

Derivation of the Boltzmann equation from particle dynamics

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Introduction

This paper is exclusively concerned with the equation of Boltzmann (or Maxwell and Boltzmann) for a gas of identical hard spheres and its caricature. In his original derivation of the equation Boltzmann made crucial use of an assumption of *molecular chaos* or so-called *stosszahlansatz* which, groundlessly introduced, is acceptable for no better reason than that it is plausible or expedient and, lacking in precision of its meaning, obscures the relations of the Boltzmann equation to the underlying dynamics — while has been recognized its significance in the kinetic theory of gases, the Boltzmann equation hardly rested on any solid foundation. It therefore was (and is) highly desirable to derive the Boltzmann equation from the Liouville equation, i.e., to derive it from particle dynamics which is completely deterministic (causal) so that all the randomness introduced into the derivation comes only through the *initial randomness* of the particle configuration in the phase space. Through arguments (though not of mathematical rigor) based on the careful analysis of the Liouville equation and its reduced equation for the one particle correlation function H . Grad (1958) afforded an excellent insight into the nature of “*stosszahlansatz*”, pointing out, among others, the crucial role played by very small parts of the phase space through which the behavior of a correlation function is determined by that of higher order ones.

He also emphasized that for the validity of the Boltzmann equation the number of particles must be kept inversely proportional to the surface area of a particle (so that the mean free path $\asymp 1$), motivating the subsequent naming of the *Grad limit* (or Boltzmann-Grad limit). After C. Cercignani (1972) had brought into an explicit and exact form the BBGKY hierarchy for the hard sphere dynamics, which, introduced in a general setting through works published around 1946 by Bogoliubov, Born & Green, Kirkwood, and Yvon, is a chain of equations to be satisfied by a sequence of correlation functions, O. E. Lanford III (1975) showed, with the same dynamics, that a series expansion of the correlation function derived from the BBGKY hierarchy approaches in the Grad limit the expansion corresponding to the Boltzmann equation for short times. Though Lanford, in his paper [9], gave only an outline of his argument and did not prove the BBGKY hierarchy or the series expansion, the details of the former was provided by F. King [8], while H. Spohn, in his notes [15], proved the latter two. (The latter problem is treated also in the appendix of [6].)

The present article primarily (i.e., except for §7) aims at giving a complete and coherent exposition to the Lanford derivation of the Boltzmann equation, including proofs of the BBGKY hierarchy and the series expansion which are somewhat different from Spohn's. This, however, will be carried out first with a model dynamics different from that of hard spheres. The model is not realistic (the momentum is not conserved through a collision) but still possesses the essential feature of hard sphere dynamics. In this model the modulus of the velocity of each particle can be assumed to be unity throughout since the collision there does not change it. This allows us to work through the all steps of the derivation of the Boltzmann equation for our model without annoyed by the unbounded factors of the velocity variable which in the hard sphere case appear in the series expansion of the correlation functions. The unboundedness of the velocity becomes essentially relevant only in the last stage of our derivation where we argue about the convergence of the series which represents the correlation function and must obtain a certain bound of each term of it. After finishing a full story with the caricature we shall get a bound for the hard sphere dynamics, which makes up the deficiency originating in our working with the caricature to ensure that the story is valid also for it.

In §0 we make a heuristic argument to formally derive the first equation of the BBGKY hierarchy and introduce the first two terms of the series expansion, which together will help one to understand the mechanism how the Boltzmann equation emerges from particle dynamics. The model dynamics mentioned above is introduced in §1 and the fundamental facts about it are deduced in §§2 and 3. It is shown especially that the set of configurations which eventually lead to a triple or higher order collision is Lebesgue null and the flow defined by the dynamics in the phase space preserves the Lebesgue measure. With the

flow being defined only up to the time when a multiple collision if any occurs, the proofs of these two assertions are intertwined in a way. The proofs of the BBGKY hierarchy and the series expansion are given in §4. There we impose, on the density f_n of the initial configuration of n particles, the continuity at almost every phase point, different from the one assumed by others ([6], [8], [9], [15] etc.), i.e., the continuity along trajectories. In §5 we study the Boltzmann equation and the Boltzmann hierarchy. The deduction of the Boltzmann equation from the series expansion is made in §6. In §7 we treat our system of particles along the idea of M. Kac [7] and H. P. McKean [12]; we shall prove that the stochastic process of a tagged particle converges (for short times) to a Markov process governed by the collision operator linearized about the solution of the Boltzmann equation. In §8 we give some comments and estimates with the help of which all the arguments made through §§1 to 7 are applied word for word or at most with a minor modification to the hard sphere dynamics.

§0. The Boltzmann equation for the hard core potential and its heuristic derivation

In this section we introduce the Boltzmann equation for the hard core potential and make a heuristic argument for the derivation of it. All the succeeding sections may be read independently of the present section.

The Boltzmann equation is an integro-differential equation which is to describe the time evolution of the single particle distribution in dynamical process of many particles. Let q and v be elements of R^3 and stand for the position and velocity of a single particle, and let $u(t, q, v)$ denote the density of the distribution of them. Then the Boltzmann equation for hard core potential without the outer force is given by

$$(0.1) \quad \frac{\partial}{\partial t} u(t, q, v) + v \cdot \frac{\partial}{\partial q} u(t, q, v) \\ = \int_{S^{(2)}} \int_{R^3} [u(t, q, v^*)u(t, q, v_1^*) - u(t, q, v)u(t, q, v_1)] |(v - v_1) \cdot l| dl dv_1 .$$

Here dl is a surface element on the two-dimensional unit sphere $S^{(2)}$ and (v^*, v_1^*) denotes the velocities of two particles after a collision in which velocities before the collision are v and v_1 and the second particle runs into the first one at a point on the first's surface in the direction l from its center, so that for each $l \in S^{(2)}$ there corresponds a (linear) transformation $A_l: (v, v_1) \in R^6 \rightarrow (v^*, v_1^*) \in R^6$ (see the beginning of §8 for details). Multiplying the both sides of (0.1) by a smooth test function $\phi = \phi(q, v)$, integrating them and making use of the measure preserving property of the transformation A_l (for each l), we get another version of (0.1):

$$(0.2) \quad \frac{d}{dt} \langle \phi, u(t) \rangle = \left\langle v \cdot \frac{\partial \phi}{\partial q} + K_{u(t)} \phi, u(t) \right\rangle$$

where $\langle \phi, u(t) \rangle = \int_{R^6} \phi(q, v) u(t, q, v) dq dv$ and

$$K_{u(t)} \phi(q, v) = \int_{R^3} \int_{S^{(2)}} (\phi(q, v^*) - \phi(q, v)) |(v - v_1) \cdot l| u(t, q, v_1) dv_1 dl.$$

Now consider the gas of n identical hard spheres (particles) of diameter ε moving in the 3-dimensional flat torus R^3/Z^3 . (It is not essential to the present problem whether the space in which particles move is a finite region (as far as it is surrounded by smooth elastic walls), the torus or the whole R^3 -space). Two particles collide with each other elastically. The n particle phases, i.e., the configurations (of positions and velocities) of n particles, which eventually lead to triple or higher order collisions form a set of Lebesgue measure zero (Theorem 3.1) and may be neglected. Starting from each phase outside this set the flow of n particles is determined for all times. Suppose that the n particle phase is randomly distributed at time zero according to a density f_n which is symmetric with respect to n particles. Let us choose any particle, tag it and then pursue it. Let $x_t = [q_t, v_t]$ stand for its phase at time t ; x_t is then a stochastic process, which, in our lacking in knowledge about the other particles, appears to change its course and speed haphazardly. Assume that the distribution of x_t has a continuous density, which we denote by $u_{n_1}(t, x)$, $x = [q, v]$. We would like to compute the derivative of $\langle u_{n_1}(t), \phi \rangle$, i.e., the derivative of the expectation $E\phi(x_t)$. If the tagged particle does not encounter with any other particle during a small time interval $(t, t + dt)$, then $v_{t+dt} = v_t$ and $q_{t+dt} = q_t + v_t dt$. What then in the case when a collision takes place? Let $(v_t - v') \cdot l > 0$. Then it collides at a time $t + s$ at a point, $q_t + sv_t + (\varepsilon/2)l$, on its surface with another particle having the velocity v' if the latter has passed the position

$$(0.3) \quad q' = q_t + s(v_t - v') + \varepsilon l$$

just at time t , provided their presumed courses of motion have not been intercepted by the remaining particles. The probability of having such a particle in $dq' dv'$ may be expressed by means of the density of conditional probability, $w(t, x'|x)$, of any remaining particle being found in its phase x' at time t given $x_t = x$. Under some continuity condition on $w(t, x'|x)$ this probability is given by $w(t, [q_t + \varepsilon l, v'] | x_t) dq' dv'$ where q' is not a free variable but determined through (0.3) so that $dq' = \varepsilon^2 (v_t - v') \cdot l dl dt$ (ds is replaced by dt). Since there are altogether $n - 1$ particles whose phase may be in $dq' dv'$ and the probability for a third particle to intercept the encounter between the two particles must be of the higher order, the difference $E\phi(x_{t+dt}) - E\phi(x_t)$ may agree with the expectation of $v_t \cdot (\partial \phi / \partial q)(x_t) dt$ plus that of

$$\varepsilon^2(n-1) \int_{(v_i-v'), l>0} (\phi(x_i^*) - \phi(x_i)) w(t, [q_i + \varepsilon l, v'] | x_i) (v_i - v') \cdot l dl dv' dt$$

up to a small order term of dt , where $x_i^* = (q_i, v_i^*)$, v_i^* being the first R^3 -component of $A_i(v_i, v')$. If $u_{n|2}(t, x, x')$ denotes the density of two particle phase distribution (two particle correlation function) at time t , then $w(t, x' | x) u_{n|1}(t, x) = u_{n|2}(t, x, x')$. Consequently

$$(0.4) \quad \frac{d}{dt} \langle \phi, u_{n|1}(t) \rangle - \left\langle v \cdot \frac{\partial}{\partial q} \phi, u_{n|1}(t) \right\rangle = \varepsilon^2(n-1) \int_{(v-v'), l>0} (\phi(x^*) - \phi(x)) u_{n|2}(t, x, [q + \varepsilon l, v']) (v - v') \cdot l dl dv',$$

where $x = [q, v]$, $x^* = [q, v^*]$, and v^* is the first R^3 -component of $A_i(v, v')$. Now take the limit of $\varepsilon \rightarrow 0$ and $n \rightarrow \infty$ in such a way that $\varepsilon^2 n$ converges to 2 (the Grad limit), assume that $u_{n|1}$ converges to a limit, $u(t)$ say, and employ the stosszahlansatz which may here be interpreted as

$$(0.5) \quad \lim u_{n|2}(t, [q, v], [q + \varepsilon l, v']) = u(t, q, v) u(t, q, v')$$

which is claimed for $(v - v') \cdot l > 0$, then you have the Boltzmann equation (0.2) (the contribution to the integral of the collision term from $(v - v') \cdot l > 0$ equals that from $(v - v') \cdot l < 0$ except for the minus sign). It is noted that the phase of two particles entering in the left hand side of (0.5) is that of the “in-coming” collision. In other words the chaos property as expressed in (0.5) is needed for and “only for particles which are about to collide” as H. Grad put it. (See [5] and Remark 6.3 of the present paper for further discussions.)

The equation (0.4) is the first (or last) of the chain of equations called BBGKY hierarchy (as it has been after Bogoliubov, Born, Green, Kirkwood and Yvon). The m -th equation ($m \geq n$) of it relates the m particle correlation function $u_{n|m}(t)$ with that of $m + 1$ particles just as (0.4) relates $u_{n|1}(t)$ with $u_{n|2}(t)$. Its integrated form is iterated to produce a series expansion of $u_{n|m}(t)$. Let $T_t^{(m)}$ denote the m particle flow and $f_{n|m}$ the initial correlation function $u_{n|m}(0)$. Then it expresses $u_{n|m}(t)$ as a sum of $f_{n|m} \circ T_{-t}^{(m)}$ and a series of $n - m$ integrals of $f_{n|k}$'s, $m < k \leq n$; the first term of the series may read

$$\varepsilon^2(n-m) \sum_{j=1}^m \int_0^t ds \int_{R^3} \int_{S^{(2)}}^* f_{n|m+1}(T_{-t+s}^{(m+1)} C_{j,l}^{v',l} T_{-s}^{(m)} \mathbf{x}) (v_j - v) \cdot l dl dv$$

where \mathbf{x} is a m particle phase and $C_{j,l}^{v',l}$ denotes the operation of adding an extra particle beside the j -th particle of a configuration on which it operates according to $C_{j,l}^{v',l} \mathbf{x} = (\mathbf{x}, [q_j + \varepsilon l, v])$, and the inner-most integral extends over those l for which the added particle shares no spatial region with the other particles. If the addition of a particle results in a configuration of “out-going” collision or

equivalently $(v_j - v) \cdot l < 0$, the operations of $T_{-t+s}^{(m+1)}$ makes it first instantaneously turn into the corresponding one of in-coming collision and then evolve backward in time. (In §6 we shall use a slightly different notation to explicitly indicate the types of collisions.) From this simplest case the form of the other integrals may be guessed: in the k -th integral the operation of the addition and the backward motion are repeated k times succeeding to the initial operation of $T_{-s_1}^{(m)}$. Then it would be seen that each integral would converge to the one corresponding to free motion of point particles, i.e., to the corresponding one in the series expansion for the Boltzmann equation. Thus the Boltzmann equation might be derived under some assumption about the convergence of $f_{n|m}$'s, if each term of the series expansion remains within a bound such that it enables us to take the limit term-wise. Such a bound is obtained for a short time interval whose length is determined according to the bound for $f_{n|m}$'s, but the reasoning for it apparently fails to work for longer times. In the two dimensional case where particles (elastic disks) move the whole R^2 , the initial distribution is a local perturbation from vacuum, and the mean free path ($\sim n \times$ the radius of a disk) is sufficiently small, Illner and Pulvirenti [6] established the validity of the derivation of the Boltzmann equation for all times. It seems however very difficult to establish it for all times in more general situations, especially in a situation where the effect of collisions does not fade out for large times (this is not the case treated in [6]).

§1. Description of the model and notations

Let $S_2 = R^2/Z^2$ the two-dimensional torus. We will study an n -particle dynamics in S_2 , which is described below. A *particle* is a circular disk of diameter ε ($0 < \varepsilon \ll 1$) whose center $q \in R^2$ represents the position of the particle. It moves with a constant velocity v of modulus one: $|v| = 1$ until it collides with the other particles. Two particles *collide* and change their velocities instantaneously when they come to touch each other. Let q and q_1 be positions of two particles at the moment of the collision and let v and v_1 be their respective velocities just before the collision.

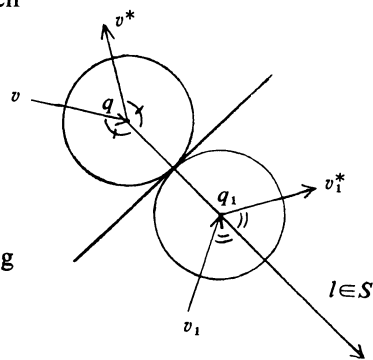
Then the velocities after the collision, denoted by v^* and v_1^* , are given by

$$v^* = v - 2(v \cdot l)l$$

$$v_1^* = v_1 + 2(v_1 \cdot l)l$$

if $(v - v_1) \cdot l \neq 0$, where l is a unit vector pointing in the same direction as $q_1 - q$:

$$l = \frac{1}{\varepsilon}(q_1 - q)$$



(see fig.). In other words, particles behave as if there is an elastic wall which is tangent to both disks at the time of collision. For the *grazing* collision:

$$(v - v_1) \cdot l = 0$$

we set $v^* = v$, $v_1^* = v_1$. By collision is not changed the modulus of velocities:

$$|v| = |v^*| \quad \text{and} \quad |v_1| = |v_1^*|.$$

In particular the energy is preserved, but not the momentum is. For all collisions possible in this dynamics it holds that

$$(v - v_1) \cdot l = - (v^* - v_1^*) \cdot l \geq 0.$$

The velocity v and the collision parameter l are understood to be elements of the unit circle, denoted by S .

