2),

$$\delta_{f_n}(X) + \frac{1}{2}\int f_n^2\, d\sigma(1) \leq \int f_n\, d\sigma(f_n)$$

by Lemma 2.1. Now,

$$\int f_n\, d\sigma(f_n) = \int_{X_n}(p_n-q_n)\, d(\mu-\nu).$$

Since $|p_n-q_n|\leq p+q$, $\int(p+q)\, d(\mu+\nu)<+\infty$ by Proposition 2.2 (cf. the proof of Corollary 2.3) and $p_n-q_n\to p-q$ $(n\to\infty)$, Lebesgue's convergence theorem implies

$$\int f_n\,d\sigma(f_n) \longrightarrow \int (p-q)\,d(\mu-\nu) = \int f\,d\sigma(f) \quad (n\to\infty).$$

Hence, for any compact set K in X,

$$\delta_f(K) + \frac{1}{2}\int_K f^2 d\sigma(1) = \lim_{n\to\infty}\left\{\delta_{f_n}(K) + \frac{1}{2}\int_K f_n^2\,d\sigma(1)\right\}$$

$$\leq \liminf_{n\to\infty}\left\{\delta_{f_n}(X) + \frac{1}{2}\int f_n^2\,d\sigma(1)\right\}$$

$$\leq \liminf_{n\to\infty}\int f_n\,d\sigma(f_n) = \int f\,d\sigma(f).$$

Hence

$$\delta_f(X) + \frac{1}{2}\int f^2\,d\sigma(1) \leq \int f\,d\sigma(f) < +\infty,$$

which implies $\delta_f(X) < +\infty$ and $\int f^2\,d\sigma(1) < +\infty$. Since

$$\delta_f + \frac{1}{2}f^2\sigma(1) + \frac{1}{2}\sigma(f^2) = f\sigma(f),$$

we obtain

$$|\sigma(f^2)|(X) \leq 2\left\{\delta_f(X) + \frac{1}{2}\int f^2 d\sigma(1)\right\} + \int |f|\,d|\sigma(f)| < +\infty$$

and Green's formula given in (i). Thus the theorem is proved.

COROLLARY 3.1. *If* $f,\ g\in\mathcal{D}_{IC}$, *then* $|\delta_{[f,g]}|(X) < +\infty$, $|\sigma(fg)|(X) < +\infty$, $\int |fg|\,d\sigma(1) < +\infty$, $\int |f|\,d|\sigma(g)| < +\infty$, $\int |g|\,d|\sigma(f)| < +\infty$ *and*

$$2\delta_{[f,g]}(X) + \int fg\,d\sigma(1) + \sigma(fg)(X) = \int f\,d\sigma(g) + \int g\,d\sigma(f).$$

COROLLARY 3.2. *If* $f,\ g\in\mathcal{D}_{IC}\cap\mathcal{D}^*_{IC}$, *then*

$$\int fg\,d\sigma(1) + \sigma(fg)(X) = \int fg\,d\sigma^*(1) + \sigma^*(fg)(X).$$

PROOF. By Theorem 2.1, $\delta_{[f,g]}(X) = \delta^*_{[f,g]}(X)$. On the other hand, $\int f\,d\sigma(g) + \int g\,d\sigma(f) = \int g\,d\sigma^*(f) + \int f\,d\sigma^*(g)$. Hence the required formula follows from the above corollary.

COROLLARY 3.3. *Let* $f_n, f\in\mathcal{D}_{IC}$ *and suppose*

$$\int (f_n - f)\,d\sigma(f_n - f) \longrightarrow 0 \quad (n\to\infty).$$

Then $\delta_{f_n-f}(X)\to 0$ *and* $\delta_{f_n}\to\delta_f$ *vaguely* $(n\to\infty)$.

PROOF. By the theorem, $\delta_{f_n-f}(X)\to 0$ $(n\to\infty)$. For any $\varphi\in C_0^+(X)$, the bilinear form $(f, g)\to\int\varphi d\delta_{[f,g]}$ is non-negative definite. Hence

$$\left[\left(\int\varphi\, d\delta_{f_n}\right)^{1/2}-\left(\int\varphi\, d\delta_f\right)^{1/2}\right]^2\leq\int\varphi\, d\delta_{f_n-f}\longrightarrow 0\quad(n\to\infty).$$

Therefore, $\delta_{f_n}\to\delta_f$ vaguely.

