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# Time change and orbit equivalence in ergodic theory

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## § 1. Introduction

The theory of the orbit equivalence has occupied an important place in ergodic theory and is closely related to the notion of the time changes. In this paper we shall investigate the relation between them.

In §2 and §3 we shall follow Totoki's paper [6] with some additional results. He constructed time changes from given flows by means of so called additive cocycles, which have the similar properties to those of the lags of the parameters in the orbit equivalence. To speak in more detail, an additive cocycle $\varphi$ with respect to a flow $(X, \mathscr{B}, \mu, \{T_t\})$ (usually on a Lebesgue probability space) is a measurable map from $\mathbf{R} \times X$ to $\mathbf{R}$, which satisfies the equation $\varphi(t+s, x) = \varphi(s, x) + \varphi(t, T_s x)$ $t, s \in \mathbf{R}$, $x \in X$, with $\varphi^x(t) \equiv \varphi(t, x)$ non-decreasing (not necessarily strictly increasing) and continuous in $t$ for $x$ in a $\{T_t\}$-invariant co-null set. Among them additive cocycles of the forms $\varphi(t, x) = \int_0^t f(T_s x) ds$ with non-negative $f$'s are important ones, as we shall see later on. For an additive cocycle $\varphi$, we shall see $\lim_{h \to 0} \varphi(h, x)/h$ exists a.e. by the same method as in the Wiener's local ergodic theorem, from which we have the Lebesgue decomposition of $\varphi$. The time change is defined as the same way as in [6] and its invariant measure is described by $\varphi$. Especially for $S$-flows, we have $S$-representations of time changes.

In §4 we shall argue the relation between these time changes and the orbit equivalence. Time change defines an equivalence relation, which is denoted by $\sim$, among ergodic flows. That is, $\{S_t\} \sim \{T_t\}$ if there exists some integrable additive cocycle $\varphi$ such that the time change $\{S_t^\varphi\}$ is isomorphic to $\{T_t\}$. It turns out that $\sim$ is an equivalence relation and we shall show, as expected naturally, that $\sim$ and the orbit equivalence $\overset{\mathcal{O}}{\sim}$ coincide. This is the main theorem in this paper.

In §5 we shall treat a special problem of isomorphic relation. The time change of an S-flow by an additive cocycle $\varphi$ is isomorphic to those by $\varphi_n$, where $\varphi_n$'s are the ones defined by the functions $f_n$'s such that $f_n \to f = \lim_{h \to 0} \varphi(h, \cdot)/h$ a.e. In this connection we have the following problem. Is the time change by $\varphi_{ac}$ isomorphic to the one by $\varphi$, where $\varphi_{ac}$ is the additive cocycle defined by $f$? Generally this is not true except for the trivial case when $\varphi$ is of integral form (defined by $f$).

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## §2. Additive cocycle

Let $(X, \mathscr{B}, \mu, \{T_t\})$ be a measure preserving flow on a complete finite measure space. We shall often use the notation such as $\{T_t\}$ for abbreviation. To make the arguments clear, we define the following two notions of measurability according to [6]. That is, $\{T_t\}$ is said to be *B-measurable* if the mapping $(t, x) \to T_t x$ is $\mathscr{B}(\mathbf{R}) \times \mathscr{B}/\mathscr{B}$-measurable, where $\mathscr{B}(\mathbf{R})$ is the $\sigma$-algebra of all Borel sets in $\mathbf{R}$, and is said to be *L-measurable* if the mapping above is $\overline{\mathscr{B}(\mathbf{R}) \times \mathscr{B}}^{\lambda\times\mu}/\mathscr{B}$-measurable, where $\lambda$ is the ordinary Lebesgue measure on $\mathbf{R}$ and ${}^{-\lambda\times\mu}$ is the completion.

DEFINITION 2.1. Let $\{T_t\}$ be a $B$-measurable (or an $L$-measurable) flow on a finite measure space $(X, \mathscr{B}, \mu)$. A map $\varphi$ from $\mathbf{R}\times X$ to $\mathbf{R}$ is said to be an *additive cocycle* with respect to $\{T_t\}$ if it satisfies the following properties (2.1)~(2.3).

(2.1) $\varphi$ is $\mathscr{B}(\mathbf{R}) \times \mathscr{B}/\mathscr{B}(\mathbf{R})$-measurable.

(2.2) There exists a $\{T_t\}$-invariant co-null set $D$ called the *domain* of $\varphi$, on which the following conditions are satisfied.

(2.2a) For each $x \in D$, the function $t \to \varphi(t, x)$ is continuous and non-decreasing. (This function is denoted by $\varphi^x$.)

(2.2b) For each $x \in D$ and $t, s \in \mathbf{R}$, the equality

$$\varphi(t+s, x) = \varphi(s, x) + \varphi(t, T_s x)$$

holds.

(2.3) Set

$$C = \{x \in D: \lim_{t\to\infty} \varphi(t, x) = \infty \text{ and } \lim_{t\to-\infty} \varphi(t, x) = -\infty\},$$

then $\mu(C) > 0$. ($C$ is called the *carrier* of $\varphi$.) □

REMARK 2.2. We may assume $\varphi^x$ is identically zero for each $x \in D \setminus C$ as mentined in [6].