The crucial feature of the transformation

$$(1.1) \quad A_l: (v, v_1) \longrightarrow (v^*, v_1^*)$$

is involved in the following lemma.

LEMMA 1.1. For all bounded measurable functions F on S^5

$$\begin{aligned} & \iiint_{(v-v_1) \cdot l > 0} F(v, v_1, v^*, v_1^*, l) (v - v_1) \cdot l \, dv \, dv_1 \, dl \\ &= \iiint_{(v-v_1) \cdot l > 0} F(v^*, v_1^*, v, v_1, -l) (v - v_1) \cdot l \, dv \, dv_1 \, dl. \end{aligned}$$

PROOF. Since the transformation (1.1) preserves the measure $dv \, dv_1$, from the identities $A_l^{-1} = A_l$ and $(v - v_1) \cdot l = - (v^* - v_1^*) \cdot l$ it follows that for each l

$$\begin{aligned} & \iint_{(v-v_1) \cdot l > 0} F(v, v_1, v^*, v_1^*, l) (v - v_1) \cdot l \, dv \, dv_1 \\ &= - \iint_{(v^*-v_1^*) \cdot l < 0} F(A_l^{-1}(v^*, v_1^*), v^*, v_1^*, l) (v^* - v_1^*) \cdot l \, dv \, dv_1 \\ &= - \iint_{(v-v_1) \cdot l < 0} F(v^*, v_1^*, v, v_1, l) (v - v_1) \cdot l \, dv \, dv_1. \end{aligned}$$

The desired equality is obtained after the integration by l where we make a change of the variable according to $l \rightarrow -l$. Q. E. D.

Let us define the dynamics of n particles. We shall assume, unless the contrary is stated, that n particles are initially located in such a way that they do not overlap one another so that the dynamics is described as a flow in the phase space

$$\Omega_n^{(\varepsilon)} = \{(\mathbf{q}, \mathbf{v}) : \mathbf{q} = (q_1, \dots, q_n) \in S_2^n, \mathbf{v} = (v_1, \dots, v_n) \in S^n, |q_i - q_j| \geq \varepsilon \text{ if } i \neq j\}.$$

($|q|$ denotes the Euclidean length of $q \in S_2$ which is understood to be represented by a point of $(-1/2, 1/2]^2$.) *Multiple* (i.e., triple and higher order) collisions are undefined. If the system happens to come into the multiple touch, then it is *stopped* (frozen) at that time. For simplifying the arguments we also *stop* the system both at the time of grazing collision and of twin collisions (where two pairs are in touch simultaneously).

The trajectory drawn in $\Omega_n^{(\varepsilon)}$ by the system has discontinuity at the time of a collision. To make things definite and clear we shall take a *left continuous version* of a trajectory.

To give formal definitions let us introduce following notations.

$$\Omega_n^0 = \Omega_n^{(0)} = S_{2n} \times S^n.$$

$[q, v] \in \Omega_1^{(\varepsilon)}$ (i.e. the square bracket is used to denote a phase point of one particle).

$$\mathbf{x} = (x_1, \dots, x_n) = (q_1, v_1, \dots, q_n, v_n) = (\mathbf{q}, \mathbf{v}) \in \Omega_n^{(\varepsilon)},$$

$$\text{where } x_i = [q_i, v_i], \mathbf{q} = (q_1, \dots, q_n), \mathbf{v} = (v_1, \dots, v_n).$$

$T_i^0 = T_i^{(0,n)}$: the flow of the n -particle free motion, i.e.,

$$T_i^0 \mathbf{x} = (\mathbf{q} + t\mathbf{v}, \mathbf{v}) = ([q_1 + tv_1, v_1], \dots, [q_n + tv_n, v_n]).$$

$$\partial\Omega_n^{(\varepsilon)} = \{\mathbf{x} \in \Omega_n^{(\varepsilon)} : |q_i - q_j| = \varepsilon \text{ for some pair } i \neq j\}.$$

$\Sigma_n^{(\varepsilon)} = \{\mathbf{x} \in \partial\Omega_n^{(\varepsilon)} : \mathbf{x} \text{ is in a multiple touch, (at least) two pairs from } \mathbf{x} \text{ are in collision, or a pair from } \mathbf{x} \text{ is in grazing collision}\}.$

Let $\mathbf{x} \in \partial\Omega_n^{(\varepsilon)} \setminus \Sigma_n^{(\varepsilon)}$. Then there is just one pair $1 \leq i < j \leq n$ for which $|q_i - q_j| = \varepsilon$. Let $l = (q_i - q_j)/\varepsilon$ and

$$(v_i^*, v_j^*) = A_l(v_i, v_j)$$

$$\mathbf{x}^* = (\dots, [q_i, v_i^*], \dots, [q_j, v_j^*], \dots)$$

(entries other than v_i and v_j are unchanged). We say \mathbf{x} (or (x_i, x_j)) is in *in-coming* collision or *out-going* collision according as

$$(v_i - v_j) \cdot l > 0 \quad \text{or} \quad (v_i - v_j) \cdot l < 0.$$

If \mathbf{x} is in in-coming [out-going] collision, then \mathbf{x}^* is in out-going [resp. in-coming] collision.

The n -particle flow $T_t = T_t^{(n,\varepsilon)}$ ($t \geq 0$) is now defined as follows. Let $\sigma(\mathbf{x})$ be the first time when the free motion starting at \mathbf{x} arrives at $\partial\Omega_n^{(\varepsilon)}$ after time zero, i.e.,

$$\sigma(\mathbf{x}) = \inf \{t > 0: T_t^0 \mathbf{x} \in \partial \Omega_n^{(\varepsilon)}\}.$$

Then in the first step set

if $\mathbf{x} \in \Sigma_n^{(\varepsilon)}$, $T_t \mathbf{x} = \mathbf{x}$ for all $t \geq 0$;

if $\mathbf{x} \in \Omega_n^{(\varepsilon)} \setminus \partial \Omega_n^{(\varepsilon)}$, $T_t = T_t^0 \mathbf{x}$ for $0 \leq t \leq \sigma(\mathbf{x})$;

if $\mathbf{x} \in \partial \Omega_n^{(\varepsilon)} \setminus \Sigma_n^{(\varepsilon)}$ and \mathbf{x} is in out-going collision, then

$$(1.2) \quad T_0 \mathbf{x} = \mathbf{x}^* \text{ and } T_t \mathbf{x} = T_t^0 \mathbf{x} \text{ for } 0 < t \leq \sigma(\mathbf{x});$$

if $\mathbf{x} \in \partial \Omega_n^{(\varepsilon)} \setminus \Sigma_n^{(\varepsilon)}$ and \mathbf{x} is in in-coming collision, then

$$(1.3) \quad T_0 \mathbf{x} = \mathbf{x} \text{ and } T_t \mathbf{x} = T_t^0 \mathbf{x}^* \text{ for } 0 < t \leq \sigma(\mathbf{x});$$

and in the succeeding steps repeat the same procedure.

Let $\sigma_n = \sigma_n(\mathbf{x})$, $n = 1, 2, \dots$, be the successive times of the collisions in the above repeated procedures. If $\{\sigma_n\}$ is unbounded, then is determined $T_t \mathbf{x}$ for all $t \geq 0$, which satisfies the semi-group property

$$T_{t+s} \mathbf{x} = T_t T_s \mathbf{x}, \quad s \geq 0, \quad t \geq 0,$$

and whose position component the Lipshitz continuity

$$(1.4) \quad |Q(T_t \mathbf{x}) - Q(T_s \mathbf{x})| \leq \sqrt{n} |t - s|,$$

where

$$Q(\mathbf{x}) = \mathbf{q} \quad \text{for } \mathbf{x} = (\mathbf{q}, \mathbf{v}).$$

If the sequence $\{\sigma_n\}$ is bounded, then either the system hits $\Sigma_n^{(\varepsilon)}$ at some σ_n , being stopped there for ever since then, or it increasingly approaches a constant, t_0 say. In the latter case the Lipshitz continuity (1.4) is valid for $0 \leq s < t < t_0$, showing that there exists $\lim_{t \uparrow t_0} Q(T_t \mathbf{x})$ which must be a spatial configuration for a multiple touch, so that the system is frozen at the time t_0 (any velocity configuration may be assigned). Thus $T_t \mathbf{x}$ is defined for all $t \geq 0$. Clearly the semi-group property and the Lipshitz continuity (1.4) valid for all $t, s \geq 0$.

Our dynamics is time-reversible in an obvious way so that $\{T_t, t \leq 0\}$ is defined in the same manner as above. Note that if $\mathbf{x} \in \partial \Omega_n^{(\varepsilon)} \setminus \Sigma_n^{(\varepsilon)}$, $T_0 \mathbf{x}$ is always in in-coming collision and does not always agree with \mathbf{x} .

Set

$$\tau = \tau(\mathbf{x}) = \inf \{t \geq 0: T_t \mathbf{x} \in \Sigma_n^{(\varepsilon)}\},$$

$$\tau^{(-)} = \tau^{(-)}(\mathbf{x}) = \sup \{t \leq 0: T_t \mathbf{x} \in \Sigma_n^{(\varepsilon)}\}.$$

Here (and below) the infimum [supremum] of the empty set is understood to be

∞ [resp. $-\infty$]. As incorporated in the discussion above the number of collisions experienced by the flow $T_t \mathbf{x}$ in the finite interval $[t_1, t_2]$ is finite if $\tau^{(-)} < t_1 < t_2 < \tau$.

LEMMA 1.2. $G = \{(t, \mathbf{x}) \in R \times \Omega_n^{(\varepsilon)} : T_t \mathbf{x} \notin \partial \Omega_n^{(\varepsilon)}\}$ is an open subset of $R \times \Omega_n^{(\varepsilon)}$ and the map: $(t, \mathbf{x}) \rightarrow T_t \mathbf{x}$ from G into $\Omega_n^{(\varepsilon)}$ is continuous.

PROOF. Let $(t, \mathbf{x}) \in G$ and $t > 0$. Since $\mathbf{x} \in \{\tau > t\}$, the flow starting at \mathbf{x} experiences at most a finite number of collisions in the interval $[0, t]$, all of which must be pairwise and proper (i.e., not grazing). Therefore we can choose a finite sequence $0 = t_0 < t_1 < \dots < t_m = t$ such that there is at most one collision between t_i and t_{i+1} and $T_{t_i} \mathbf{x} \notin \partial \Omega_n^{(\varepsilon)}$, $i = 1, \dots, m$. It is easy to show that the map $T_{t_{i+1}-t_i}$ is continuous at $T_{t_i} \mathbf{x}$ for $i = 1, 2, \dots, m-1$ and the map: $(s, \mathbf{y}) \rightarrow T_{t_1+s} \mathbf{y}$ is continuous at $(0, \mathbf{x})$. These together with the semigroup property of T_t , $t \geq 0$ implies that the map $(t', \mathbf{x}') \rightarrow T_{t'} \mathbf{x}'$ is continuous at (t, \mathbf{x}) . One can similarly proceed in case when $t \leq 0$. In particular G is open. Q. E. D.

NOTATIONS. Throughout the paper we shall use the following notations and conventions in addition to those introduced above.

$$U_n^{(\varepsilon)}(t)\phi(\mathbf{x}) := \phi(T_{-t}^{(n,\varepsilon)} \mathbf{x}) \text{ for } \phi \text{ a function on } \Omega_n^{(\varepsilon)}.$$

$$Q_i^{(n,\varepsilon)}(t) = Q_i^{(n,\varepsilon)}(t, \mathbf{x}): \text{ the } i\text{-th position component of } T_t^{(n,\varepsilon)} \mathbf{x}.$$

$$V_i^{(n,\varepsilon)}(t) = V_i^{(n,\varepsilon)}(t, \mathbf{x}): \text{ the } i\text{-th velocity component of } T_t^{(n,\varepsilon)} \mathbf{x}.$$

$$X_i^{(n,\varepsilon)}(t) := [Q_i^{(n,\varepsilon)}(t), V_i^{(n,\varepsilon)}(t)].$$

$$Q^{(n,\varepsilon)}(t) := (Q_1^{(n,\varepsilon)}(t), \dots, Q_n^{(n,\varepsilon)}(t)) \text{ and similarly for } V^{(n,\varepsilon)}(t) \text{ and } X^{(n,\varepsilon)}(t).$$

$\mathbf{F}[\mathbf{x}, I]$ ($\mathbf{x} \in \Omega_n^{(\varepsilon)}$, I is an interval of R) denotes a trajectory in $\Omega_n^{(\varepsilon)}$ of the flow $T_t \mathbf{x}$ during the time interval I , i.e. the history of the function $t \in I \rightarrow T_t \mathbf{x} \in \Omega_n^{(\varepsilon)}$.

$$D_n = D_n^{(\varepsilon)} := \{\mathbf{q} \in S_2^n : |q_i - q_j| \geq \varepsilon, \text{ for all } i \neq j\}.$$

$$\langle f, \phi \rangle := \int_{\Omega_n} f \phi d\mathbf{x} \text{ for } f, \phi \text{ bounded measurable functions on } \Omega_n; d\mathbf{x} := dx_1 \cdots dx_n \text{ (} dx_i = dq_i dv_i \text{ is the Lebesgue measure of } S_2 \times S \text{)}.$$

It is convenient to introduce an *extra point* ∂ in the following way. When we are concerned with the flow $T_t^{(n,\varepsilon)}$, all the points of $S_2^n \times S^n \setminus \Omega_n^{(\varepsilon)}$ are identified with ∂ (a single point) and ∂ is added to $\Omega_n^{(\varepsilon)}$ as an *isolated point*; then set

$$(1.5) \quad T_t^{(n,\varepsilon)} \mathbf{x} = T_t^{(n,\varepsilon)} \partial = \partial \quad \text{for } \mathbf{x} \in S_2^n \times S^n \setminus \Omega_n^{(\varepsilon)}$$

for all t . Any function f on $\Omega_n^{(\varepsilon)}$ will be automatically extended to the function on $S_2^n \times S^n \simeq \Omega_n^{(\varepsilon)} \cup \{\partial\}$ by setting $f(\partial) = 0$ so that if f is a continuous function

on $\Omega_n^{(\varepsilon)}$, then it is considered to be a function on $S_2^n \times S^n$, which is still continuous though it is identically zero outside $\Omega_n^{(\varepsilon)}$.

The super- or sub-scripts n and ε in $T_i^{(n,\varepsilon)}$, $\Omega_n^{(\varepsilon)}$, $X_i^{(n,\varepsilon)}$ etc. will often be omitted if doing this gives rise to no fear of misunderstanding.

REMARK 1.1. i) In the model introduced above $v^*[v_1^*]$ does not depend on v_1 [resp. v]. This fact however will not be used at all in this paper.

ii) It is emphasized that Lemmas 1.1 and 1.2 together with their proofs and all the arguments in Sections 2 to 4, where the BBGKY hierarchy for correlation functions will be derived, are applicable to the dynamics of the hard sphere model with little modification. This is because the conservation of energy allows us to assume that the velocities are bounded as far as the number of particles are fixed. Even when the number of particles goes to infinity so that such boundedness of velocities can not be assumed, it is justified by using Lemma 8.1 of this paper that we may formally apply the arguments for the present model given in §§5 to 7 (with minor modifications) to deduce corresponding results for the hand sphere case.

iii) It is only to make the situation simple and transparent and to focus our attention on the essential part of the problem treated in this paper that we adopt the two dimensional torus S_2 as the space on which particles move. In fact it is easy to extend the present arguments to the case where S_2 is replaced by a d -dimensional vessel which is surrounded by smooth elastic walls.

§2. The two-particle system

In this section we are concerned with the dynamics of two particles. We denote the phases of the first and second particle by $x = [q, v]$ and by $x_1 = [q_1, v_1]$, respectively, and write $\mathbf{x} = (x, x_1)$.