§4. Gradient measures for the heat equation — an example of the general theory

In this section, we consider the heat equation

$$Lu\equiv\frac{\partial u}{\partial t}-\Delta_x u=0$$

and the adjoint heat equation

$$L^*u\equiv-\frac{\partial u}{\partial t}-\Delta_x u=0$$

on $\boldsymbol{R}^{d+1}=\{(x, t)\,|\,x\in\boldsymbol{R}^d, t\in\boldsymbol{R}\}$ $(d\geq 1)$. For any domain X in $\boldsymbol{R}^{d+1}$, the sheaf $\mathscr{H}=\mathscr{H}_L$ (resp. $\mathscr{H}^*=\mathscr{H}_L^*$) of solutions of $Lu=0$ (resp. $L^*u=0$) on X defines a P-harmonic space on X (cf. [2; §3.3]). Furthermore, $\mathscr{H}_L$ and $\mathscr{H}_L^*$ are mutually adjoint harmonic structures on X with a Green function $G(x, y)$ satisfying the conditions given in §1 (cf. e.g., [10] and [3; 1, XVII]). By [10; §15], every $\mathscr{H}_L$- (resp. $\mathscr{H}_L^*$-) superharmonic function u is locally integrable with respect to the Lebesgue measure $dxdt$ on $\boldsymbol{R}^{d+1}$ and $\mu=Lu$ (resp. L^*u) in the distribution sense is a non-negative measure. Thus, for any $f\in\mathscr{R}(U)$ (resp. $\mathscr{R}^*(U)$), U: open in X, Lf (resp. L^*f) in the distribution sense is a signed measure on U and in fact, $\sigma(f)=Lf$ (resp. $\sigma^*(f)=L^*f$). Conversely, if $f\in C(U)$ and $\nu=Lf$ (resp. L^*f) in the distribution sense is a signed measure such that $G(|\nu|\,|_K)$ (resp. $G^*(|\nu|\,|_K)$) is continuous for any compact set K in U, then $f\in\mathscr{R}(U)$ (resp. $\mathscr{R}^*(U)$). In particular, if $f(x, t)$ on U is C^1 in t and C^2 in x, then $f\in\mathscr{R}(U)\cap\mathscr{R}^*(U)$, and in this case we easily see that the gradient measure δ_f of f is given by

$$\delta_f=|\nabla_x f(x, t)|^2 dxdt\ \left(=\sum_{j=1}^d\left(\frac{\partial f}{\partial x_j}\right)^2(x, t)\,dxdt\right).$$

PROPOSITION 4.1. *Let* Ω *be a domain in* $\boldsymbol{R}^d$, $T>0$ *and let* $X=\Omega\times(0, T)$. *Then, for the harmonic space* $(X, \mathscr{H}_L)$,

$$\mathscr{D}_{IC}\subset L^2(0, T; H_0^1(\Omega))\cap L^\infty(0, T; L^2(\Omega))$$

and

$$\delta_f = |\nabla_x f(x, t)|^2 dxdt \tag{4.1}$$

for $f \in \mathscr{Q}_{IC}$, *where the gradient* $\nabla_x f$ *consists of the generalized derivatives of* f. (For the spaces $L^2(0, T; H_0^1(\Omega))$ and $L^\infty(0, T; L^2(\Omega))$, one may refer, e.g., to [5].)

PROOF. (I) The case where $f = G\mu$ with a non-negative measure μ having a C^∞-density φ on Ω with compact support in Ω. In this case, it is known (e.g., [3; 1, XVII, 6]) that f is C^∞ in (x, t), so that (4.1) holds with $\nabla_x f$ in the classical sense.

Let $\{\Omega_n\}$ be an exhaustion of Ω by smooth domains such that Supp $\varphi \subset \Omega_1 \times (0, T)$, and let G_n be the Green function for L on $\Omega_n \times (0, T)$. Again by [3; 1, XVII, 6], $G_n\mu$ is C^∞ in (x, t) and $L(G_n\mu) = \varphi$ on $\Omega_n \times (0, T)$ in the classical sense. Let $f_n = G_n\mu$ on $\Omega_n \times (0, T)$ and $=0$ on $(\Omega \setminus \Omega_n) \times (0, T)$. Then, $f_n \uparrow f$ on X. By the boundary regularity of the Green function on smooth domains (e.g., [3; 1, XV, 7]), we see that $G_n\mu(\cdot, t)$ has continuous partial derivatives on $\overline{\Omega_n}$ for each $t \in (0, T)$ and f_n is continuous on X. It follows that $f_n(\cdot, t) \in H_0^1(\Omega)$ for any $t \in (0, T)$. Furthermore, by Green's formula, for each $t \in (0, T)$, we have

$$\begin{aligned} \int_{\Omega_n} |\nabla_x f_n(x, t)|^2 dx &= -\int_{\Omega_n} f_n(x, t) \Delta_x f_n(x, t)\, dx \\ &= -\int_{\Omega_n} f_n(x, t) \frac{\partial f_n}{\partial t}(x, t)\, dx + \int_{\Omega_n} f_n(x, t)\varphi(x, t)\, dx \\ &= -\frac{1}{2}\frac{\partial}{\partial t}\int_{\Omega_n} [f_n(x, t)]^2 dx + \int_{\Omega_n} f_n(x, t)\varphi(x, t)\, dx. \end{aligned}$$