Furthermore an additive cocycle satisfying the following condition (2.4) is called an *integrable* one.

(2.4) The function $x \longrightarrow \varphi_1(x) \equiv \varphi(1, x)$ is integrable.

Let $(X, \mathscr{B}, \mu, \{T_t\})$ be a $B$-measurable (or an $L$-measurable) flow and $f$ be a non-negative measurable function which is strictly positive on a set of positive measure and satisfies the following condition.

(2.5) $\int_0^t f(T_s x)ds$ is finite for each $t \in \mathbf{R}$ and for each $x$ in some $\{T_t\}$-invariant co-null set.

Then it can be shown that $\varphi(t, x) = \int_0^t f(T_s x)ds$ is well-defined and $\varphi$ is an additive cocycle. (See [6].) An additive cocycle of this form is called that of *integral form* (defined by $f$). The reason why additive cocycle of this form is important is that we can get the concrete form of the induced measure for the time change. Especially if $f$ is integrable, $f$ satisfies the condition (2.5) and it is easy to see that $\varphi$ defined above is an integrable additive cocycle.

Suppose $\varphi$ is an additive cocycle of integral form with the defining function $f$ integrable. Then by the Wiener's local ergodic theorem, $\lim_{h \to 0} \varphi(h, x)/h = f(x)$ a.e. In general we obtain the next results.

LEMMA 2.3. *For any additive cocycle* $\varphi$, $\lim_{h \to 0} \varphi(h, x)/h$ *exists a.e. and this limit* $f(x)$ *satisfies* (2.5). *Furthermore if* $\varphi$ *is integrable, then* $f$ *is also integrable.*

PROOF. As in the local ergodic theorem, we first define

$$E = \left\{(t, x) \in \mathbf{R} \times X \colon \lim_{h \to 0} \frac{\varphi(t+h, x) - \varphi(t, x)}{h} \text{ exists}\right\}.$$

Then we can easily see that $E \in \mathscr{B}(\mathbf{R}) \times \mathscr{B}$. From the property of (2.2a) of $\varphi$, we have

$$\lambda(E_x{}^c) = 0, \quad \text{where} \quad E_x = \{t \in \mathbf{R} \colon (t, x) \in E\}.$$

Then the argument similar to the local ergodic theorem leads us to the first conclusion of the lemma.

Next, since $f(T_t x)$ is the derivative of the non-decreasing function $\varphi^x$ at $t$, we have

$$\left| \int_0^t f(T_s x)ds \right| \leqq |\varphi(t, x)|, \quad \text{for all} \quad t \in \mathbf{R}.$$

Hence $f$ satisfies the property (2.5). If $\varphi$ is integrable, by Fatou's lemma we have

$$\int f d\mu = \int \lim_{n \to \infty} n\varphi(1/n, x)d\mu(x) \leqq \lim_{n \to \infty} n \int \varphi(1/n, x)d\mu(x)$$
$$= \|\varphi_1\| < \infty,$$

where $\| \quad \|$ is the $L^1$-norm. (The last equality follows from (2.2b).) q. e. d.

Though it may cause confusions, we shall call the function obtained in Lemma 2.3 the *derivative* of the additive cocycle $\varphi$. That is, the value of the derivative of $\varphi$ is nothing but the derivative of the real function $\varphi^x$ at zero in the sense of ordinary integration theory.

THEOREM 2.4. *Let $\varphi$ be an additive cocycle with the derivative $f$. Then $\varphi$ admits the unique decomposition*

$$\varphi = \varphi_{ac} + \varphi_{sing} \tag{2.6}$$

*where $\varphi_{ac}$ is the additive cocycle defined by $f$ and $\varphi_{sing}$ is the one such that $(\varphi_{sing})^x$ is singular with respect to the Lebesgue measure on $\mathbf{R}$ for each $x$ in some $\{T_t\}$-invariant co-null set. Here 0 is also regarded as an additive cocycle.*

PROOF. Since $f(T_t x)$ is the derivative of $\varphi^x$ at $t$, it is clear that

$$\varphi^x(t) - (\varphi_{ac})^x(t) = \varphi^x(t) - \int_0^t f(T_s x)ds$$

is singular for each $x \in D$. ($D$ is the intersection of the domain of $\varphi$ and $\varphi_{ac}$.) The non-decreasing property of $(\varphi_{sing})^x$ is shown by

$$\begin{aligned}\varphi_{sing}(t+h, x) - \varphi_{sing}(t, x) &= \varphi_{sing}(h, T_t x)\\ &= \varphi(h, T_t x) - \int_0^h f(T_s \circ T_t x)ds \geqq 0.\end{aligned}$$

The positivity of measures of the carriers are shown in the same way as in Lemma 2.1 of [6] unless either of them is identically zero. Other conditions can easily be checked. q. e. d.