Let $t > 0$ and E_t denote the set of all configurations $\mathbf{x} \in \Omega_2 = \Omega_2^{(\varepsilon)}$ such that the flow starting from \mathbf{x} at least once experiences a collision in the time interval $[0, t)$. By introducing the parameter s which stands for the time of the first collision, we see

$$E_t = \{ \mathbf{x} \in \Omega_2 : q_1 - q = \varepsilon l + s(v - v_1) \text{ and } (v - v_1) \cdot l \geq 0 \text{ for some } 0 \leq s < t \text{ and } l \in S \}.$$

For points of E_t , fixing q, v and v_1 , we can consider q_1 as a function of s and l by the equality in the braces above, having

$$(2.1) \quad dq_1 = \varepsilon(v - v_1) \cdot l \, ds$$

and

$$(2.2) \quad \int_{E_t} \phi(T_t \mathbf{x}) f(\mathbf{x}) d\mathbf{x} = \int_{\Omega_1} d\mathbf{x} \int_s dv_1 \int_0^t ds \int_{(v-v_1) \cdot l > 0} \phi([q+sv+(t-s)v^*, v^*], [q+sv+\varepsilon l+(t-s)v_1^*, v_1^*]) \\ \times f(\mathbf{x}, [q+\varepsilon l+s(v-v_1), v_1]) \varepsilon(v-v_1) \cdot l dl$$

for any bounded measurable functions ϕ and f on Ω_2 , provided that t is small enough to the size of the torus (it suffices that $t < \varepsilon + 1/2$).

LEMMA 2.1. *The Lebesgue measure on $\Omega_2 = \Omega_2^{(\varepsilon)}$ is preserved by the flow $T_t = T_t^{(\varepsilon, 2)}$.*

PROOF. In the integral on the right-hand side of (2.2) with $f \equiv 1$ we first carry out the integration w.r.t. q by changing the variable according to

$$q \longrightarrow q + tv^* + s(v-v^*)$$

to see

$$\int_{E_t} \phi(T_t \mathbf{x}) d\mathbf{x} = \int dv \int dv_1 \int_0^t ds \int_{(v-v_1) \cdot l > 0} dl \int \varepsilon(v-v_1) \cdot l \\ \times \phi([q, v^*], [q+\varepsilon l-(t-s)(v^*-v_1^*), v_1^*]) dq.$$

Then by applying Lemma 1.1 and using an analogue of (2.2) we get

$$(2.3) \quad \int_{E_t} \phi(T_t \mathbf{x}) d\mathbf{x} = \int_{E_{-t}} \phi(\mathbf{x}) d\mathbf{x}.$$

Here E_{-t} is defined in the same way as E_t but for the time-reversed flow. Since

$$T_t^0(\Omega_2 \setminus E_t) = \Omega_2 \setminus E_{-t},$$

(2.3) yields

$$\int_{\Omega_2} \phi(T_t \mathbf{x}) d\mathbf{x} = \int_{\Omega_2 \setminus E_t} \phi(T_t^0 \mathbf{x}) d\mathbf{x} + \int_{E_t} \phi(T_t \mathbf{x}) d\mathbf{x} = \int_{\Omega_2} \phi(\mathbf{x}) d\mathbf{x}.$$

Q. E. D.

§ 3. The n -particle system

The purpose of this section is to prove Theorem 3.1 below and to prepare Lemma 3.4 which will play crucial role in the next section. In the course of proofs of these results we shall define certain sets and functions by means of T_t , and one may raise questions of whether they are measurable. We shall not discuss such questions until Appendix I where will be proved the measurability of T_t as well as that of sets and functions appearing in this section.

Let τ and $\tau^{(-)}$ be the arrival time at $\Sigma_n^{(\varepsilon)}$ as defined in Section 1. The parameter ε will be suppressed from the notations in below. The Lebesgue measure on Ω_n (as a subspace of $[0, 1)^{2n} \times [0, 2\pi]^n$) is denoted by $|\cdot|$.

THEOREM 3.1. i) $\tau = -\tau^{(-)} = \infty$ a.e. on Ω_n . ii) For each t the transformation T_t preserves the Lebesgue measure on Ω_n .

For the proof of Theorem 3.1 we first prove

LEMMA 3.1. For $t \geq 0$ and a bounded Borel function ϕ on Ω_n

$$\int_{\tau > t} \phi(T_t \mathbf{x}) d\mathbf{x} = \int_{-\tau^{(-)} > t} \phi(\mathbf{x}) d\mathbf{x}.$$

REMARK 3.1. As a corollary of Lemma 3.1 we have that if $A \subset \{-\tau^{(-)} > t\}$ is a Borel set, then $|T_t^{-1}A| = |A|$, because the premiss of this statement implies that $T_t^{-1}A \subset \{\tau > t\}$ and one can take $\phi = \chi_A$ (the indicator function of A) in Lemma 3.1.

For the proof of Lemma 3.1 we prepare

LEMMA 3.2. Let A be a Borel set contained in $\{\tau > t\}$ ($t > 0$) and ϕ a Borel measurable function defined on $T_t A$. If the number of collisions in $F[\mathbf{x}, [0, t]]$ is at most one for every $\mathbf{x} \in A$, then

$$\int_A \phi(T_t \mathbf{x}) d\mathbf{x} = \int_{T_t A} \phi(\mathbf{x}) d\mathbf{x}.$$

PROOF. Let

$A^{i,j} = \{\mathbf{x} \in A : \text{there occurs a (unique) collision between the } i\text{-th and the } j\text{-th particle in } F[\mathbf{x}, [0, t]]\}$

$A_0 = \{\mathbf{x} \in A : \text{there is no collision in } F[\mathbf{x}, [0, t]]\}.$

Then

$$A = \sum_{i < j} A^{i,j} + A_0 \quad (\text{disjoint union}).$$

Let $T_t^{(i,j)}$ denote the flow in which the particles other than the i -th and the j -th make free motion. Since $|\partial\Omega_n| = 0$ and

$$A^{i,j} = \{\mathbf{x} : T_t^{(i,j)} \mathbf{x} \in T_t A^{i,j}\} \setminus (\partial\Omega_n \setminus A^{i,j}),$$

we then have

$$\int_{A^{i,j}} \phi(T_t \mathbf{x}) d\mathbf{x} = \int_{A^{i,j}} \phi(T_t^{(i,j)} \mathbf{x}) d\mathbf{x} = \int g(T_t^{(i,j)} \mathbf{x}) d\mathbf{x}, \quad g = \phi \cdot \chi_{T_t A^{i,j}};$$

the last integral, by Lemma 2.1, equals

$$\int g(\mathbf{x})d\mathbf{x} = \int_{T_t A^{i,j}} \phi(\mathbf{x})d\mathbf{x}.$$

Thus

$$\int_A \phi(T_t \mathbf{x})d\mathbf{x} = \sum_{i < j} \int_{A^{i,j}} \phi(T_t \mathbf{x})d\mathbf{x} + \int_{A_0} \phi(T_t \mathbf{x})d\mathbf{x} = \int_{T_t A} \phi(\mathbf{x})d\mathbf{x}.$$

Q. E. D.

PROOF OF LEMMA 3.1. If $\tau(\mathbf{x}) > t$ and $t > 0$, the number of collisions in $F[\mathbf{x}, [0, t]]$ is finite. Therefore if we set

$$(3.1) \quad A_m = \{\mathbf{x}: \tau(\mathbf{x}) > t \text{ and the flow starting at } \mathbf{x} \text{ experiences at most one collision in each time interval } [kt/2^m, (k+1)t/2^m], k=0, \dots, 2^m-1\},$$

then $A_1 \subset A_2 \subset \dots$ and

$$(3.2) \quad \{\tau > t\} = \bigcup_{m=1}^{\infty} A_m,$$

so that

$$\int_{\tau > t} \phi(T_t \mathbf{x})d\mathbf{x} = \lim_{m \rightarrow \infty} \int_{A_m} \phi(T_t \mathbf{x})d\mathbf{x}$$

By Lemma 3.2 and the group property of T_t the integral under the limit equals

$$\int_{T_\delta A_m} \phi(T_{t-\delta} \mathbf{x})d\mathbf{x}, \quad \delta = t/2^m$$

and, transformed step by step, finally becomes

$$\int_{T_t A_m} \phi(\mathbf{x})d\mathbf{x}.$$

This proves Lemma 3.1, because $\bigcup_m T_t A_m = T_t \{\tau > t\} = \{-\tau^{(-)} > t\}$. Q. E. D.

PROOF OF THEOREM 3.1. On account of Lemma 3.1 it suffices to prove the first half of the theorem. For $t > 0$ set

$$B(t) = \{\mathbf{x} \in \Omega_n: \text{either there are at least two collisions in } F[\mathbf{x}, [0, t]]; \\ \text{or } \tau(\mathbf{x}) \leq t \text{ and there is no collision before } \tau(\mathbf{x})\}.$$

To have two collisions in the time interval $[0, t]$ there must be either three particles which are located within the distance $\varepsilon + 2t$ of each other or two pairs of particles such that two particles of each pair are located within the distance $\varepsilon + 2t$. The Lebesgue measure of all such configurations from Ω_n is of the order $O(t^2)$ as t goes to zero. Since the set of \mathbf{x} such that in the flow starting at \mathbf{x} the first collision

takes place in Σ_n is a Lebesgue null set, we accordingly have

$$(3.3) \quad |B(t)| = O(t^2) \quad \text{as } t \downarrow 0.$$

On the other hand

$$\Omega_n \setminus A_m \subset \cup_{k=0}^{2^m-1} \{x \in \Omega_n: T_{k\delta}x \in B(\delta) \text{ and } \tau > k\delta\}$$

where $\delta = t/2^m$ and A_m is defined by (3.1). Applying Lemma 3.1,

$$|\Omega_n \setminus A_m| \leq 2^m |B(t/2^m)| = O(2^{-m}).$$

Hence (3.2) proves $|\{\tau \leq t\}| = 0$ as desired.

Q. E. D.

REMARK 3.2. An analogue of the first half of Theorem 3.1 is proved by Alexander [2] for the hard sphere dynamics, and by Aizenman [1] under a more general setting. The proof given above is simialr to that of [2].

For convenience of later applications we here state a lemma which slightly generalizes (3.3). Two particles, with labels i and j , are said to make a *shadow collision* in $\mathbf{F}[x, [0, t]]$, if there exists an interval $[r, s] \subset [0, t]$ such that $(X_i^\eta(r), X_j^\eta(r)) \in E_{s-r}$ (i.e. there is a collision in $\mathbf{F}[(X_i^\eta(r), X_j^\eta(r)), [0, s-r]]$), but there is no collision between them in $\mathbf{F}[x, [r, s]]$.

LEMMA 3.3. Let $\tilde{B}(t)$ be the set of such configurations $x \in \Omega_n$ that either the total number of collisions and shadow collisions in $\mathbf{F}[x, [0, t]]$ is greater than or equal to two; or $\tau(x) \leq t$. Then $|\tilde{B}(t)| = O(t^2)$ as $t \rightarrow 0$. ($O(t^2)$ may depend on n and ε .)

The proof of Lemma 3.3 is the same as that of (3.3) on account of Theorem 3.1.

LEMMA 3.4. If $A \subset \Omega_n$ and $|A| = 0$, then

$$(3.4) \quad T_t x \notin A \quad \text{for a.a. (almost all) } (x, t) \in \partial\Omega_n \times R,$$

where a set of x is measured by the induced Lebesgue measure on the hypersurface $\partial\Omega_n = \partial D_n \times S^n$.

COROLLARY. i) The map $(x, t) \in \Omega_n \times R \rightarrow T_t x \in \Omega_n$ is continuous at a.a. points of $\partial\Omega_n \times R$. ii) $-\tau^{(-)} = \tau = \infty$ a.e. on $\partial\Omega_n = \partial D_n \times S^n$.

DEDUCTION OF COROLLARY FROM LEMMA 3.4. i) is immediate from Lemma 3.4 and Lemma 1.2 (take $A = \partial\Omega_n$). As for ii) one has only to note that the set $A = \{\tau < \infty \text{ or } \tau^{(-)} > -\infty\}$ is invariant under T_t .

We continue to consider the n -particle system with n and ε fixed, and before

proceeding to the proof of Lemma 3.4 we introduce the following notations:

$$z = z_n^{(\varepsilon)}(\mathbf{x}', l, v) = (x_1, \dots, x_{n-1}, [q_{n-1} + \varepsilon l, v])$$

$$\text{for } \mathbf{x}' = (x_1, \dots, x_{n-1}) \in \Omega_{n-1}^{(\varepsilon)}, \quad l \in S, \quad v \in S,$$

$$N_n^{+(-)} = \{(\mathbf{x}', l, v) \in \Omega_{n-1}^{(\varepsilon)} \times S \times S : z(\mathbf{x}', l, v) \in \Omega_n^{(\varepsilon)}, (v_{n-1} - v) \cdot l > 0 (< 0)\},$$

$$N_n = N_n^+ \cup N_n^-.$$

$N_n^+[N_n^-]$ consists of triples (\mathbf{x}', l, v) for which the n -th and $(n-1)$ -th particles from $z(\mathbf{x}', l, v)$ are in in-coming [resp. out-going] collision. It is noted that if $z(\mathbf{x}', l, v)$ is in in-coming [out-going] collision, then $z(\mathbf{x}', -l, v)$ is in out-going [resp. in-coming] collision. By the symmetry of the role of particles (3.4) is equivalently expressed as

$$(3.4') \quad T_t z(\mathbf{x}', l, v) \notin A \quad \text{for a.a. } (\mathbf{x}', l, v, t) \in N_n \times \mathbb{R}.$$

PROOF OF LEMMA 3.4. Let $|A|=0$. First we shall prove

$$(3.5) \quad T_s z(\mathbf{x}', l, v) \notin A \quad \text{for a.a. } (\mathbf{x}', l, v, s) \in N_n^- \times (0, \infty).$$

Put

$$B = \{\mathbf{x} \in \Omega_n \setminus \partial\Omega_n : \text{the first collision in the time reversed flow } T_{-t}\mathbf{x}, t > 0, \\ \text{is pair-wise and occurs between the } n\text{-th and the } (n-1)\text{-th particle}\}$$

and

$$\hat{B} = \{(\mathbf{x}', l, v, s) : (\mathbf{x}', l, v) \in N_n^-, 0 \leq s < \sigma_1(z(\mathbf{x}', l, v))\}$$

where $\sigma_1(\mathbf{x})$ is the first collision time after time 0:

$$\sigma_1(\mathbf{x}) = \inf \{t > 0 : T_t \mathbf{x} \in \partial\Omega_n\}.$$

Clearly B is a Borel set and $B = \{T_s^0 z(\mathbf{x}', l, v) : (\mathbf{x}', l, v, s) \in \hat{B}\}$. Making use of (2.1) we then have that for a bounded Lebesgue measurable function ϕ on Ω_n

$$(3.6) \quad \int_B \phi(\mathbf{x}) d\mathbf{x} = \int_{\hat{B}} \varepsilon(v - v_{n-1}) \cdot l \phi(T_s z(\mathbf{x}', l, v)) ds d\mathbf{x}' dl dv.$$

Applying this relation to $\phi(\mathbf{x}) = \chi_A(T_t \mathbf{x})$, which is Lebesgue measurable for each t , we see that if $|A|=0$, then for each $t \geq 0$

$$T_{t+s} z(\mathbf{x}', l, v) \notin A \quad \text{for a.a. } (\mathbf{x}', l, v, s) \in \hat{B}.$$

Since $\sigma_1 > 0$ a.e. on N_n^- , this implies (3.5). In view of Lemma 3.5 below (3.5) implies the corresponding relation for $N_n^+ \times (0, \infty)$. The relation for $N_n \times (-\infty, 0)$ is proved in the same way. Thus we have (3.4'). The proof of Lemma 3.4 is complete. Q. E. D.

LEMMA 3.5. For every bounded Borel function ϕ on Ω_n

$$\int_{N_n^+} \phi(z(\mathbf{x}', l, v)) d\mathbf{x}' dl dv = \int_{N_n^-} \phi(z^*(\mathbf{x}', l, v)) d\mathbf{x}' dl dv,$$

where z^* is a configuration in Ω_n obtained from z by $*$ -operation.