Hence, noting that $f_n(x, t) = 0$ for $0 \le t < \varepsilon$ for some $\varepsilon > 0$, we have

$$\begin{aligned} \int_{\Omega_n \times (0,s)} |\nabla_x f_n(x, t)|^2 dxdt + \frac{1}{2}\int_{\Omega_n} [f_n(x, s)]^2 dx \\ = \int_{\Omega_n \times (0,s)} f_n\, d\mu \le \int_X f\, d\mu < +\infty \end{aligned} \tag{4.2}$$

for any $s \in (0, T)$. Since $f - f_n$ is $\mathscr{H}_L$-harmonic on $\Omega_n \times (0, T)$ and $f - f_n \downarrow 0$ $(n \to \infty)$, we see that $\nabla_x f_n \to \nabla_x f$ locally uniformly on $\Omega \times (0, T)$. Hence, by Fatou's lemma,

$$\int_{\Omega_n \times (0,s)} |\nabla_x f(x, t)|^2 dxdt + \frac{1}{2}\int_{\Omega} [f(x, s)]^2 dx \le \int_X f\, d\mu < +\infty \tag{4.3}$$

for any $s \in (0, T)$. Therefore, $f \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$. Furthermore, (4.2) also shows that $\{f_n\}$ is bounded in $L^2(0, T; H^1(\Omega))$. Since $f_n \to f$ on X, it follows that $f_n \to f$ weakly in $L^2(0, T; H^1(\Omega))$. Since $f_n \in L^2(0, T; H_0^1(\Omega))$ for each n and $L^2(0, T; H_0^1(\Omega))$ is weakly closed in $L^2(0, T; H^1(\Omega))$, it follows that $f \in L^2(0, T; H_0^1(\Omega))$.

(II) Next, consider the case where $f=G\mu$ with a non-negative measure μ such that $\mathrm{Supp}\,\mu$ is compact in X. Note that $f\in\mathscr{P}_{BIC}$ in this case. Let $\{\eta_n\}$ be a sequence of mollifiers on $\boldsymbol{R}^{d+1}$ tending to the Dirac measure and put $\mu_n=\mu*\eta_n$, the convolution of μ and η_n. We may assume that $\mathrm{Supp}\,\mu_n\subset X$ for all n. Then, each μ_n is a measure of the type considered in (I) and $\mu_n\to\mu$ vaguely as $n\to\infty$. Let $f_n=G\mu_n$.

We can write (cf. [3; pp. 331–332])

$$G((x, t), (y, s)) = W(x-y, t-s) - H(x, y; t, s),$$

where

$$W(x, t) = \begin{cases} (4\pi t)^{-d/2}e^{-|x|^2/4t}, & t>0 \\ 0, \quad t\leq 0 \end{cases}$$

and $H(x, y; t, s)$, as a function of (x, t), is the generalized Dirichlet solution for $Lu=0$ in X with the boundary values $W(\xi-y, \tau-s)$ for $(\xi, \tau)\in\partial X$. Suppose $\mathrm{Supp}\,\mu_n\subset K$ with a compact set K in X. We see easily that the family $\{H(x, y; t, s)\}_{(x,t)\in X}$ is uniformly equicontinuous on K. It then follows that

$$u_n(x, t) \equiv \int H(x, y; t, s)\, d\mu_n(y, s) \longrightarrow \int H(x, y; t, s)\, d\mu(y, s) \equiv u(x, t)$$

uniformly in $(x, t)\in X$. Since u_n and u are $\mathscr{H}_L$-harmonic on X, it follows that $\nabla_x u_n\to\nabla_x u$ locally uniformly on X. Since $f=W*\mu-u$, the continuity of f implies the continuity of $W*\mu$. Therefore, $W*\mu*\eta_n\to W*\mu$ $(n\to\infty)$ locally uniformly on $\boldsymbol{R}^{d+1}$. Hence, $f_n\to f$ locally uniformly on X, so that

$$\int_{\Omega\times(0,T)} f_n\, d\mu_n \longrightarrow \int_{\Omega\times(0,T)} f\, d\mu < +\infty \quad (n\to\infty).$$