COROLLARY 2.5. *An additive cocycle $\varphi$ is of integral form if and only if there is $\{T_t\}$-invariant null set $N$ such that $\varphi^x$ is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}$ for each $x \in N^c$. Furthermore in this case the function which defines $\varphi$ is the derivative of $\varphi$.*

## § 3. Time change

From now on we treat only Lebesgue probability spaces. Let $(X, \mathscr{B}, \mu, \{T_t\})$ be an $L$-measurable flow and $\varphi$ be its additive cocycle. Suppose $\mathscr{B}_0$ is a $\sigma$-algebra which is introduced in [6], i.e. a countably generated $\sigma$-algebra with respect to which $\{T_t\}$ becomes $B$-measurable and $\varphi$ becomes $\mathscr{B}(\mathbf{R}) \times \mathscr{B}_0 / \mathscr{B}(\mathbf{R})$-measurable and whose completion is precisely $\mathscr{B}$. Define $X^\varphi$, $\mathscr{B}_0^\varphi$, $\mu_0^\varphi$, $\varphi^{-1}$, $\{T_t^\varphi\}$ as follows.

(3.1) $$X^{\varphi} = \{x \in C\colon \varphi(t, x) > 0 \text{ for all } t > 0\},$$

where $C$ is the carrier of $\varphi$.

(3.2) $$\mathscr{B}_0^{\varphi} = \{B \cap X^{\varphi}\colon B \in \mathscr{B}_0\}.$$

(3.3) $$\mu^{\varphi}(B) = \int\left(\int_0^1 \chi_B(T_t x) d\varphi(t, x)\right) d\mu(x), \qquad \text{for} \quad B \in \mathscr{B}_0^{\varphi}.$$

(3.4) $$\varphi^{-1}(t, x) = \sup\{s\colon \varphi(s, x) \leqq t\}, \qquad \text{for} \quad x \in X^{\varphi}.$$

(3.5) $$T_t^{\varphi} x = T_{\varphi^{-1}(t,x)} x, \qquad \text{for} \quad x \in X^{\varphi} \quad \text{and} \quad t \in \mathbf{R}.$$

Note that it can be shown $\varphi^{-1}(t, x)$ is finite for all $x \in X^{\varphi}$ and $t \in \mathbf{R}$ by the same argument as in [6]. It is easy to see that for an additive cocycle $\varphi$, $\mu^{\varphi}$ is finite if and only if $\varphi$ is integrable and that for the additive cocycle $\varphi$ defined by a function $f$, $\mu^{\varphi} = f\mu_{|\mathscr{B}_0^{\varphi}}$.

THEOREM 3.1 ([6]). *$(X^{\varphi}, \mathscr{B}_0^{\varphi}, \mu^{\varphi})$ is a $\sigma$-finite measure space and $\{T_t^{\varphi}\}$ is a B-measurable flow on it.*

Suppose $\mathscr{B}^{\varphi}$ is the completion of $\mathscr{B}_0^{\varphi}$ with respect to $\mu^{\varphi}$. Then the $B$-measurable flow above is uniquely extended to an $L$-measurable flow $(X^{\varphi}, \mathscr{B}^{\varphi}, \mu^{\varphi}, \{T_t^{\varphi}\})$ which we shall call the *time change* of $(X, \mathscr{B}, \mu, \{T_t\})$ by $\varphi$. If the additive cocycle $\varphi$ is integrable we treat the time change with the normalized induced measure unless otherwise stated. We call it the *time change* by $\varphi$ and denote it by $(X^{\varphi}, \mathscr{B}^{\varphi}, \mu^{\varphi}, \{T_t^{\varphi}\})$ or simply $\{T_t^{\varphi}\}$ in this case too. That is, we take the following induced measure (3.3)′ instead of (3.3).

(3.3)′ $$\mu^{\varphi}(B) = \frac{1}{\|\varphi_1\|}\int\left(\int_0^1 \chi_B(T_t x) d\varphi(t, x)\right) d\mu(x), \qquad \text{for} \quad B \in \mathscr{B}_0^{\varphi}.$$

Let $S$ be an automorphism (i.e. an invertible measure preserving transformation) on a finite Lebesgue space $(X, \mathscr{B}, \mu)$ and $\alpha$ be a strictly positive integrable function on $X$. As usual we define an $S$-flow (or a special flow, a flow build under a function) $(S, \alpha)$ on $(X_\alpha, \mathscr{B}_\alpha, \mu_\alpha)$ where $X_\alpha = \{(x, u)\colon x \in X, 0 \leqq u < \alpha(x)\}$, $(\mathscr{B}_\alpha, \mu_\alpha)$ is the restriction of $(\overline{\mathscr{B} \times \mathscr{B}(\mathbf{R})}^{\mu \times \lambda}, \mu \times \lambda)$ to $X_\alpha$. (See [1].) We call $X$, $S$ and $\alpha$ the base space, the base automorphism and the ceiling function of the $S$-flow $(S, \alpha)$ respectively. If a flow $\{S_t\}$ is isomorphic to an $S$-flow $(S, \alpha)$, we call $(S, \alpha)$ an $S$-representation (or a special representation) of $\{S_t\}$. In this case the base automorphism $S$ is often called a cross-section of $\{S_t\}$.