PROOF. Immediate from Lemma 1.1.

§4. The BBGKY hierarchy and the series expansion of correlation functions

Throughout this section we fix $0 < \varepsilon \ll 1$ and n . Let $f_n = f_n(\mathbf{x})$ be a bounded Borel function of $\Omega_n = \Omega_n^{(\varepsilon)}$, and $u_n(t) = u_n(t, d\mathbf{x})$ ($t \in R$) be the image measure of $f_n(\mathbf{x}) d\mathbf{x}$ under T_t . Since T_t preserves the Lebesgue measure, $u_n(t)$ has a density which is given by

$$u_n(t, \mathbf{x}) = f_n(T_{-t}\mathbf{x}) \quad \mathbf{x} \in \Omega_n.$$

Though $u_n(t, \mathbf{x})$ is not continuous in \mathbf{x} (nor in t) even if f_n is continuous, we shall need some continuity property of $u_n(t, \mathbf{x})$ in the following discussions. What we impose on f_n is the continuity at almost all $\mathbf{x} \in \Omega_n$:

(4.1) There is a subset B of Ω_n of full measure such that $f_n(\mathbf{x})$ is continuous at every point of B .

(This condition will be removed later.) To make our arguments clear and simple we shall assume also the condition that $f_n = 0$ on $\Sigma_n^{(\varepsilon)}$, which together with the Theorem in Appendix I guarantees that $u_n(t, \mathbf{x})$ is Borel measurable in (t, \mathbf{x}) . By Lemma 1.2 the property (4.1) is inherited by $u_n(t, \cdot)$ for all t . Moreover, by applying Lemma 3.4 in addition, the condition (4.1) implies

(4.2) the function $u_n(t, \mathbf{x})$ on $R \times \Omega_n$ is continuous at a.a. (almost all) points $(t, \mathbf{x}) \in R \times \partial\Omega_n$;

in particular, for a.a. t , the function $u_n(t, \cdot)$ on Ω_n satisfies

(4.2') $u_n(t, \mathbf{x})$ (as a function on Ω_n) is continuous at a.a. $\mathbf{x} \in \partial\Omega_n$.

Throughout this section we shall assume the condition (4.1) to hold.

The purpose of this section is to derive a system of equations (BBGKY hierarchy) for correlation functions $u_{n|m}(t) = u_{n|m}(t, x_1, \dots, x_m)$, $1 \leq m \leq n$, which is defined by $u_{n|n} = u_n$ and for $m < n$

$$u_{n|m}(t, x_1, \dots, x_m) = \int_{\Omega_{n-m}} u_n(t, \mathbf{x}) dx_{m+1} \cdots dx_n$$

if $(x_1, \dots, x_m) \notin \Sigma_m^{(\varepsilon)}$ and $u_{n|m} = 0$ otherwise. (Recall that $u_n(t, \mathbf{x}) = 0$ if $\mathbf{x} \in \Omega_n^0 \setminus$

$\Omega_n^{(\varepsilon)}$ by our convention (1.5). From our definition of T_t (especially (1.2) and (1.3)) it trivially follows that for all t and $m=1, 2, \dots, n$

$$(4.3) \quad u_{n|m}(t, \mathbf{x}) = u_{n|m}(t, \mathbf{x}^*), \quad \mathbf{x} \in \partial\Omega_m^{(\varepsilon)} \setminus \Sigma_m^{(\varepsilon)}.$$

(Don't confuse this with the continuity of $u_{n|m}$ along the trajectories.)

We define an operator $K_{m,m+1} = K_{m,m+1}^{(\varepsilon)}$ which transforms a bounded measurable function g on $\Omega_m^{(\varepsilon)}$ into a bounded measurable function on $\Omega_m^{(\varepsilon)}$ by

$$K_{m,m+1}g(x_1, \dots, x_m) = \sum_{j=1}^m \int_{(v_j-v), l > 0} dl dv (v_j - v) \cdot l \\ \times \{g(\dots, x_{j-1}, [q_j, v_j^*], x_{j+1}, \dots, x_m, [q_j - \varepsilon l, v^*]) - g(\dots, x_m, [q_j + \varepsilon l, v])\}$$

where $(v_j^*, v^*) = A_l(v_j, v)$. Though the right-hand side above is determined only by the values of g on the boundary $\partial\Omega_{m+1}$, for g and h equal to each other a.e. on Ω_{m+1} we have $K_{m,m+1}g = K_{m,m+1}h$ a.e. on Ω_m if the both of them are continuous at a.a. points of $\partial\Omega_{m+1}$. On account of Lemma 4.1, which will come after Theorems 4.1 and 4.2 below, (4.2') implies that the continuity condition of g above is satisfied by $u_{n|m+1}(t)$ for a.a. t .

Now we can describe the BBGKY hierarchy. One might heuristically derives its original form

$$(4.4) \quad \frac{\partial}{\partial t} u_{n|m}(t) = \mathcal{L}_m u_{n|m}(t) + \varepsilon(n-m)K_{m,m+1}u_{n|m+1}(t)$$

where \mathcal{L}_m is the Liouville operator of m -particle dynamics. However $\mathcal{L}_m u_{n|m}(t)$ has no meaning as a usual function (see Remark 4.2 i given at the end of this section) and one seeks the integrated version of (4.4):

THEOREM 4.1 (BBGKY hierarchy). *Let $1 \leq m < n$ and let f_n satisfy the condition (4.1) and be symmetric in x_{m+1}, \dots, x_n (i.e., invariant under any permutation of these variables). Then for all $t \in R$*

$$(4.5) \quad u_{n|m}(t) = U_m^{(\varepsilon)}(t)f_{n|m} + \varepsilon(n-m) \int_0^t U_m^{(\varepsilon)}(t-s)K_{m,m+1}u_{n|m+1}(s)ds \\ \text{a.e. on } \Omega_m^{(\varepsilon)}.$$

The relation (4.5) can be iterated, as will be proved later, to yield a series expansion of $u_{n|m}(t)$. To state the result let us introduce an operator $\mathcal{U}_m^{(\varepsilon)}$ which transforms a function $f(t, \mathbf{x})$, $t \in R$, $\mathbf{x} \in \Omega_m^{(\varepsilon)}$, which is Borel measurable and locally bounded and vanishes on Σ_m into a function of the same kind defined by

$$\mathcal{U}_m^{(\varepsilon)}f(t, \mathbf{x}) = \int_0^t f(s, T_{-t+s}^{(m,\varepsilon)}\mathbf{x})ds.$$

THEOREM 4.2. *Let m and f_n be as in Theorem 4.1. Then for each $t \in R$ it holds that for a.a. $\mathbf{x} \in \Omega_m^{(\varepsilon)}$*

$$(4.6) \quad u_{n|m}(t, \mathbf{x}) = \sum_{k=0}^{n-m} (n-m)_k \varepsilon^k (\mathcal{U}K)^k \{U_{m+k}(\cdot) f_{n|m+k}\}(t, \mathbf{x})$$

where $(n)_0 = 1$, $(n)_k = n(n-1)\dots(n-k+1)$, $k \geq 1$ and $(\mathcal{U}K)^k$ is an abbreviation for the k -time iteration of $\mathcal{U}_j^{(\varepsilon)} K_{j,j+1}^{(\varepsilon)}$ ($m \leq j \leq m+k-1$):

$$(\mathcal{U}K)^k = \mathcal{U}_m^{(\varepsilon)} K_{m,m+1}^{(\varepsilon)} \dots \mathcal{U}_{m+k-1}^{(\varepsilon)} K_{m+k-1,m+k}^{(\varepsilon)}.$$

Moreover the equality (4.6) holds for a.a. $(t, \mathbf{x}) \in R \times \partial\Omega_m^{(\varepsilon)}$.

REMARK 4.1. Applying Lemma 1.1 and the relations (4.3) and $A_l = A_{-l}$ we see that in Theorems 4.1 and 4.2 $K_{m,m+1}$ can be replaced by $K'_{m,m+1}$ where

$$K'_{m,m+1} g(x_1, \dots, x_m) = \sum_{j=1}^m \int_{S^2} g(x_1, \dots, x_m, [q_j + \varepsilon l, v]) (v - v_j) \cdot l \, dl \, dv.$$

REMARK 4.2. Spohn [15] gave proofs of the BBGKY hierarchy (4.5) and the series expansion for the hard sphere dynamics, assuming that $f_n = u_n(0)$ is continuous along trajectories. As for the BBGKY the starting point of his proof is the expression like (4.10) which follows and what is carried out is to justify the procedure of formally taking limit in it to obtain an equation which is expected, as we shall do in our proof of (4.8) below but in a different way. Another approach is made by Illner and Pulvirenti [6]: making use of the special flow representation, which is employed also by Takahashi [13] in the same context without details and will be incorporated in our second proof of (4.8), and arguments of Remark 4.4 that follow they deduced the BBGKY hierarchy in a differential form similar to (4.27) with f_n and ϕ (a test function) both smooth along trajectories, and for getting the series expansion resorted to the uniqueness of the solution to it.

Put $\Omega'_n = \Omega_n \cap \{\tau = -\tau^{(-)} = \infty\}$. Then the “continuity-along-trajectory” means the continuity of $f_n(T_t^{(n)} \mathbf{x})$ as a function of t for each $\mathbf{x} \in \Omega'_n$, which is implied by both of the boundary condition $f_n(\mathbf{x}^*) = f_n(\mathbf{x})$ for $\mathbf{x} \in \partial\Omega_n$ and the continuity of f_n with respect to the space variable \mathbf{q} . This continuity property lightens our task not so much as it appears to do, because it may be deteriorated by operating $K_{m,m+1}$ (repeatedly). Corollary ii) of Lemma 3.4 assures that $z(\mathbf{x}', v, l) \in \Omega'_n$ for a.a. $(\mathbf{x}', l, v) \in N_n$, while though plausible it is not clear whether this holds for a.a. (l, v) for each $\mathbf{x}' \in \Omega_{n-1}$. (The latter fails to hold for a discrete velocity model as treated in [16].) In Spohn [15] this kind of troublesome things are disposed of by proving that the correlation functions $u_{n|m}(t)$ inherit the continuity-along-trajectory property and are also continuous in t for each $\mathbf{x} \in \Omega'_m$, and that $\int_0^t U_m(t-s) K_{m,m+1} \tilde{u}(s) ds$ satisfies these two continuity properties if $\tilde{u}(t)$ does.

LEMMA 4.1. Let X and Y be measurable spaces, μ and ν finite measures on X and Y respectively, and $\mu \otimes \nu$ the direct product measure of μ and ν . Suppose that X is a subset of a topological space X' . Let f be a bounded function on $X' \times Y$ such that for every $x' \in X'$ the function $f(x', \cdot)$ of Y is ν -measurable (i.e., measurable with respect to the completion of σ -field of Y). If

$$(4.7) \quad \lim_{\substack{x' \rightarrow x \\ x' \in X'}} f(x', y) = f(x, y) \quad \text{for } \mu \otimes \nu\text{-a.a. } (x, y) \in X \times Y,$$

then $g(x') = \int_Y f(x', y) \nu(dy)$, $x' \in X'$, is continuous at μ -a.a. $x \in X$.

PROOF. The proof may be a standard exercise from the measure theory. Let A be a set of all points $(x, y) \in X \times Y$ at which the equality in (4.7) holds, and denote by $A(x)$ an x -section of A . Then by Fubini's theorem together with the assumption of the lemma $Y \setminus A(x)$ is ν -null set for μ -a.a. x ; hence for μ -a.a. x

$$g(x') = \int_{A(x)} f(x', y) \nu(dy) \quad \text{for all } x' \in X'.$$

Since f is continuous in x' at (x, y) if $y \in A(x)$, an application of the bounded convergence theorem shows that g is continuous at x for which the above equality holds. Thus g is continuous at a.a. x . Q. E. D.

REMARK 4.3. A simple application of Lemma 4.1 shows that the condition (4.1) on f_n implies the same one on $u_{n|m}(t)$ for all t (in the latter Ω_n of course must be replaced by Ω_m). By taking $X = (0, \infty) \times \partial\Omega_m$, $X' = (0, \infty) \times \Omega_m$ and $Y = \Omega_{n-m}$ in Lemma 4.1 one sees a similar implication of the condition (4.2).

PROOF OF THEOREM 4.1. It suffices to prove that for all t and for all $\phi \in C(\Omega_m^0)$ the function

$$y(s) = \langle u_{n|m}(t-s), U_m(-s)\phi \rangle, \quad s \in \mathcal{R}$$

is absolutely continuous and its Radon-Nikodim derivative is given by

$$(4.8) \quad \frac{d}{ds} y(s) = -\varepsilon(n-m) \langle K_{m,m+1} u_{n|m+1}(t-s), U_m(-s)\phi \rangle.$$

In fact by integrating (4.8) we have

$$(4.9) \quad \begin{aligned} \langle u_{n|m}(t), \phi \rangle &= \langle U_m(t) f_{n|m}, \phi \rangle \\ &+ \varepsilon(n-m) \int_0^t \langle U_m(t-s) K_{m,m+1} u_{n|m+1}(s), \phi \rangle ds. \end{aligned}$$

Since the function $U_m(t-s) K_{m,m+1} u_{n|m+1}(s)(\mathbf{x})$ is measurable in (s, \mathbf{x}) , Fubini's theorem is applied to ensure that (4.9) implies (4.5) of Theorem 4.1. The two

proofs of (4.8) will be given below. The first one is rather faithful to the heuristic argument made in §0 to derive the first equation of the BBGKY hierarchy. The idea of the second proof is to make use of the formula (3.6) which is used in the proof of Lemma 3.4. The continuity-along-trajectory condition on f_n even if assumed is hardly helpful in the first proof, while in the second one it is a little helpful (yet dispensable).

THE FIRST PROOF OF (4.8). Let us write $\mathbf{x}=(\mathbf{x}', \mathbf{x}'')$ with $\mathbf{x}' \in \Omega_m$ and $\mathbf{x}'' \in \Omega_{n-m}$, and put

$$\phi_s(\mathbf{x}) = \phi(T_s^{(m)}\mathbf{x}'), \quad \mathbf{x} \in \Omega_m \times \Omega_{n-m}.$$

Then, by recalling the convention $u_n(t, \partial)=0$, we get $y(s)=\langle u_n(t-s), \phi_s \rangle$ and for $h \in R$

$$\begin{aligned} y(s+h) - y(s) &= \int_{\Omega_n} \{u_n(t-s, T_h\mathbf{x}) - u_n(t-s, T_h\mathbf{x}', T_h\mathbf{x}'')\} \phi_{s+h}(\mathbf{x}) d\mathbf{x} \\ &\quad + \int_{\Omega_n} \{u_n(t-s, T_h\mathbf{x}', T_h\mathbf{x}'') \phi_{s+h}(\mathbf{x}) - u_n(t-s, \mathbf{x}) \phi_s(\mathbf{x})\} d\mathbf{x} \\ &= I(h) + II(h) \quad (\text{say}), \end{aligned}$$

where $T_h\mathbf{x}' = T_h^{(m)}\mathbf{x}'$ and $T_h\mathbf{x}'' = T_h^{(n-m)}\mathbf{x}''$.

In order to compute these integrals we shall divide the range of integration into appropriate sets on each of which they will have expressions like (2.2), and thereby find out the limits of $I(h)/h$ and $II(h)/h$ as $h \rightarrow 0$. Heuristic procedure for this is rather simple but we must be careful in taking limit, since the integrals on Ω_n are reduced to that on the boundary $\partial\Omega_n$ in the limit and the functions involved are not continuous on $\partial\Omega_n$.