Since (4.3) holds for f_n, it follows that $\{f_n\}$ is bounded in $L^2(0, T; H_0^1(\Omega))$ as well as in $L^\infty(0, T; L^2(\Omega))$. Since $f_n\to f$, it then follows that $f\in L^2(0, T; H_0^1(\Omega))\cap L^\infty(0, T; L^2(\Omega))$, and $f_n\to f$ weakly in $L^2(0, T; H_0^1(\Omega))$, which implies (4.3) for the present f. Furthermore, since $\partial f/\partial x_j\in L^2_{\mathrm{loc}}(X)$, $\partial(W*\mu)/\partial x_j\in L^2_{\mathrm{loc}}(X)$ $(j=1,\dots, d)$. Hence, $\partial(W*\mu*\eta_n)/\partial x_j\to\partial(W*\mu)/\partial x_j$ in $L^2_{\mathrm{loc}}(X)$, so that $\partial f_n/\partial x_j\to\partial f/\partial x_j$ in $L^2_{\mathrm{loc}}(X)$ $(j=1,\dots, d)$. Therefore, $|\nabla_x f_n|^2dxdt\to|\nabla_x f|^2dxdt$ vaguely on X. On the other hand, $\delta_{f_n}=|\nabla_x f_n|^2dxdt$ and $\delta_{f_n}\to\delta_f$ vaguely on X by Corollary 3.3, since $\int(f_n-f)\,d(\mu_n-\mu)\to 0$. Therefore $\delta_f=|\nabla_x f|^2dxdt$.

(III) Now, let $f=G\mu\in\mathscr{P}_{IC}$. Let $\{X_n\}$ be an exhaustion of X and let $\mu_n=\mu|_{X_n}$ and $f_n=G\mu_n$, $n=1, 2,\dots$. Then each f_n is of the type considered in (II), so that $f_n\in L^2(0, T; H_0^1(\Omega))\cap L^\infty(0, T; L^2(\Omega))$, (4.3) holds for f_n and $\delta_{f_n}=|\nabla_x f_n|^2\cdot dxdt$. Since $f_n\uparrow f$, we see, as in (II), that $f\in L^2(0, T; H_0^1(\Omega))\cap L^\infty(0, T; L^2(\Omega))$ and (4.3) holds for f. Thus (4.3) also holds for $f-f_n\in\mathscr{P}_{IC}$ for each n. Since f_n

$\uparrow f$ and $f \in \mathscr{P}_{IC}$, we see that $\int (f-f_n)\, d(\mu-\mu_n) \to 0$ $(n \to \infty)$. Hence $\delta_{f_n} \to \delta_f$ vaguely on X by Corollary 3.3, and (4.3) for f_n, $n=1, 2,\dots$, imply that $f_n \to f$ in $L^2(0, T; H_0^1(\Omega))$. It then follows that $|\nabla_x f_n|^2 dxdt \to |\nabla_x f|^2 dxdt$ vaguely, which implies $\delta_f = |\nabla_x f|^2 dxdt$.

Finally, if $f \in \mathscr{Q}_{IC}$, then $f = f_1 - f_2$ with $f_1, f_2 \in \mathscr{P}_{IC}$. Hence we obtain the proposition.

COROLLARY 4.1. *Let X be a domain in $\boldsymbol{R}^{d+1}$ and let $f \in \mathscr{R}(X)$ for the harmonic space $(X, \mathscr{H}_L)$. Then $\partial f/\partial x_j \in L^2_{\mathrm{loc}}(X)$, $j=1,\dots, d$, and*

$$\delta_f = |\nabla_x f(x, t)|^2 dxdt \quad \text{on} \quad X.$$

REMARK 4.1. In view of [7; Lemme II–6] and the above proposition, if $f \in \mathscr{Q}_{IC}$ for the harmonic space $(X, \mathscr{H}_L)$ with $X = \Omega \times (0, T)$, then

$$\delta_f(\Omega \times (0, s)) + \frac{1}{2} \int_\Omega [f(x, s)]^2 dx = \int_{\Omega \times (0, s)} f \, d\sigma(f)$$

for any $s \in (0, T)$. Thus, by Theorem 3.1, we have

$$\sigma(f^2)(X) = \lim_{s \to T} \int_\Omega [f(x, s)]^2 dx$$

in this case.

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