THEOREM 3.2 ([6]). *Let $(S, \alpha)$ be an S-flow on $X_\alpha$ made by the base automorphism $(X, \mathscr{B}, \mu, S)$ and the positive integrable function $\alpha$ on $X$. For convenience we assume $\int \alpha d\mu = 1$. Let $\varphi$ be an integrable additive cocycle with respect ot $(S, \alpha)$. Then*

$$(S, \alpha)^{\varphi} \simeq (S_E, h)$$

*where the left hand side is the (normalized) time change,* $\simeq$ *a metrical isomorphic relation,* $S_E$ *the induced automorphism,*

$$(E, \mathscr{B}_E, \mu/\|\varphi_1\|, S_E), \quad E = \{x \in X: \varphi(\alpha(x), x, 0) > 0\}$$

*and h is the function*

$$h(x) = \varphi(\alpha(x), x, 0), \quad x \in E.$$

*(We substituted* $\varphi(t, x, u)$ *for* $\varphi(t, (x, u))$.*)*

REMARK 3.3. Since any induced automorphism of an ergodic automorphism is also ergodic and an $S$-flow is ergodic if and only if the base automorphism is ergodic, we can conclude that any time change of an ergodic flow on a Lebesgue probability space is also ergodic by virtue of Theorem 3.2 and the special representation theorem ([1]).

## §4. Orbit equivalence

Throughout this section flows are all assumed to be $L$-measurable flows on Lebesgue probability spaces.

DEFINITION 4.1. Let $\{S_t\}$ and $\{T_t\}$ be two ergodic flows. We denote $\{S_t\} \precsim \{T_t\}$ if there exists an integrable additive cocycle $\varphi$ with respect to $\{S_t\}$ such that $\{S_t^{\varphi}\} \simeq \{T_t\}$. □

Though the relation above seems to be only a one-way relation, we shall see that it defines an equivalence relation among ergodic flows. We first prepare the following lemma.

LEMMA 4.2. *Let* $\{S_t\}$ *be an ergodic flow and* $\varphi$ *be an integrable additive cocycle. Then there exists an integrable additive cocycle* $\psi$ *such that*

$$\{S_t^{\psi}\} \simeq \{S_t^{\varphi}\} \quad \textit{and}$$

(4.1) $\psi^x$ *is piecewise linear with positive slopes for each* $x \in C$, *where* $C$ *is the common carrier of* $\varphi$ *and* $\psi$.

PROOF. We may assume that $\{S_t\}$ is an $S$-flow made by an automorphism $(X, \mathscr{B}, \mu, S)$ and a positive integrable function $\alpha$, because every ergodic flow on a Lebesgue space is isomorphic to an $S$-flow.

Set

$$E = \{x \in X: \varphi(\alpha(x), x, 0) > 0\}.$$

Then $\mu(E)>0$, so that we can consider $S_E$. Since the flow is ergodic, $(S, \alpha)\simeq(S_E, \alpha_E)$, with an isomorphism $\Phi$ from $X_\alpha$ to $E_{\alpha_E}$, where $\alpha_E$ is the function on $E$ defined by

$$\alpha_E(x) = \sum_{k=0}^{r_E(x)-1} \alpha(S^k x).$$

($r_E$ is the recurrence time function for $E$.)

Define the additive cocycle $\tilde{\psi}$ with respect to $(S_E, \alpha_E)$ by

$$\tilde{\psi}(t, x, u) = \int_0^t f((S_E, \alpha_E)_v(x, u))dv$$

where $f(x, u)=\varphi(\alpha(x), x, 0)/\alpha_E(x)$ $(x\in E,\ 0\leqq u<\alpha_E(x))$. Then by Theorem 3.2,

$$(S_E, \alpha_E)^{\tilde{\psi}} \simeq (S_E, h), \quad (h(x)=\varphi(\alpha(x), x, 0)).$$

Since again by Theorem 3.2, $(S, \alpha)^\varphi\simeq(S_E, h)$, we have

$$(S_E, \alpha_E)^{\tilde{\psi}} \simeq (S, \alpha)^\varphi.$$

Then the additive cocycle $\psi$ defined by

$$\psi(t, x, u) = \tilde{\psi}(t, \Phi(x, u)), \quad (x, u)\in X_\alpha$$

is the desired one. q.e.d.

PROPOSITION 4.3. *The relation* $\precsim$ *has the following properties.*

(4.2) $\{S_t\}\precsim\{T_t\}$ *implies* $\{T_t\}\precsim\{S_t\}$.

(4.3) $\{S_t\}\precsim\{T_t\}$, $\{T_t\}\precsim\{U_t\}$ *implies* $\{S_t\}\precsim\{U_t\}$.

*Therefore* $\precsim$ *defines an equivalence relation* $\sim$ *among ergodic flows.*

PROOF. Let $\{S_t\}$, $\{T_t\}$, $\{U_t\}$ be ergodic flows on $(X, \mathscr{B}, \mu)$, $(Y, \mathscr{C}, \nu)$, $(Z, \mathscr{D}, \rho)$ respectively.

Suppose $\{S_t^\varphi\}\simeq\{T_t\}$ for some $\varphi$. By Lemma 4.2, we may assume $\varphi$ has the property (4.1). Then it is easy to see that $\varphi^{-1}$ defined by (3.4) is an additive cocycle. In fact, since $(\varphi^{-1})^x$ is just the inverse function of $\varphi^x$, $(\varphi^{-1})^x$ is also piecewise linear with positive slopes, which means $\varphi^{-1}$ is the one of integral form defined by the reciprocal of the function defining $\varphi$. Therefore, if $\mu^\varphi=a\mu$ $(a\in L_+^1)$, then $(\mu^\varphi)^{\varphi^{-1}}=\mu^\varphi/a=\mu$.