Let $E_h^{i,j}$ for $h > 0$ [resp. $h < 0$] denote the set of configurations $\mathbf{x} \in \Omega_n$ such that there is a collision between the i -th and j -th particle in $F[\mathbf{x}, (0, h)]$ [resp. $F[\mathbf{x}, (h, 0)]$]. Clearly the range of integration for $I(h)$ can be reduced to the union of $E_h^{i,j}$ over such pairs (i, j) that $i \leq m < j$. Observing that $|E_h^{i,j} \cap E_h^{k,l}| = O(h^2)$ if $(i, j) \neq (k, l)$ (see (3.3)) and noting the assumed symmetry of f_n , we see that $I(h)$ equals

$$(4.10) \quad (n-m) \sum_{j=1}^m \int_{E_h^{j,m+1}} \{u_n(t-s, T_h\mathbf{x}) - u_n(t-s, T_h\mathbf{x}', T_h\mathbf{x}'')\} \phi_{s+h}(\mathbf{x}) d\mathbf{x}$$

plus an $O(h^2)$ term. If h is very small as compared with ε , then $(T_h\mathbf{x}', T_h\mathbf{x}'') \notin \Omega_n$ for $(\mathbf{x}', \mathbf{x}'') \in E_h^{j,m+1} \setminus B_h$ where B_h is a certain set with $|B_h| = o(h)$ (this magnitude depends only on ε). Let $h > 0$. Recalling $u_n(t, \partial)=0$ again, we consequently observe that the m -th summand (as a typical one among the m terms) of (4.10) equals

$$(4.11) \int_{\Omega_m} d\mathbf{x}' \int_0^h dr \int_{(v_m-v) \cdot l > 0} dl dv \varepsilon(v_m-v) \cdot l \phi(T_{s+h}\mathbf{x}') \\ \times u_{n|m+1}(t-s, T_h\tilde{\mathbf{x}}, [q_m+rv_m+(h-r)v_m^*, v_m^*], [q_m+\varepsilon l+rv_m+(h-r)v^*, v^*])$$

plus an $o(h)$ term, where $\tilde{\mathbf{x}}=(x_1, \dots, x_{m-1})$. Since $u_{n|m+1}$ satisfies the condition (4.2') with Ω_{m+1} in place of Ω_n , for a.a. s the ratio of this integral to h converges to

$$\int_{\Omega_m} d\mathbf{x}' \int_{(v_m-v) \cdot l > 0} dl dv \varepsilon(v_m-v) \cdot l \phi(T_s, \mathbf{x}') \\ \times u_{n|m+1}(t-s, \tilde{\mathbf{x}}, [q_m, v_m^*], [q_m+\varepsilon l, v^*])$$

as $h \downarrow 0$. The limit as $h \uparrow 0$ is similarly taken, resulting in the same limiting expression. Clearly we have $I(h)=O(h)$ as $h \rightarrow 0$ (uniformly in s). Therefore the relation (4.8) follows if we show $II(h)=O(h)$ and for a.a. s

$$(4.12) \lim_{h \rightarrow 0} II(h)/h = -\varepsilon(n-m) \int_{\Omega_m} \left\{ \sum_{j=1}^m \int_{(v_j-v) \cdot l > 0} dl dv (v_j-v) \cdot l \right. \\ \left. \times u_{n|m+1}(t-s, \mathbf{x}', [q_j-\varepsilon l, v]) \right\} \phi(T_s \mathbf{x}') d\mathbf{x}'.$$

(Apply (4.3) to identify the limit of $(I(h)+II(h))/h$ thus obtained as the right hand side of (4.8): note that the configurations appearing in $u_{n|m+1}(t-s)$ above are in out-going collision.)

For the proof of (4.12) put $B_h = \{(\mathbf{x}', \mathbf{x}'') \in \Omega_m \times \Omega_{n-m} \setminus \Omega_n : (T_h \mathbf{x}', T_h \mathbf{x}'') \in \Omega_n\}$. Then the integral defining $II(t)$ equals the same integral over $\Omega_m \times \Omega_{n-m}$ minus that over B_h , of which the former one vanishes since $\phi_{s+h}(T_h^{-1} \mathbf{x}', T_h^{-1} \mathbf{x}'') = \phi_s(\mathbf{x}, \mathbf{x}'')$. Hence

$$II(h) = - \int_{B'_h} u(t-s, \mathbf{x}', \mathbf{x}'') \phi(T_s \mathbf{x}') d\mathbf{x}' d\mathbf{x}'',$$

where $B'_h = \{(\mathbf{x}', \mathbf{x}'') \in \Omega_n : (T_{-h} \mathbf{x}', T_{-h} \mathbf{x}'') \in \Omega_m \times \Omega_{n-m} \setminus \Omega_n\}$. The last integral agrees with $(n-m)$ times the integral over $\cup_{j=1}^m E_{-h}^{j, n-m+1}$ up to an $o(h)$ term. In the same way as above we deduce (4.12) from this expression. The first proof of (4.8) is complete.

THE SECOND PROOF OF (4.8). Let ϕ_s be as in the first proof. First of all we seek for another expression of the right-hand side of (4.8): the one by the surface integral over $\partial\Omega_n$. Let $\mathbf{n} = \mathbf{n}(\mathbf{q})$ be a unit normal vector of ∂D_n pointing inward of D_n , which, at $\mathbf{q} \in \partial D_n$ where a unique pair particles, the k -th and the j -th say, are in touch, has entries given by $n_i = 0$ for $i \neq k, j$ and

$$n_j = (\varepsilon\sqrt{2})^{-1}(q_j - q_k), \quad n_k = (\varepsilon\sqrt{2})^{-1}(q_k - q_j).$$

Put $F_s(\mathbf{x}) = u_n(t-s, \mathbf{x}) \phi_s(\mathbf{x})$. If $\mathbf{x} \in \partial\Omega$ and it is only a pair of particles either

both from those labeled 1 to m or both from those labeled $m + 1$ to n that are in touch, then $F_s(\mathbf{x}^*) = F_s(\mathbf{x})$ for all s . This (together with Remark 4.1) shows the following identity, in the right-hand side integral of which the range of integration is enlarged but the added part cancels within itself:

$$(4.13) \quad \begin{aligned} \varepsilon(n-m) \langle K_{m,m+1} u_{n|m+1}(t-s), U_m(-s)\phi \rangle \\ = \sqrt{2} \varepsilon \int_{\partial\Omega_n} u_n(t-s, \mathbf{y}) \phi_s(\mathbf{y}) d\pi(\mathbf{y}), \end{aligned}$$

where for $\mathbf{y} = (\mathbf{q}, \mathbf{v}) \in \partial\Omega_n = S^n \times \partial D_n$

$$d\pi(\mathbf{y}) := \mathbf{n}(\mathbf{q}) \cdot \mathbf{v} \times \text{surface element of } \partial D_n.$$

Let $\sigma_1 = \sigma_1(\mathbf{x})$ be the first collision time as in the proof of Lemma 3.4. One can easily see that $\sigma_1 < \infty$ a.e. (In some other models this is not true. A modification of the following argument which may be needed for such models will be indicated in Remark 4.5 at the end of this section.) Let us write the relation (3.6) in the following form: for a bounded measurable function F on Ω_n

$$(4.14) \quad \int_{\Omega_n} F(\mathbf{x}) d\mathbf{x} = \sqrt{2} \varepsilon \int_{\partial\Omega_n^{\text{out}}} d\pi(\mathbf{y}) \int_0^{\sigma_1(\mathbf{y})} F(T_r \mathbf{y}) dr$$

where $\partial\Omega_n^{\text{out}} = \{\mathbf{y} = (\mathbf{q}, \mathbf{v}) \in \partial\Omega_n : \mathbf{n} \cdot \mathbf{v} > 0\}$ (the set of configurations in out-going collision). Substituting $F \circ T_{-h}$ in place of F , noticing $\int_0^{\sigma_1} F(T_{-h+t} \mathbf{y}) dt = \int_{-h}^{\sigma_1-h} F(T_r \mathbf{y}) dt$ and taking the right derivative at $h=0$, one sees that if $F(T_r \mathbf{y})$ is left-continuous in t both at $t=0$ and $t=\sigma_1$ for a.a. $\mathbf{y} \in \partial\Omega_n^{\text{out}}$, then

$$\int_{\partial\Omega_n^{\text{out}}} (F(T_0 \mathbf{y}) - F(T_{\sigma_1} \mathbf{y})) d\pi(\mathbf{y}) = 0;$$

or, by applying Lemma 1.1,

$$(4.15) \quad \int_{\partial\Omega_n^{\text{out}}} F(T_{\sigma_1} \mathbf{y}) d\pi(\mathbf{y}) = - \int_{\partial\Omega_n^{\text{in}}} F(\mathbf{y}) d\pi(\mathbf{y})$$

where $\partial\Omega_n^{\text{in}}$ is the set of configurations in the in-coming collision. The relation is true for every F that is continuous near the boundary $\partial\Omega_n$. Therefore it shows that T_{σ_1} as a mapping from $\partial\Omega_n^{\text{in}}$ into itself preserves the (negative) measure $d\pi(\mathbf{y})$ (note that in the left-hand side of (4.15) replacing $\partial\Omega_n^{\text{out}}$ by $\partial\Omega_n^{\text{in}}$ amounts to multiplying -1).

Let us apply (4.14) to $F_s(\mathbf{x}) = u_n(t-s, \mathbf{x}) \phi_s(\mathbf{x})$. Since $T_r^{(m)} \mathbf{y}' = (T_r^{(n)} \mathbf{y})'$ for $0 < r \leq \sigma_1(\mathbf{y})$ (recall that $\mathbf{y} = (\mathbf{y}', \mathbf{y}'')$ with $\mathbf{y}' \in \Omega_m, \mathbf{y}'' \in \Omega_{n-m}$), it gives

$$(4.16) \quad y(s) = \sqrt{2} \varepsilon \int_{\partial\Omega_n^{\text{out}}} d\pi(\mathbf{y}) \int_s^{s+\sigma_1(\mathbf{y})} u_n(t, T_r \mathbf{y}) \phi(T_r^{(m)} \mathbf{y}') dr.$$

By Lemma 3.4 and the assumption (4.1), $u_n(t, T_s y)$ and $\phi(T_s^{(m)} y')$ are continuous at a.a. $(s, y) \in (0, t) \times \partial\Omega_n^{out}$. The same is true also with $T_{\sigma_1} y$ in place of y , since T_{σ_1} preserves the (negative) measure $d\pi$ as already noted. Clearly from (4.16) $y(s)$ is absolutely continuous, and, by $T_{s+\sigma_1} y' = T_s(T_{\sigma_1} y)'$ (for $y \in \partial\Omega_n^{out}$), its Radon-Nikodym derivative equals

$$\frac{d}{ds} y(s) = \sqrt{2}\varepsilon \int_{\partial\Omega_n^{out}} \{u_n(t-s, T_{\sigma_1} y)\phi_s(T_{\sigma_1} y) - u_n(t-s, y)\phi_s(y)\} d\pi(y).$$

Finally, applying (4.15), we see that this agrees with -1 times the expression on the right-hand side of (4.13). Thus (4.8) has been proved. Q. E. D.

The equality in (4.5), if it to be iterated to yield the series expansion (4.6), would have at least to hold a.e. on $\partial\Omega_m$ for a.a. t . The next lemma is somewhat stronger than what we need. (We shall not apply the continuity with respect to the time variable.)

LEMMA 4.2. *Let m and f_n be as in Theorem 4.1. Then for each $0 \leq k < m+1$ the function of $(t_0, t_1, \dots, t_k, \mathbf{x}) \in [0, \infty)^{k+1} \times \Omega_{m+1-k}$ defined by*

$$(4.17) \quad U_{m+1-k}(t_k)K_{m+1-k, m+2-k} \cdots U_m(t_1)K_{m, m+1}u_{n|m+1}(t_0)(\mathbf{x})$$

is continuous at a. a. points of $[0, \infty)^{k+1} \times \partial\Omega_{m+1-k}$.

PROOF. We give a proof only in the case $k=1$; the remaining case $k>1$ may be proved similarly. Thus we consider a function g of $(s, t, \mathbf{x}) \in [0, \infty)^2 \times \Omega_m$ defined by

$$(4.18) \quad g(s, t; \mathbf{x}) = U_m(t)K_{m, m+1}u_{n|m+1}(s)(\mathbf{x}),$$

and prove

$$(4.19) \quad g \text{ is continuous at } (s, t, z(\mathbf{x}', l', v'))$$

for a.a. $(s, t, \mathbf{x}', l', v') \in [0, \infty)^2 \times N_m$

where $z(\mathbf{x}', l, v)$ and N_m are introduced previous to the proof of Lemma 3.4. Corresponding to the sum in the definition of $K_{m, m+1}$, $g(s, t, \mathbf{x})$ is the sum of m integrals alike, the last of which can be expressed

$$(4.20) \quad \int_{S \times S} u_{n|m+1}(s, z(T_{-i}^{(m)} \mathbf{x}, l, v))(v-v_m) \cdot l dl dv$$

(see Remark 4.1). First let us consider this expression with $t=0$. Since the function $u_{n|m+1}(s, y)$ of $(s, y) \in (0, \infty) \times \Omega_{m+1}$ is continuous at $(s, z(\mathbf{x}, l, v))$ for a.a. $(s, \mathbf{x}, l, v) \in (0, \infty) \times \Omega_m \times S \times S$ (cf. Remark 4.3) and, for each (l, v) , the mapping: $\mathbf{x} \rightarrow z(\mathbf{x}, l, v)$ is continuous at a.a. $\mathbf{x} \in \Omega_m$ (observe that these state-

ments are valid even when $z(\mathbf{x}, l, v)$ is outside Ω_{m+1} to be identified with ∂ , it holds that

$$(4.21) \quad \text{the mapping: } (s, \mathbf{x}) \longrightarrow u_{n|m+1}(s, z(\mathbf{x}, l, v)) \text{ is continuous at } (s, \mathbf{x}) \\ \text{for a.a. } (s, \mathbf{x}, l, v).$$

Consequently, by Lemma 4.1, the function of (s, \mathbf{x}) expressed in (4.20) with $t=0$ is continuous at a.a. (s, \mathbf{x}) ; hence so is $g(s, 0, \mathbf{x})$. Now Lemma 3.4 may be applied to see that $g(s, t, \mathbf{x}) = g(s, 0, T_{-t}^{(m)}\mathbf{x})$ satisfies (4.19). Q. E. D.

PROOF OF THEOREM 4.2. Since the equality (4.5) holds a.e. (with t fixed), it must hold for all points at which both sides of (4.5) are continuous. But by Lemma 4.2 they are continuous at a.a. points of $\partial\Omega_m$ for a.a. t . The equality (4.5) is therefore correct in the same sence, so that it can be iterated untill we arrive at (4.6). Lemma 4.2 shows also that each term of the sum in (4.6) has the continuity property as stated in (4.2); in particular (4.6) holds a.e. on $R \times \partial\Omega_m$. Q. E. D.

THEOREM 4.3. *The condition (4.1) may be removed from the assumptions of Theorems 4.1 and 4.2.*

The proof of Theorem 4.3 will be given not in this section but in §6.