Let $\Phi$ be an isomorphism from $\{S_t^\varphi\}$ to $\{T_t\}$. Then for the additive cocycle defined by

$$\psi(t, y) = \varphi^{-1}(t, \Phi^{-1}y), \quad t\in\mathbf{R}, \quad y\in Y,$$

we have

$$\{T_t^\psi\} \simeq \{(S^\varphi)_t^{\varphi^{-1}}\} = \{S_t\}.$$

Next suppose that for $\varphi$ and $\psi$,

$$\{S_t^\varphi\} \simeq \{T_t\}, \tag{4.4}$$

$$\{T_t^\psi\} \simeq \{U_t\}. \tag{4.5}$$

Here we may again assume that both $\varphi$ and $\psi$ have the property (4.1) in Lemma 4.2. Let $\Phi$ and $\Psi$ be isomorphisms from the left hand sides to the right hand sides in (4.4) and (4.5) respectively. Then the additive cocycle $\theta$ defined by

$$\theta(t, x) = \psi(\varphi(t, x), \Phi x), \quad t \in \mathbf{R}, \quad x \in X$$

satisfies the relation $\{S_t^\theta\} \simeq \{U_t\}$. In fact, since $\theta^{-1}(t, x) = \varphi^{-1}(\psi^{-1}(t, \Phi x), x)$, we have

$$\begin{aligned} S_t^\theta x &= S_{\theta^{-1}(t,x)}x = S_{\varphi^{-1}(\psi^{-1}(t,\Phi x),x)}x = S^\varphi_{\psi^{-1}(t,\Phi x)}x \\ &= \Phi^{-1}\circ T_{\psi^{-1}(t,\Phi x)}\circ\Phi x = \Phi^{-1}\circ T_t^\psi\circ\Phi x = \Phi^{-1}\circ\Psi^{-1}\circ U_t\circ\Psi\circ\Phi x, \end{aligned}$$

and since

$$\lim_{h\to 0}\frac{\theta(h, x)}{h} = \lim_{h\to 0}\frac{\varphi(h, x)}{h}\cdot\frac{\psi(\varphi(h, x), \Phi x)}{\varphi(h, x)} = a(x)b(\Phi x),$$

where $a$ and $b$ are the derivatives of $\varphi$ and $\psi$ respectively, we have

$$\begin{aligned} \int_X f d\mu^\theta &= \int_X fa(b\circ\Phi)d\mu = \int_X f(b\circ\Phi)d\mu^\varphi = \int_Y (f\circ\Phi^{-1})bdv \\ &= \int_Y f\circ\Phi^{-1}dv^\psi = \int_Z f\circ\Phi^{-1}\circ\Psi^{-1}d\rho, \quad \text{for all} \quad f \in L^1(X, d\mu). \end{aligned}$$

(Note that $\theta$ is of integral form.) q.e.d.

Next we consider the relation between the equivalence $\sim$ and the orbit equivalence.

DEFINITION 4.4. Let $(X, \mathscr{B}, \mu, \{S_t\})$ and $(Y, \mathscr{C}, v, \{T_t\})$ be two ergodic flows. They are said to be mutually *orbit equivalent* if there exists an onto, one to one, bimeasurable map $\Phi$: $X \to Y$ such that $v\circ\Phi$ is equivalent to $\mu$, and $\Phi\circ S_t x = T_{\varphi(t,x)}\circ\Phi x$ for all $t \in \mathbf{R}$, all $x$ in some $\{T_t\}$-invariant co-null set, where $\varphi$ is $\mathscr{B}(\mathbf{R}) \times \mathscr{B}/\mathscr{B}(\mathbf{R})$-measurable with $\varphi^x(t) \equiv \varphi(t, x)$ strictly increasing and continuous in $t$ for all $x$ in the set mentioned above. In this case we use the notation $\{S_t\} \mathrel{\mathscr{Q}} \{T_t\}$ and $\Phi$ is called an *orbit map*. □

REMARK 4.5. It is easy to see that $\varphi$ in Definition 4.4 is an additive cocycle in the sense of Definition 2.1. Furthermore $\varphi$ is an integrable one as we shall see later on (Theorem 4.9).

Next result, which is fundamental to introduce the Kakutani equivalence, was given in [4] without proof (see also [5]).

PROPOSITION 4.6. *Suppose that two ergodic flows* $(\tilde{X}, \tilde{\mathscr{B}}, \tilde{\mu}, \{\tilde{S}_t\})$ *and* $(\tilde{Y}, \tilde{\mathscr{C}}, \tilde{\nu}, \{\tilde{T}_t\})$ *are mutually orbit equivalent and* $\tilde{\Phi}$ *is an orbit map such that* $\tilde{\Phi}\circ\tilde{S}_t x = \tilde{T}_{\tilde{\varphi}(t,x)}\circ\tilde{\Phi}x$. *Then* $\tilde{\varphi}^x$ *is absolutely continuous for a.e.* $x$ *in* $\tilde{X}$.

Before going into the proof of Proposition 4.6, we prepare the next lemma.

LEMMA 4.7. *Suppose that an ergodic flow* $(\tilde{X}, \tilde{\mathscr{B}}, \tilde{\mu}, \{\tilde{S}_t\})$ *is orbit equivalent to an ergodic S-flow* $(Y_\beta, \mathscr{C}_\beta, \nu_\beta, \{(T, \beta)_t\})$. *Then there exists an S-representation* $(X_\alpha, \mathscr{B}_\alpha, \mu_\alpha, \{(S, \alpha)_t\})$ *of* $\{\tilde{S}_t\}$ *and an orbit map* $\Phi$ *from* $X_\alpha$ *to* $Y_\beta$ *(note that* $(S, \alpha)$ *is orbit equivalent to* $(T, \beta)$*) such that*

$$\Phi(X\times\{0\}) = Y\times\{0\}, \tag{4.6}$$

*i.e.* $\Phi$ *maps the base space of* $X_\alpha$ *onto that of* $Y_\beta$.

PROOF. Let $\tilde{\Phi}$ be an orbit map from $\tilde{X}$ to $Y_\beta$ such that

$$\tilde{\Phi}\circ\tilde{S}_t x = (T, \beta)_{\tilde{\varphi}(t,x)}\circ\tilde{\Phi}x.$$

Let $X=\tilde{\Phi}^{-1}(Y\times\{0\})$, $\mathscr{B}=\bar{\Phi}^{-1}(\mathscr{C})$, and define for $x$ in $X$

$$\alpha(x) = \tilde{\varphi}^{-1}(\beta\circ\bar{\Phi}x, x),$$

where $\bar{\Phi}=\pi\circ\tilde{\Phi}$: $X\to Y$ and $\pi$ is the natural projection from $Y_\beta$ to $Y$. Then it can be shown that $\alpha$ is measurable with respect to $\mathscr{B}$. Next define a transformation $S$ on $X$ by

$$Sx = \tilde{S}_{\alpha(x)}x.$$

Then one can show in the same way as in the special representation theorem of ergodic flows that

$$(S, \alpha) \simeq \{\tilde{S}_t\}$$

with the mapping

$$\Psi(x, u) = \tilde{S}_u x, \qquad \text{for} \quad (x, u)\in X_\alpha,$$

an isomorphism with respect to the $S$-invariant base measure

$$\mu(B) = \int_{B^*}\frac{1}{\alpha(x)}\,d(\tilde{\mu}\circ\Psi\circ\tilde{\pi}^{-1})(C_x), \qquad \text{for} \quad B\in\mathscr{B},$$

where $\tilde{\pi}$ is the natural projection from $X_\alpha$ to the factor space by the partition $\{C_x\colon x\in X\}$, $C_x=\{(x, u)\colon 0\leqq u<\alpha(x)\}$ and $B^*=\{C_x\colon x\in B\}$. It is clear that the

$S$-representation $(X_\alpha, \mathscr{B}_\alpha, \mu_\alpha, \{(S, \alpha)_t\})$ and the orbit map $\Phi = \tilde{\Phi} \circ \Psi$ from $(S, \alpha)$ to $(T, \beta)$ satisfy (4.6). q.e.d.

PROOF OF PROPOSITION 4.6. By the special representation theorem of ergodic flows, we may assume that $(\tilde{Y}, \tilde{\mathscr{C}}, \tilde{\nu}, \{\tilde{T}_t\})$ is an $S$-flow $(Y_\beta, \mathscr{C}_\beta, \nu_\beta, \{(T, \beta)_t\})$. Let $(X_\alpha, \mathscr{B}_\alpha, \mu_\alpha, \{(S, \alpha)_t\})$ be the $S$-representation of $\{\tilde{S}_t\}$ under the isomorphism $\Psi$ and $\Phi = \tilde{\Phi} \circ \Psi$ be the orbit map from $(S, \alpha)$ to $(T, \beta)$ in Lemma 4.7. Then we have the following equality.

$$\Phi \circ (S, \alpha)_t(x, u) = (T, \beta)_{\varphi(t,x,u)} \circ \Phi(x, u),$$

for all $t \in \mathbf{R}$ and all $(x, u)$ in a $(S, \alpha)$-invariant co-null set, where $\varphi(t, x, u) \equiv \tilde{\varphi}(t, \Psi(x, u))$. As for the measure $\nu_\beta \circ \Phi$, the following equality holds;

$$\nu_\beta \circ \Phi(B) = \int \lambda(\varphi^{(x,0)}(B_x)) d(\nu \circ \bar{\Phi})(x), \qquad \text{for} \quad B \in \mathscr{B}_\alpha,$$

where $B_x = \{u : (x, u) \in B\}$ and $\bar{\Phi}$ is the restriction of $\pi \circ \Phi$ to $X$ as before. From this it is easy to see that

$$\nu \circ \bar{\Phi} \sim \mu. \tag{4.7}$$

To show the assertion of the theorem, it is enough to see that $\varphi(t, x, u)$ is absolutely continuous for a.e. $(x, u)$ and this is assured by showing that $\varphi(t, x, 0)$ is absolutely continuous in $t$ on the interval $[0, \alpha(x))$ for $\mu$-a.e. $x$ in $X$. Suppose the theorem fails to hold. Then there exists $B \in \mathscr{B}$, $\mu(B) > 0$, such that $\varphi^{(x,0)}$ has a non-zero singular part in $[0, \alpha(x))$ for each $x \in B$. Therefore for each $x \in B$, there exists a $\lambda$-null set $N_x \subset [0, \alpha(x))$ such that

$$\bar{\varphi}^x_{sing}(N_x) > 0, \quad \text{where} \quad \bar{\varphi}^x_{sing} = \lambda \circ \varphi^{(x,0)}_{sing}.$$