REMARK 4.4. Throughout this remark f_n is a bounded Borel function on Ω_n which satisfies the continuity condition (4.1) and ϕ is a continuous function on Ω_n which has a bounded gradient $V_q \phi$ on $\Omega_n \setminus \partial\Omega_n$, where V_q denotes the gradient with respect to q . Through i) to iii) below another proof of Theorem 4.2 will be given.

i) In a manner similar to that proving Theorem 4.1 we can show that $\langle u_n(t), \phi \rangle$ is absolutely continuous in t and

$$(4.22) \quad D_t^{\mathbb{R}^N} \langle u_n(t), \phi \rangle = \langle u_n(t), V \cdot V_q \phi \rangle + \sqrt{2} \varepsilon \int_{\partial\Omega_n} u_n(t, \mathbf{y}) \phi(\mathbf{y}) d\pi(\mathbf{y})$$

where $D_t^{\mathbb{R}^N}$, applied to an absolutely continuous function, denotes the Radon-Nikodym derivative, $V(\mathbf{x}) = \mathbf{v}$, and $d\pi(\mathbf{y})$ is the same as in (4.13). The integral of the second term on the right-hand side can also be expressed as follows

$$(4.23) \quad \int_{\partial\Omega_n^{\text{in}}} [u_n(t, \mathbf{y}) \phi(\mathbf{y}) - u_n(t, \mathbf{y}^*) \phi(\mathbf{y}^*)] d\pi(\mathbf{y}).$$

ii) We here (and only here) assume that $f_n(\mathbf{x})$ has a bounded gradient $V_q f_n$ off $\partial\Omega_n$ and satisfies the boundary condition $f_n(\mathbf{y}^*) = f_n(\mathbf{y})$ for $\mathbf{y} \in \partial\Omega_n$ (in iii below the role of f_n here will be played by a test function ϕ). Then, by extending the argument made in the proof of Lemma 1.2, we have

$$(4.24) \quad \frac{\partial}{\partial t} u_n(t, \mathbf{x}) = -\mathbf{v} \cdot \nabla_q u_n(t, \mathbf{x}) \quad \text{for } (t, \mathbf{x}) \in G, \quad \mathbf{x} \notin \partial\Omega_n$$

and

$$(4.25) \quad \frac{\partial}{\partial t} u_n(t, \mathbf{x}) = -(V \cdot \nabla_q f_n)(T_{-t}, \mathbf{x}) \quad \text{for } (t, \mathbf{x}) \in G.$$

Let $B^h = \{\mathbf{x} \in \Omega_n: T_{-t+s}\mathbf{x} \in \partial\Omega_n \text{ for some } -h < s < h\}$ for $h > 0$ with t fixed. Since (4.25) shows that $(\partial/\partial t)u_n(t, \mathbf{x})$ is bounded and since $|B^{2h}| = O(h)$ as $h \rightarrow 0$, (4.24) shows that for every bounded measurable function ψ on Ω_n

$$(4.26) \quad \frac{d}{dt} \langle u_n(t), \psi \rangle = -\langle V \cdot \nabla_q u_n(t), \psi \rangle,$$

where the boundary condition on f_n must be employed; otherwise a certain integral of $(f_n(\mathbf{y}^*) - f_n(\mathbf{y}))\psi(T_s, \mathbf{y})$ over $\partial\Omega_n^{\text{in}}$ must be added to the right-hand side. It is noted that (4.22) is easily deduced from (4.26). In fact applying first (4.14) to the right-hand side of (4.26) and then (4.25) with $u_n(t)$ in place of f_n we see

$$\frac{d}{dt} \langle u_n(t), \phi \rangle = -\int_{\partial\Omega_n^{\text{out}}} d\pi(\mathbf{y}) \int_0^{\sigma_1} \left[\frac{\partial}{\partial s} u_n(t, T_s \mathbf{y}) \right] \phi(T_s \mathbf{y}) ds.$$

The assumed continuity along the trajectory allows us to carry out the usual partial integration for the inner integral above, which after applications of (4.14) and (4.15) results in (4.22). Such deduction may also be done by formally applying Gauss' theorem, although because of the discontinuity of $u_n(t)$ caused by the multiple collision a straightforward application of it does not seem to be readily justified.

iii) Let us write the right-hand side of (4.22) symbolically $\langle \mathcal{L}_n u_n(t), \phi \rangle$ so that $D_t^{\text{RN}} \langle u_n(t), \phi \rangle = \langle \mathcal{L}_n u_n(t), \phi \rangle$. Suppose that f_n is symmetric in x_{m+1}, \dots, x_n and ϕ is independent of these variables ($1 \leq m < n$). Then the equation (4.22) turns into

$$(4.27) \quad D_t^{\text{RN}} \langle u_{n|m}(t), \phi \rangle = \langle \mathcal{L}_m u_{n|m}(t), \phi \rangle + \varepsilon(n-m) \langle K_{m,m+1} u_{n|m+1}(t), \phi \rangle,$$

and gives a precise meaning to (4.4). The integral (4.23) vanishes if $\phi(\mathbf{x}^*) = \phi(\mathbf{x})$ on $\partial\Omega_n$ and by the same argument as has been made to verify (4.26) for each s we can replace ϕ by $\psi := U_n(s)\phi$ in (4.22) with a.e. defined $\mathbf{v} \cdot \nabla_q \psi$. Applying these two facts with m in place of n we get a simple expression for the first term on the right-hand side of (4.27) (with ϕ replaced by $U_m(s)\phi$), so that for each s

$$(4.28) \quad D_t^{\text{RN}} \langle u_{n|m}(t), U_m(s)\phi \rangle \\ = \langle u_{n|m}(t), V \cdot \nabla_q (U_m(s)\phi) \rangle + \varepsilon(n-m) \langle K_{m,m+1} u_{n|m+1}(t), U_m(s)\phi \rangle.$$

We conclude this remark by verifying the series expansion (4.6) from (4.28).

First observe that the right-hand side of (4.6), which we denote by $\tilde{u}_{n|m}(t)$, satisfies (4.28) and is continuous as a function of t taking values in $L_1(\Omega_m, dx)$. This continuity is possessed also by $u_{n|m}(t)$. Since $u_{n|m}(t) = \tilde{u}_{n|m}(t) = u_n(t)$ by the very definition, the problem is reduced to showing that if a continuous function $u: t \rightarrow u(t) \in L_1(\Omega_m, dx)$ satisfies $D_t^{\text{RN}} \langle u(t), U_m(s)\phi \rangle = \langle u(t), V \cdot \nabla_q (U_m(s)\phi) \rangle$ together with $u(0) = 0$, then $u = 0$. But an application of (4.26) with m and $U_m(t)\phi$ in place of n and $u_n(t)$, respectively, shows that the premiss of this claim implies $D_t^{\text{RN}} \langle u(t_0 - t), U_m(t)\phi \rangle = 0$, so that $\langle u(t_0), \phi \rangle = 0$ for all t_0 , proving the claim and hence the series expansion (4.6).

REMARK 4.5. In the second proof of (4.8) we have applied the condition that $\sigma_1 < \infty$ a.e., which is valid for the present model but may not be for other models: it clearly does not hold, e.g., if our disks move in the whole space R^2 in stead of S_2 . Although for this example can be applied a simple device of approximating R^2 by a large torus or a large compact domain bounded by an elastic smooth curve, we here indicate a straightforward way which dispenses with the use of this special situation.

Let $\sigma_1^{(-)}$ be the last collision time before time 0, and put $\hat{\Omega}_n = \{\sigma_1 - \sigma_1^{(-)} < \infty\}$, $\partial\hat{\Omega}_n^{\text{out}} = \{y \in \partial\Omega_n^{\text{out}} : \sigma_1(y) < \infty\}$, $\partial\hat{\Omega}_n^{\text{in}} = \{y \in \partial\Omega_n^{\text{in}} : \sigma^{(-)}(y) > -\infty\}$, $\Omega_n^+ = \{\sigma_1 = \infty, \sigma_1^{(-)} > -\infty\}$ and $\Omega_n^- = \{\sigma_1 < \infty, \sigma_1^{(-)} = -\infty\}$. Then (4.14) is valid if we put $\hat{\cdot}$ on Ω in both places where it appears in the formula. Arguing as before but with this modified version of (4.14) we find the limit of I_h/h as $h \downarrow 0$, where $I_h := \int_{\hat{\Omega}_n} (F(T_{-h}x) - F(x))dx$. On the other hand, approximating the difference $T_{-h}\hat{\Omega}_n \setminus \hat{\Omega}_n$ [resp. $\hat{\Omega}_n \setminus T_{-h}\hat{\Omega}_n$] by the set $\cup_{0 < r < h} T_{-r}\partial\hat{\Omega}_n^{\text{out}[\text{in}]}$ we directly obtain

$$\lim_{h \downarrow 0} I_h/h = \sqrt{2}\varepsilon \left[\int_{\partial\hat{\Omega}_n^{\text{out}}} F(T_0y)d\pi(y) + \int_{\partial\hat{\Omega}_n^{\text{in}}} F(y)d\pi(y) \right].$$

From two expressions for $\lim I_h/h$ thus obtained it follows that (4.15) is true if $\hat{\cdot}$ is put on Ω in the both integrals of it. Now, divide the integral which defines $y(s)$ according to the partition of Ω_n (its range of integration) into $\hat{\Omega}_n$, Ω_n^\pm and the rest, and let $y(s) = \hat{y}(s) + y_+(s) + y_-(s) + R(s)$ be the corresponding decomposition. Then we have (4.16) with $\hat{\cdot}$ on y as well as on Ω ; and the similar expression for $y^+(s)$ [resp. $y^-(s)$] in which $\partial\Omega_n^{\text{out}}$ is replaced by $\partial\Omega_n^+ := \partial\Omega_n^{\text{out}} \setminus \partial\hat{\Omega}_n^{\text{out}}$ [resp. $\partial\Omega_n^- := \partial\Omega_n^{\text{in}} \setminus \partial\hat{\Omega}_n^{\text{in}}$] and $s + \sigma_1(y)$ (the upper limit of the inner integral) by ∞ [resp. $-\infty$]. Since the set $\Omega_n \setminus (\hat{\Omega}_n \cup \Omega_n^+ \cup \Omega_n^-)$ is invariant under T_s and $\phi_s(x) = \phi(T_s^{(m)}x) = \phi(T_s^{(n)}x)$ for x from it, $R(s)$ is independent of s . It is now immediate to see that the Radon-Nikodim derivatives of $\hat{y}(s)$, $y^+(s)$ and $y^-(s)$ agree with $\sqrt{2}\varepsilon$ times the integrals which together constitute the decomposition of the integral on the right-hand side of (4.13) that corresponds to the decomposition $\partial\Omega = (\partial\hat{\Omega}_n^{\text{out}} + \partial\hat{\Omega}_n^{\text{in}}) + \partial\Omega_n^+ + \partial\Omega_n^-$, proving (4.8).

§5. The Boltzmann equation and a factorization property of the Boltzmann hierarchy

For measurable functions $f=f(x)$ on Ω_m^0 and $h=h(t, x)$ on $[0, t_0] \times \Omega_m^0$ we set

$$U_m^0(t)f(x) = f(T_{-t}^0 x),$$

$$K_{m-1, m}^0 f(x) = \sum_{i=1}^{m-1} \int_{(v_i - v_m) \cdot l > 0} \{f(\dots, [q_i, v_i^*], \dots, [q_i, v_m^*]) - f(\dots, [q_i, v_m])\} (v_i - v_m) \cdot l \, dl \, dv_m \quad (m \geq 2),$$

$$\mathcal{Q}_m^0 h(t, x) = \int_0^t U_m^0(t-s)h(s, \cdot)(x) \, ds.$$

In this section we are concerned with the Boltzmann equation

$$(5.1) \quad \frac{\partial}{\partial t} u(t, x) = -v \cdot \frac{\partial}{\partial q} u(t, x) + K_{1,2}^0 u(t) \otimes u(t)$$

or its weak version

$$(5.2) \quad \frac{d}{dt} \langle u(t), \phi \rangle = \left\langle u(t), v \cdot \frac{\partial}{\partial q} \phi \right\rangle + \langle K_{1,2}^0 u(t) \otimes u(t), \phi \rangle, \quad \phi \in C^1(\Omega_1^0).$$

Here \otimes denotes the outer product of functions; e.g., for two functions $f(x)$ and $g(y)$, $f \otimes g$ denotes the function which sends (x, y) to $f(x)g(y)$.

First of all we note that (5.2) with the initial condition $u(0)=f$ is equivalent to the integral equation

$$(5.3) \quad \langle u(t), \phi \rangle = \langle U_1^0(t)f, \phi \rangle + \int_0^t \langle K_{1,2}^0 u(s) \otimes u(s), U_1^0(-t+s)\phi \rangle \, ds,$$

provided that $u(t, x)$ is (t, x) -measurable and $\int_{\Omega_1^0} |u(t, x)| \, dx$ is locally bounded.

Next let us observe that a measurable solution of (5.3) having the property that

$$(5.4) \quad \text{ess sup}_q \int |u(t, q, v)| \, dv \text{ is locally bounded}$$

is *uniquely determined* (if it exists) by $u(0)=f$. The proof is easy. In fact, by noticing that for f and g functions on Ω_1^0 and function $\phi = \phi(v)$

$$\begin{aligned} & \int (K_{1,2}^0 f \otimes g)(q, v) \phi(v) \, dv \\ &= \int_{(v-v_1) \cdot l > 0} \{\phi(v^*) - \phi(v)\} f(q, v) g(q, v_1) (v - v_1) \cdot l \, dl \, dv \, dv_1 \end{aligned}$$

(for a.a. $q \in S_2$), we see that if $w = u - \tilde{u}$ is a difference of two solutions u and \tilde{u} of (5.3), then

$$\begin{aligned} h(t, q) &:= \int |w(t, q, v)| dv \\ &= \sup_{\|\phi\|_\infty \leq 1} \left| \int w(t, q, v) \phi(v) dv \right| \\ &\leq \int_0^t \sup_{\|\phi\|_\infty \leq 1} \left| \int K_{1,2}^0(w \otimes u + \tilde{u} \otimes w)(s, q, v) \phi(v) dv \right| ds \\ &\leq 4(2\pi)^2 \int_0^t \left\{ \text{ess sup}_q \int |u(s, q, v)| dv + \text{ess sup}_q \int |\tilde{u}(s, q, v)| dv \right\} h(s, q) ds \end{aligned}$$

so that $h \equiv 0$. In what follows we shall consider only solutions for (5.1), (5.2) or (5.3) which satisfy (5.4).

THEOREM 5.1. *Let f be a continuous function on Ω_1^0 . Then there exists a unique solution $u(t, x)$ of (5.2) with $u(0) = f$ for $0 \leq t < (8\|f\|_\infty)^{-1}$. $u(t, x)$ is continuous in (t, x) . For $m = 1, 2, \dots$ the m -fold outer product of $u(t)$ is expanded in the series:*

$$(5.5) \quad u(t)^{m \otimes} (x) = \sum_{k=0}^\infty (\mathcal{Q}^0 K^0)^k (U_{m+k}^0(\cdot) f^{(m+k \otimes)})(t, x)$$

which converges uniformly in $0 \leq t \leq t_0, x \in \Omega_m^0$ if $t_0 < (8\|f\|_\infty)^{-1}$. (Here $(\mathcal{Q}^0 K^0)^k$ is the iteration of $\mathcal{Q}_j^0 K_{j,j+1}^0, j = m+k-1, \dots, m$, as in Theorem 4.2.) If $f \in C^1(\Omega_1^0)$ then $u(t, x)$ admits partial derivatives $\partial u / \partial t$ and $\partial u / \partial q$ which are continuous in (t, x) ; in particular u satisfies (5.1).

The following generalization of Theorem 5.1 will be used in §§6 and 7.

THEOREM 5.2. *Let $p = 1, 2, \dots$. Let f and $f^{(i)}, i = 1, \dots, p$ be continuous functions on Ω_1^0 , and $u(t)$ the solution of (5.2) in the interval $[0, (8\|f\|_\infty)^{-1})$ with $u(0) = f$. Then there exists a (unique) solution $u^{(i)}(t), 0 \leq t < (8\|f\|_\infty)^{-1}$, of*

$$(5.6) \quad \frac{d}{dt} \langle u^{(i)}(t), \phi \rangle = \left\langle u^{(i)}(t), v \cdot \frac{\partial}{\partial q} \phi \right\rangle + \langle K_{1,2} u^{(i)}(t) \otimes u(t), \phi \rangle$$

($\phi \in C^1(\Omega_1^0)$) with $u^{(i)}(0) = f^{(i)}, i = 1, \dots, p$. $u^{(i)}(t, x)$ are continuous in (t, x) . Moreover the following analogue of (5.5) holds:

$$(5.7) \quad \prod_{i=1}^p u^{(i)}(t, x_i) = \sum_{k=0}^\infty (\mathcal{Q}^0 K^0)^k \{ U_{p+k}^0(\cdot) (f^{(1)} \otimes \dots \otimes f^{(p)} \otimes f^{k \otimes}) \}(t, x)$$

where the convergence of the series on the right-hand side is uniform in $0 \leq t \leq t_0, x \in \Omega_p^0$ if $0 \leq t_0 < (8\|f\|_\infty)^{-1}$.