Since $\bar{\varphi}^x_{sing}$ is concentrated on the set $E^c_x$, where

$$E = \{(x, u) : \lim_{h \to 0} \varphi(h, x, u)/h \text{ exists}\},$$

(note that $E^c_x = \{t \in [0, \alpha(x))$: the derivative of $\varphi^{(x,0)}$ at $t$ does not exist$\}$), we may take $N_x = E^c_x$. Let

$$F = \bigcup_{x \in B} E^c_x.$$

Then $F \subset E^c$, which shows the measurability of $F$. (Note that $E^c$ is a $\mu_\alpha$-null set as we have seen in the proof of Lemma 2.3.) By the definition of $F$, $\mu_\alpha(F) = 0$. But since $\nu \circ \bar{\Phi}(B) > 0$ by (4.7), we have

$$\nu_\beta \circ \Phi(F) = \int_B \bar{\varphi}^x_{sing}(E^c_x) d(\nu \circ \bar{\Phi})(x) > 0,$$

which shows a contradiction to the assumption that $\nu_\beta \circ \Phi$ is equivalent to $\mu_\alpha$.

q. e. d.

REMARK 4.8. In the proof of Proposition 4.6, the transformation $S$ coincides with $\bar{\Phi} \circ T \circ \bar{\Phi}^{-1}$. Hence we have that the measure $\nu \circ \bar{\Phi}$ is also $S$-invariant. But since $S$-invariant $\mu$ is equivalent to $\nu \circ \bar{\Phi}$ and any measurable cross-section of the ergodic fow $\{\tilde{S}_t\}$ is also ergodic, we can conclude that $\mu = c\nu \circ \bar{\Phi}$ for some constant $c$.

THEOREM 4.9. *$\{S_t\} \mathcal{Q} \{T_t\}$ if and only if $\{S_t\} \sim \{T_t\}$.*

PROOF. We may assume that $\{S_t\} = (S, \alpha)$ and $\{T_t\} = (T, \beta)$ respectively on $X_\alpha$ and $Y_\beta$ for some ergodic $S$, $T$ and positive $\alpha$, $\beta$.

Suppose that $(S, \alpha) \mathcal{Q} (T, \beta)$. Then there exists an orbit map $\Phi$ such that $\Phi \circ (S, \alpha)_t(x, u) = (T, \beta)_{\varphi(t,x,u)} \circ \Phi(x, u)$. As we have seen in Proposition 4.6, $\varphi^{(x,u)}$ is absolutely continuous for a.e. $(x, u)$, we have $\varphi(t, x, u) = \int_0^t f((S, \alpha)_s(x,u))ds$ for some positive function $f$ by Theorem 2.4. Therefore the measure $f\mu_\alpha$ is the induced measure by $\varphi$ which is $\sigma$-finite and $(S, \alpha)^\varphi$-invariant. But it can be easily seen that $\nu_\beta \circ \Phi$, which is equivalent to $\mu_\alpha$, is also $(S, \alpha)^\varphi$-invariant. Hence $f\mu_\alpha = c\nu_\beta \circ \Phi$ for some constant $c$ by the ergodicity of $(S, \alpha)^\varphi$, which shows the integrability of $f$ and this is equivalent to that of $\varphi$. Therefore we can consider the normalized time change, which is also denoted by $(S, \alpha)^\varphi$ as mentioned before, and it is easy to see that $(S, \alpha)^\varphi$ is isomorphic to $(T, \beta)$ under $\Phi$.

Conversely suppose that $(S, \alpha) \sim (T, \beta)$. Then there exists an additive cocycle $\varphi$ with the property (4.1) such that $(S, \alpha)^\varphi \simeq (T, \beta)$. It is a routine work to check that an isomorphism $\Phi$ between them is an orbit map from $(S, \alpha)$ to $(T, \beta)$.

q. e. d.

## § 5. Non-equivalence of the time changes by $\varphi$ and $\varphi_{ac}$

In this section, we shall treat a special topic in connection with Theorem 2.4 and Theorem 3.2.

Let $(S, \alpha)$ be an $S$-flow on $X_\alpha$ and $\varphi$ be an additive cocycle whose domain is $D$. Define $f_n$ ($n \in \mathrm{N}$) as follows.

$$f_n(x, u) = \frac{\varphi\left(\frac{k}{n}\alpha(x), x, 0\right) - \varphi\left(\frac{k-1}{n}\alpha(x), x, 0\right)}{\alpha(x)/n},$$

$$\text{if } x \in E \text{ and } \frac{k-1}{n}\alpha(x) \leqq u < \frac{k}{n}\alpha(x),$$

$$= 0, \quad \text{if} \quad x \in E^c,$$

where $E=\{x \in D: \varphi(\alpha(x), x, 0)>0\}$.

Let $\varphi_n$ ($n \in \mathbf{N}$) be the additive cocycle defined by $f_n$. Then it is easily shown by Theorem 3.2 that

$$(5.1) \qquad (S, \alpha)^{\varphi_n} \simeq (S, \alpha)^{\varphi}, \quad \text{for all} \quad n \in \mathbf{N}.$$

Let $f$ be the derivative of $\varphi$. Then a little computation shows

$$(5.2) \qquad f_n \longrightarrow f \quad \text{a.e.}$$

Comparing (5.1) and (5.2), it will be interesting to ask whether $(S, \alpha)^{\varphi_{ac}}$ is isomorphic to $(S, \alpha)^{\varphi}$ or not, where $\varphi_{ac}$ is the absolute continuous part of the decomposition of $\varphi$ in Theorem 2.4.

One way to give an answer to this question is to try to compare the entropies of the time changes. With the aid of Abramov's formula, one can compute the entropies of the time changes.

THEOREM 5.1 ([6]). *Let $(S, \alpha)$ be an ergodic S-flow whose base automorphism is $(X, \mathscr{B}, \mu, S)$ ($\int \alpha d\mu = 1$). Then for integrable $\varphi$*

$$h((S, \alpha)^{\varphi}) = h(S)\mu(X)/\|\varphi_1\|$$

*where the left hand side is the entropy of the normalized time change and $h(S)$ is that of the normalized automorphism $(X, \mathscr{B}, \mu/\mu(X), S)$.*

We also prepare the next lemma.

LEMMA 5.2. *Let $(X, \mathscr{B}, \mu, \{T_t\})$ be a flow, $\varphi$ and $\psi$ be additive cocycles with the common domain D, such that $\varphi \leqq \psi$ in the sense that*

$$\varphi(t, x) \leqq \psi(t, x), \quad \textit{for all} \quad t \geqq 0, \quad x \in D.$$

*Suppose that for each $x \in D$, if there exists $t_0$ such that*

$$(5.3) \qquad \varphi(t_0, x) = \psi(t_0, x)$$

*holds, then we have*

$$(5.4) \qquad \varphi(t, x) = \psi(t, x), \quad \textit{for all} \quad t \in [0, t_0] \quad \textit{and all} \quad x \in D.$$

PROOF. Let $0 \leqq t \leqq t_0$. Then from the additivity of $\varphi$ and $\psi$, we have

$$(5.5) \qquad \varphi(t_0, x) = \varphi(t, x) + \varphi(t-t_0, T_t x),$$

$$(5.6) \qquad \psi(t_0, x) = \psi(t, x) + \psi(t-t_0, T_t x).$$

The equation (5.4) follows from (5.3), (5.5), (5.6) and the condition $\varphi \leqq \psi$.

Though we restrict ourselves to simple cases, we have the next results.

THEOREM 5.3. *Let $(S, \alpha)$ be an ergodic S-flow with positive finite entropy, whose base automorphism is $(X, \mathscr{B}, \mu, S)$, and $\int \alpha d\mu = 1$. Let $\varphi$ and $\psi$ be integrable additive cocycles such that*

$$\varphi \leqq \psi. \tag{5.7}$$

*Suppose that*

$$(S, \alpha)^{\varphi} \simeq (S, \alpha)^{\psi}. \tag{5.8}$$

*Then $\varphi = \psi$.*

PROOF. The relation (5.8) and the condition on the entropy imply that

$$0 < h(S)\mu(X)/\|\varphi_1\| = h(S)\mu(X)/\|\psi_1\| < \infty$$

by Theorem 5.1. Then we have $\|\varphi_1\| = \|\psi_1\|$ and furthermore $\varphi_1 = \psi_1$ a.e. by (5.7).

It follows that $\varphi_n = \psi_n$ for all $n \in \mathbf{N}$, a.e. where $\varphi_n(x, u) \equiv \varphi(n, x, u)$ etc. Hence Lemma 5.2 shows that

$$\varphi_t = \psi_t, \quad \text{for all} \quad t > 0, \quad \text{a.e.} \quad \text{where} \quad \varphi_t(x, u) \equiv \varphi(t, x, u) \quad \text{etc.} \tag{5.9}$$

Similarly we have

$$\varphi_t = \psi_t, \quad \text{for all} \quad t < 0, \quad \text{a.e.} \tag{5.10}$$

The excluded set in (5.9) and (5.10) may be assumed to be $(S, \alpha)$-invariant, so that we have $\varphi_t = \psi_t$ for all $t \in \mathbf{R}$ with an elimination of some $(S, \alpha)$-invariant null set. This implies $\varphi = \psi$. q.e.d.

In particular, we have

COROLLARY 5.4. *Let $(S, \alpha)$ be the same as in Theorem 5.3 and $\varphi$ be an additive cocycle. Then $(S, \alpha)^{\varphi}$ cannot be isomorphic to $(S, \alpha)^{\varphi_{ac}}$ unless the trivial case $\varphi = \varphi_{ac}$, that is, unless $\varphi$ is of integral form.*

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