If a continuous solution u of (5.3) exists, we have for $m = 1, 2, \dots$

$$(5.8) \quad u(t)^{m\otimes} = U^0(t)f^{m\otimes} + \mathcal{Q}_m^0 K_{m,m+1}^0 u(\cdot)^{(m+1)\otimes}(t)$$

which can be iterated and leads to (5.5). Thus one may expect that $u(t)$ defined by the series on the right-hand side of (5.5) with $m=1$ should solve (5.3). But to prove this we need the relation (5.5) for $m=2$ which can be rewritten as follows: for $k=0, 1, 2, \dots$

$$(5.9) \quad (\mathcal{Q}^0 K^0)^k \{U_{k+2}^0(\cdot)f^{(k+2)\otimes}\} = \sum_{k_1+k_2=k} F_{k_1} \otimes F_{k_2},$$

where

$$(5.10) \quad F_k = F_k(t, x) = (\mathcal{Q}^0 K^0)^k \{U_{k+1}^0(\cdot)f^{(k+1)\otimes}\} \quad (k \geq 0).$$

The next lemma is concerned with a generalization of this identity.

LEMMA 5.1. Let $f, f^{(i)}, i=1, \dots, p$ be continuous functions on Ω_1^0 and F_k defined by (5.10). Set $F_0^{(i)}(t) = U_1^0(t)f^{(i)}$ and

$$F_k^{(i)}(t) = F_k^{(i)}(t, x) = (\mathcal{Q}^0 K^0)^k \{U_{k+1}^0(\cdot)(f^{(i)} \otimes f^{k\otimes})\}(t, x).$$

Then for $m=0, 1, 2, \dots, v=p+1, p+2, \dots$

$$(5.11) \quad (\mathcal{Q}^0 K^0)^m \{U_{m+v}^0(\cdot)(f^{(1)} \otimes \dots \otimes f^{(p)} \otimes f^{(v-p+m)\otimes})\}(t) \\ = \sum_{k_1+\dots+k_v=m} F_{k_1}^{(1)}(t) \otimes \dots \otimes F_{k_p}^{(p)}(t) \otimes \bigotimes_{j=p+1}^v F_{k_j}(t),$$

where $\bigotimes_{j=1}^k f_j = f_1 \otimes \dots \otimes f_k$.

REMARK 5.1. i) Lemma 5.1 is algebraic in nature: what we make use of in its proof are the linearity of $U_m^0(t)$ and $K_{m,m+1}^0$, the semi-group property $U(t-s)U(s) = U(t)$, the factorization $U_m^0(t)[f_1 \otimes \dots \otimes f_m] = U_1^0(t)f_1 \otimes \dots \otimes U_1^0(t)f_m$, and the relation

$$K_{m-1,m}^0 f_1 \otimes \dots \otimes f_m = \sum_{i=1}^{m-1} f_1 \otimes \dots \otimes f_{i-1} \otimes K_{1,2}(f_i \otimes f_m) \otimes f_{i+1} \otimes \dots \otimes f_{m-1}.$$

Therefore Lemma 5.1 is valid e.g. to the model for hard spheres moving in bounded or unbounded region with an elastic wall; hence analogues to Theorems 5.1 and 5.2 are proved along the same lines as they are proved after this remark, if one applies an estimate obtained in §8.

ii) The usual method of successive approximation can be applied to construct a continuous solution $u(t, x)$ of (5.3) (for the present model the construction is very simple into details). Once $u(t, x)$ is obtained, the linear equation (5.6) for $u^{(i)}$ is easy to solve. By the uniqueness of the solution and a remark as made previous to Lemma 5.1 these in turn prove (5.9) as well as the factorization, so that Lemma 5.1 is dispensable. Still it would be interesting and

sometimes useful to directly prove (5.11) without resorting to the uniqueness.

PROOF OF LEMMA 5.1. We proceed by induction on m . For simplicity we put

$$f^{(i)} := f \quad \text{and} \quad F_k^{(i)} := F_k \quad \text{for} \quad i = p + 1, \dots;$$

0 and j in \mathcal{W}_j^0 , $K_{j,j+1}^0$ and U_j^0 will be omitted. Let (5.11) hold for $1, \dots, m$. Then

$$\begin{aligned} J_{m+1} &= (\mathcal{W}K)^{m+1} \{ U(\cdot) (f^{(1)} \otimes \dots \otimes f^{(p)} \otimes f^{(v-p+m+1)}) \otimes \} \\ &= \mathcal{W}K \left\{ \sum_{k_1 + \dots + k_{v+1} = m} \left(\bigotimes_{i=1}^v F_{k_i}^{(i)} \right) \otimes F_{k_{v+1}} \right\} \\ &= \sum_{j=1}^v \sum_{k_1 + \dots + k_{v+1} = m} \mathcal{W} \{ F_1^{(1)} \otimes \dots \otimes K(F_{k_j}^{(j)}) \otimes F_{k_{v+1}} \otimes \dots \otimes F_{k_v}^{(v)} \}. \end{aligned}$$

Set

$$G_k^{(i)} := \sum_{k_1 + k_2 = k} F_{k_1}^{(i)} \otimes F_{k_2} \quad k = 0, 1, 2, \dots,$$

and then for $0 \leq s \leq t$ with t fixed

$$g_k^{(i)}(s) = U(t-s)KG_{k-1}^{(i)}(s), \quad k = 1, 2, \dots;$$

and carry out first the summation on k_j and k_{v+1} under the constraint $k_j + k_{v+1} = k - 1$ in the inner sum of the last expression of J_{m+1} and put $k = k_j$ afresh to get

$$(5.12) \quad J_{m+1} = \sum_{j=1}^v \sum_{\substack{k_1 + \dots + k_v = m+1 \\ k_j \geq 1}} \int_0^t \left(\bigotimes_{i=1}^{j-1} U(t-s)F_{k_i}^{(i)}(s) \right) \otimes g_{k_j}^{(j)}(s) \\ \otimes \left(\bigotimes_{i=j+1}^v U(t-s)F_{k_i}^{(i)}(s) \right) ds.$$

We may put $g_0^{(i)} = 0$, so that the constraint $k_j \geq 1$ in the inner sum above may be deleted, which allows us to change the order of the double sum. On the other hand, letting $p=1$ and $v=2$ in (5.11) and then having $\mathcal{W}K$ operate on the both sides of it, we obtain

$$F_k^{(i)} = \mathcal{W}KG_{k-1}^{(i)},$$

which is valid for $1 \leq k \leq m+1$ because of the induction hypothesis. Hence, by the semi group property of $U(t)$,

$$U(t-s)F_k^{(i)}(s) = \int_0^s g_k^{(i)}(r) dr;$$

in particular

$$(5.13) \quad F_k^{(i)}(t) = \int_0^t g_k^{(i)}(r) dr.$$

We have also $U(t-s)F_0^{(i)}(s) = F_0^{(i)}(t)$. Substituting these relations in the right-hand side of (5.12) and applying (5.13) together with the identity

$$\begin{aligned} \sum_{j=1}^k \int_0^t ds_j \int_{[0, s_j]^{k-1}} h(s_1, \dots, s_k) ds_1 \cdots ds_{j-1} ds_{j+1} \cdots ds_k \\ = \int_{[0, t]^k} h(s_1, \dots, s_k) ds_1 \cdots ds_k \end{aligned}$$

which is valid if h is integrable, we have

$$J_{m+1} = \sum_{k_1 + \dots + k_v = m+1} F_{k_1}^{(1)} \otimes \dots \otimes F_{k_v}^{(v)}.$$

The proof of Lemma 5.1 is complete. Q. E. D.

PROOF OF THEOREM 5.1. As mentioned after Theorem 5.2 we define $u(t)$ by (5.5) with $m=1$. Using

$$(5.14) \quad \int_{(t-v_1), t > 0} (v-v_1) \cdot l dl dv = 4 \quad \text{for every } v_1 \in S$$

we see that the p -th term of the series in (5.5) is dominated by

$$(8Mt)^k \binom{m+k-1}{k} M^m \quad (M = \|f\|_\infty)$$

which equals to M^m multiplied by the k -th term of the binomial expansion of $(1-8Mt)^{-m}$. Thus the series in (5.5) converges uniformly in $t \leq t_0$ and $x \in \Omega_m$. An application of Lemma 5.1 proves (5.5) for all m . The special case $m=2$ of (5.5) proves (5.2). Q. E. D.

The proof of Theorem 5.2 is similar and omitted.

§ 6. Convergence to a solution of the Boltzmann equation

Let $\Omega_m^0 = \Omega_m^{(0)}$ and

$$d_m = \{x \in \Omega_m^0 : q_i = q_j \text{ for some } i \neq j\}$$

and define, for $t_- \leq 0 \leq t_+$

$$J_m^-(t_-) := \{x \in \Omega_m^0 : T_s^0 x \notin d_m \text{ for all } t_- \leq s \leq 0\}$$

$$J_m^-(t_-, t_+) := \{x \in J_m^-(t_-) : T_s^0 x \in d_m \text{ for some } 0 < s < t_+\}.$$

The condition $x \in J_m^-(t_-, t_+)$ consists of the two: the one concerned with the

past — no pair of (point) particles meet (i.e., occupy a same position) in $[t_-, 0]$ in the process of free motion starting from x — and the other with the future — at least one pair meet in $(0, t_+)$. Given f_n a function on $\Omega_n^{(\varepsilon)}$, let $u_n(t)$ be defined as in §4 but with $\varepsilon = 1/n$. We write $\|f_{n|m}\|_\infty$ for $\text{ess sup } |f_{n|m}|$.

THEOREM 6.1. *Let f_n be such a sequence that the n -th entry f_n is a Borel measurable function on $\Omega_n^{(1/n)}$ which is symmetric in x_1, x_2, \dots, x_n . Assume the following conditions i) and ii):*

i) *there are constants C and M such that*

$$\|f_{n|m}\|_\infty \leq CM^m \quad \text{for } m \leq n;$$

ii) *there are a pair of numbers $t_- \leq 0 < t_+$, a non-negative integer n_0 , and a continuous function f on Ω_1^0 such that*

$$(6.1) \quad \lim_{n \rightarrow \infty} f_{n|m} = f^{m \otimes} \quad \text{a.e. on } \Omega_m^0 \quad \text{for } m \leq n_0; \text{ and}$$

$$(6.2) \quad \text{if } x_n \in \Omega_m^{(1/n)}, x \in J_m(t_-, t_+) \text{ and } x_n \rightarrow x, \text{ then}$$

$$\lim_{n \rightarrow \infty} f_{n|m}(x_n) = f^{m \otimes}(x), \quad \text{for } m \geq 2.$$

Let $\tilde{u}_{n|m}$ denote the right-hand side of the series expansion (4.6). Then for $0 < t < (1/8M) \wedge t_+$

$$(6.3) \quad \|u_{n|m}(t)\|_\infty = \|\tilde{u}_{n|m}(t)\|_\infty \leq C \left[\frac{M}{1-8tM} \right]^m \quad \text{for all } m;$$

$$(6.4) \quad \lim_{n \rightarrow \infty} u_{n|m}(t) = \lim_{n \rightarrow \infty} \tilde{u}_{n|m}(t) = u(t)^m \quad \text{a.e. on } \Omega_m^0 \quad \text{for } m \leq n_0; \text{ and}$$

$$(6.5) \quad \text{if } x_n \in \Omega_m^{(1/n)}, x \in J_m(t_- - t, t_+ - t) \text{ and } x_n \rightarrow x, \text{ then}$$

$$\lim_{n \rightarrow \infty} \tilde{u}_{n|m}(t, x_n) = u(t)^{m \otimes}(x), \quad \text{for } m \geq 2.$$

where $u(t)$ is a unique solution of the Boltzmann equation (5.2) starting with $u(0) = f$; moreover, for $m \geq 2$ and $1 \leq j \leq m$, if $x \in J_m^-(t_- - t)$, then

$$(6.6) \quad \lim_{n \rightarrow \infty} \tilde{u}_{n|m+1} \left(t, x, \left[q_j + \frac{l}{n}, v \right] \right)$$

$$= \begin{cases} \left[\prod_{k=1, k \neq j}^m u(t, x_k) \right] u(t, q_j, v_j) u(t, q_j, v) & \text{for } (v_j - v) \cdot l > 0 \\ \left[\prod_{k=1, k \neq j}^m u(t, x_k) \right] u(t, q_j, v_j^*) u(t, q_j, v^*) & \text{for } (v_j - v) \cdot l < 0 \end{cases}$$

except for a finite number of $v \in S$ (which may depend on x).

REMARK 6.1. i) In Theorem 6.1 we exactly set $\varepsilon =$ the diameter of a particle of n -particle system $= 1/n$. This can be replaced by

$$(6.7) \quad \lim_{n \rightarrow \infty} n\varepsilon = 1.$$

Under this scaling the chance the one specified particle experiences a collision per unit time is asymptotically constant (> 0), provided the dynamics is in equilibrium state (i.e. $f_n = 1/|\Omega_n|$). The limit under (6.7) is called the *Boltzmann-Grad limit*.
 ii) The uniform bound (6.3) follows from the condition i) independently of the condition ii).

REMARK 6.2. The condition ii) of Theorem 6.1 is satisfied if the convergence in (6.1) is uniform on every compact set of $\Omega_m^0 \setminus d_m$, an example of $\{f_n\}$ which satisfies the latter condition being given in Appendix III. It is noted that one can replace the condition (ii) by the uniform convergence on every compact set of $J_m^-(t_-)$ with the corresponding uniformity of convergence in the conclusion of Theorem 6.1. (See Appendix II.)

REMARK 6.3. Some comments on (6.1) to (6.5) would be to be given. If $n_0 = 0$, the condition (6.1) becomes empty. If $n_0 = 1$, (6.4) says that the solution of the Boltzmann equation is obtained as a limit of the first marginal density $u_{n|1}(t)$, which in a way is a result that we set out to seek for; the condition like (6.1) with $n_0 = 1$ is surely indispensable for such a result. Assume $t_- = 0$ to simplify the discussion below. Then the condition (6.2) imposes a kind of uniformity on the convergence of $f_{n|m}$ to $f^{m\otimes}$ along the set $J_m(0, t_+)$ which is very thin (definitely Lebesgue null) and concerns only the future behavior of particles. This may be understood in view of the series expansion (4.6): each point of $\partial\Omega_{j+1}$, which the operator $K_{j,j+1}$ involved in it is concerned in, leads to a configuration of future collision after operating the flow backward in time (what comes up from repeating these will clearly be seen in the expression (4.8) that follows).

Now we turn to (6.5). The constraint $x \in J_m(0, t_+ - t)$ involved in it inevitably ensues from the corresponding part of (6.2): such constraint is needed for and only for the first term of the series in (4.6). To explain the meaning of the other constraint that $x \in J_m^-(t)$ we let f_n be a probability density so that $X^n(t)$ is considered as a stochastic process taking values in Ω_n and $u_{n|m}(t)$ gives the probability density of $(X_1^n(t), \dots, X_m^n(t))$. We assume $\lim f_{n|m} = f^{m\otimes}$ locally uniformly in $\Omega_m^0 \setminus d_m$. In the Boltzmann-Grad limit the contribution to $u_{n|1}(t)$ of initial configurations x such that in $F[x, [0, t]]$ the first particle collides with none of the other $n-1$ particles remains away from zero. But for $(x_1^0, x_2^0) \notin J_2^-(t)$ such contribution vanishes because of a collision between the first and the second particles and therefore must substantially destroy the independence of the first and the second particles at the time t which was approximately valid at time zero under the present assumption, strongly suggesting that $u_{n|2}(t, x_1^0, x_2^0)$ would not be factorized.

Of the two chaos properties (6.4) and (6.5) the latter is established inde-

pendently of the former as it stands in the statement of Theorem 6.1, while in order to prove the former (with $n_0 > 0$) we shall have to make use of both assumptions (6.1) and (6.2). Nevertheless (6.5) can be considered more stringent than (6.4) in a sense, as suggested by an evidence observed with a model in which the velocity takes discrete values (cf. [16]). For such a model an analogue of (6.4) holds with $\lim u_{n|1}(t)$ replacing $u(t)$, but no analogue of (6.5) does except for special choices of f ; the limit of $u_{n|1}(t)$ does not solve the Boltzmann equation corresponding to the model, being consistent with the observation that for the validity of its derivation the relation like (6.5) must be essential.

REMARK 6.4. The fact that the Boltzmann equation is time-irreversible does not contradict the time-reversibility of the underlying dynamics. Let (6.2*) be the condition obtained by modifying (6.2) corresponding to the reversal of time. Assume $n_0 \geq 1$ in (6.1). Then replacing (6.2) by (6.2*), we have $\lim_{n \rightarrow \infty} u_{n|1}(t, x) = u(t, x)$, $-t_0 < t \leq 0$ and $u(t)$ solves

$$(B. eq^*) \quad \frac{d}{dt} \langle u(t), \phi \rangle = \left\langle u(t), v \cdot \frac{\partial}{\partial q} \phi \right\rangle - \langle K_{1,2}^0 u(t) \otimes u(t), \phi \rangle.$$

Assume that the conditions i), (6.1) and (6.2*) are satisfied. Let $-t_0 < t_1 < 0$ and set $\hat{f}_n(x) = u_n(-t_1, x)$ and

$$\hat{u}_n(t) = \hat{f}_n(T_{-t}x), \quad \hat{u}(t) = u(t - t_1), \quad 0 \leq t \leq t_1.$$

$u_{n|1}(t)$ converges to $\hat{u}(t)$, while $\hat{u}(t)$ solves (B.eq*), but, in general, does not the B. eq. (5.2). This does not contradict Theorem 6.1, but rather shows that Theorem 6.1 cannot be applied to $\{\hat{f}_n\}$, though $\{\hat{f}_n\}$ satisfies the condition of the theorem other than (6.2) (even when $n_0 = \infty$) — $\hat{f}_{n|m}$ is not factorized in the limit on the set $J(0, t_1)$ in general. A reasoning similar to the above is advanced by Lanford [10] as an answer to the longstanding question of why the time-irreversible equation of Boltzmann can be derived from a time-reversible dynamics of classical particles.

Let us introduce some notations to give a more or less transparent expression for each term of the series in (4.6). For $x \in \Omega_m^{(\varepsilon)}$, $l, v \in S$ and $j = 1, 2, \dots, m$ set

$$C_{j;0}^{v;l}x = (x_1, \dots, x_{j-1}, [q_j, v_j^*], x_{j+1}, \dots, x_m, [q_j - \varepsilon l, v^*])$$

$$C_{j;1}^{v;l}x = (x_1, \dots, x_m, [q_j + \varepsilon l, v]),$$

if $(v_j - v) \cdot l > 0$; and

$$C_{j;0}^{v;l}x = C_{j;1}^{v;l}x = \partial \quad \text{if } (v_j - v) \cdot l \leq 0 \quad \text{and} \quad C_{j;\sigma}^{v;l}\partial = \partial \quad (\sigma = 0, 1)$$

where ∂ is the extra point introduced before. Then

$$K_{m,m+1}g(x_1, \dots, x_m) = \sum_{j=1}^m \int_{S \times S} [g(C_{j,0}^{v_j; l} \mathbf{x}) - g(C_{j,1}^{v_j; l} \mathbf{x})] (v_j - v) \cdot l \, dl \, dv.$$

(Points outside $\Omega_{m+1}^{(\varepsilon)}$ are identified with ∂ and $g(\partial) = 0$ by convention.) For $\varepsilon \geq 0, k = 1, 2, \dots, n - m, \mathbf{x} \in \Omega_m^{(\varepsilon)}$ and a set of multivariables where

$$\Delta = (s, l, v, \sigma, j)$$

$$s = (s_1, \dots, s_k) \in [0, \infty)^k \quad \text{with} \quad s_1 < s_2 < \dots < s_k$$

$$l = (l_1, \dots, l_k) \in S^k$$

$$v = (v_{m+1}, \dots, v_{m+k}) \in S^k$$

$$\sigma = (\sigma_1, \dots, \sigma_k) \in \{0, 1\}^k$$

$$j = (j_1, \dots, j_k) \quad \text{with} \quad 1 \leq j_p \leq m + p - 1 \quad (p = 1, \dots, k),$$

we set $M_{0,\Delta}^{(\varepsilon)} \mathbf{x} = \mathbf{x}$ and

$$M_{k,\Delta}^{(\varepsilon)} \mathbf{x} = C_{j_k, \sigma_k}^{v_{m+k}, l_k} T_{-s_k + s_{k-1}}^{(m+k)} \dots C_{j_1, \sigma_1}^{v_{m+1}, l_1} T_{-s_1 + s_0}^{(m)} \mathbf{x}$$

where $s_0 = 0$. By writing $|\sigma| = \sum_j \sigma_j$, (4.6) can be written as

$$\begin{aligned} (6.8) \quad & \tilde{u}_{n|m}(t, \mathbf{x}) \\ &= f_{n|m}(T_{-t} \mathbf{x}) + \sum_{k=1}^{n-m} \sum_{\sigma} \sum_{j_k=1}^{m+k-1} \dots \sum_{j_1=1}^m (-1)^{|\sigma|} (n-m)_k \varepsilon^k \times \\ & \quad \times \int_0^t ds_k \int_0^{s_k} \dots \int_0^{s_2} ds_1 \int_{S^{2k}} f_{n|m+k}(T_{-t+s_k} M_{k,\Delta}^{(\varepsilon)} \mathbf{x}) \\ & \quad \times \prod_{p=1}^k (V^{(p)} - v_{m+p}) \cdot l_p \, dl \, dv \end{aligned}$$

where

$$V^{(p)} = V^{(p)}(\Delta, \mathbf{x}) := V_{j_p}(T_{-s_p + s_{p-1}}^{(m+p)} \dots C_{j_1, \sigma_1}^{v_{m+1}, l_1} T_{-s_1 + s_0}^{(m)} \mathbf{x})$$

($V^{(p)}$ is the velocity of the j_p -th particle at the time when the p -th addition of a particle is about to be made in the evolution of $M_{k,\Delta}^{(\varepsilon)} \mathbf{x}$).

LEMMA 6.1. *Let $t_- \leq 0, t > 0, k = 1, 2, \dots$ and Δ be as above. Let $\Gamma_k(t)$ be the set of $s: 0 < s_1 < \dots < s_k < t$. If $\mathbf{x} \in J_m^-(t_- - t)$ then there is a Borel set B of $\Gamma_k(t) \times S^{2k}$ such that $\Gamma_k(t) \times S^{2k} \setminus B$ is a Lebesgue null set and for each $(s, l, v) \in B$ and each σ, j*

$$\lim_{\varepsilon \downarrow 0} T_{-t+s_k}^{(\varepsilon)} M_{k,\Delta}^{(\varepsilon)} \mathbf{x} = T_{-t+s_k}^0 M_{k,\Delta}^{(0)} \mathbf{x},$$

the limit being in $J_{m+k}^-(t_-)$ unless it equals ∂ . (Note that the right-hand side of the above equality equals ∂ if and only if l is wrongly chosen, i.e., $(v_{m+p} - V^{(p)}(\Delta, \mathbf{x})) \cdot l_p \leq 0$ for some p .)

PROOF. Consider an orbit drawn in the space $\cup_{n=m}^{m+k} \Omega_n^{(\varepsilon)}$ by the system starting at $\mathbf{x} \in J_m^-(t_- - t)$ at time zero and arriving at $T_{-t+s_k}^{(\varepsilon)} M_{k,\Delta}^{(\varepsilon)} \mathbf{x}$ at time t in which particles evolves by the time reversed flow $T_{-s}^{(\varepsilon)}$ except that k new particles are added at times s_1, \dots, s_k according to the operation of $C_{j,\sigma}^{l,v}$'s. We denote this orbit by $w^{(\varepsilon)}(s), 0 \leq s \leq t$, which is determined by t, Δ and \mathbf{x} . It is not difficult to see that there is a Borel subset B of $\Gamma_k(t) \times S^{2k}$ such that $|\Gamma_k \times S^{2k} \setminus B| = 0$ and that if $(s, l, v) \in B$ and $w^{(0)}(s_k) \neq \partial$, then in the orbit $w^{(0)}(s), 0 \leq s \leq t$, no pair of particles occupies the same position at the same time except such simultaneous occupancies of the v -th added particle (i.e., the $(v+m)$ -th particle) and the j_v -th particle as caused by the operation of $C_{j,\sigma}^{l,v}$, and $w^{(0)}(t) \in J_{m+k}^-(t_-)$. (The problem is reduced to the case $k=m=1$ by induction on k .) It now would be almost clear that if $(s, l, v) \in B$ then $w^{(\varepsilon)}(s)$ converges to $w^{(0)}(s)$ for $0 \leq s \leq t$ (the convergence is uniform); in particular $\lim w^{(\varepsilon)}(t) = w^{(0)}(t)$. Q. E. D.

The next lemma is prepared to remove the condition (4.1) introduced in §4.

LEMMA 6.2. For each $t > 0, \sigma$, and j the mapping θ of $\Gamma_k(t) \times S^{2k} \times \Omega_m$ into Ω_{m+k} defined by

$$\theta(s, l, v, \mathbf{x}) = T_{-t+s_k} M_{\Delta} \mathbf{x}$$

is nonsingular in the sense that if A is a null set of Ω_{m+k} , then $\theta^{-1}A$ is also null.

PROOF. We prove only the case $k=1$; the general case $k > 1$ can be dealt with by induction in view of the relation $T_{-t+s} M_{\Delta} = T_{-t+s} C_{j,\sigma}^{v,l} T_{-s} (T_{s_{k-1}} M_{\Delta'})$ where we write $s = s_k, v = v_k$, etc. and $\Delta' = (s', \dots), s' = (s_1, \dots, s_{k-1})$, etc. Let $k=1$ and put

$$B = \{(s, l, v, \mathbf{x}) \in [0, t) \times S^2 \times \Omega_m : T_{-t+s} C_{j,\sigma}^{v,l} \mathbf{x} \in A\}$$

Then the (s, l, v) -section, $B'(s, l, v)$, of $B' := \theta^{-1}A$ is given by $T_{-s}^{-1}B(s, l, v)$, which is null for a.a. (s, l, v) since, by Lemma 3.4, B is null. Consequently B' is null. Q. E. D.

By virtue of Lemma 6.2 the integrals on the right-hand side of (6.8) have a definite meaning even when the Borel function f_n does not satisfy the continuity property (4.1); by Fubini's theorem the same is true for the expressions in (4.5) and (4.6). Now, approximating f_n by a.e.-convergent sequence of continuous functions, we can easily prove Theorem 4.3, i.e., that the condition (4.1) on f_n can be removed from the assumptions of Theorems 4.1 and 4.2.

PROOF OF THEOREM 6.1. Since the absolute value of the k -th term of the sum over k in (6.8) with $\varepsilon = 1/n$ is dominated by

$$\begin{aligned}
& C2^k 4^k (n-m)_k (m+k-1)_k \frac{t^k}{k!} M^{k+m} n^{-k} \\
& \leq C(8Mt)^k \binom{m+k-1}{k} M^m
\end{aligned}$$

and the infinite sum of the last quantity over k equals $C(M/(1-8tM))^m$ ($0 \leq t < 1/8M$), we have (6.3). This ensures that we can take the term-wise limit in the right-hand side of (6.8). But an application of Lemma 6.1 with the assumption ii) of Theorem 6.1 shows that each term of the sum converges in the required manner to the corresponding one of the right-hand side of (5.5). Q. E. D.

The following generalizations of Theorem 6.1 (Theorem 6.2 and 6.3) are useful in the next section where we are concerned with correlations for the distributions of $T_t \mathbf{x}$ between different times.

THEOREM 6.2. *Let p be a positive integer and $\{f_n\}$ be as in Theorem 6.1 except that f_n is supposed to be symmetric not in all labels 1 to n but only in x_{p+1}, \dots, x_n . The conditions i) and ii) are assumed but in ii) the limiting relations (6.1) and (6.2) are claimed for $m \geq p$ with the limit*

$$(6.9) \quad \prod_{i=1}^p f^{(i)}(x_i) f^{(m-p)\otimes}(x_{p+1}, \dots, x_m)$$

in place of $f^{m\otimes}(\mathbf{x})$, where $f^{(i)}$, $i=1, \dots, p$, are some continuous functions of Ω_p^0 . Then all of (6.3) to (6.6) hold with an appropriate modification: (6.4) to (6.6) are claimed for $m \geq p$ with the limit function

$$(6.10) \quad \prod_{i=1}^p u^{(i)}(t, x_i) u^{(m-p)\otimes}(x_{p+1}, \dots, x_m).$$

Here $u(t)$ is the same as in Theorem 6.1 and $u^{(i)}(t)$ is a unique solution of (5.6), $i=1, \dots, p$.

The proof of Theorem 6.2 is the same as that of Theorem 6.1 and omitted.

Given a continuous function f on Ω_1^0 and a solution $u(t)$, $0 \leq t < t_0$ of the Boltzmann equation (5.2) with $u(0)=f$, let us denote by H_t^s : $0 \leq s \leq t < t_0$ a family of linear transformations on $C(\Omega_1^0)$: for $\phi \in C(\Omega_1^0)$ and $0 \leq s < t_0$, $g_t = H_t^s\{\phi\}$ is a solution of the evolution equation in the interval $s \leq t < t_0$

$$(6.11) \quad \begin{cases} \frac{d}{dt} \langle g_t, \psi \rangle = \langle g_t, \mathfrak{A}_{u(t)} \psi \rangle & \text{for all } \psi \in C^1(\Omega_1^0) \\ g_s = \phi, \end{cases}$$

which is uniquely solved. Here

$$\mathfrak{A}_{u(t)}\psi(x) = v \cdot \frac{\partial \psi}{\partial q}(x) + \int_{(v-v_1) \cdot l > 0} \{\psi(q, v^*) - \psi(q, v)\} u(t, q, v_1)(v-v_1) \cdot l dl dv_1.$$

For a finite sequence of times $t=(t_1, \dots, t_k, t)$ with $0 < t_1 < \dots < t_k < t < t_0$ and for $\phi = \{\phi_j\}_{j=0}^k, \phi_j \in C(\Omega_1^0)$ we set

$$(6.12) \quad g^{u(t)}\{x; t, \phi\} = H_t^{t_k}\{\phi_k H_{t_k}^{t_{k-1}}\{\dots\{\phi_1 H_{t_1}^0\{\phi_0 u(0)\}\}\dots\}\}.$$

For a sequence $g = \{g_j\}_{j=p}^n$ whose j -component is a Borel measurable function on $\Omega_m^{(1/n)}$ we denote by $L_{t_1}^{n,p}\{g\}$ the sequence of the same kind whose j -component is

$$\sum_{k=0}^{n-j} (n-j)_k e^{k(\mathcal{Q}^{(1/n)} K^{(1/n)})^k} (U_{j+k}^{(1/n)}(\cdot) g_{j+k})(t, x);$$

and for ψ a Borel function on $\Omega_m^{(1/n)}$, ψg denotes a sequence whose j -component is $(\psi g)_j(x_1, \dots, x_j) = \psi(x_1, \dots, x_m) g_j(x_1, \dots, x_j) (m \leq p)$.

THEOREM 6.3. *Let $p=1, 2, \dots, k=0, 1, \dots$ and $\{f_n\}$ be as in Theorem 6.2 and satisfy hypotheses of it with $f^{(i)} = \phi_0^{(i)} \cdot f, \phi_0^{(i)} \in C(\Omega_1^0)$. For $i=1, \dots, k$ (if $k \geq 1$) let $\psi_i(x_1, \dots, x_p) = \prod_{j=1}^p \phi_i^{(j)}(x_j)$ where $\phi_i^{(j)} \in C(\Omega_1^0)$. Then for $0 < t_1 < \dots < t_k < t < (1/8M) \wedge t_+$ and for $v=p, \dots, n, n \geq p$, it holds that a.e. on $\Omega_v^{(1/n)}$*

$$(6.13) \quad \int_{\Omega_{n-p}^{(1/n)}} \psi_1(T_{-t+t_1}^{(1/n)} \mathbf{x}) \dots \psi_k(T_{-t+t_k}^{(1/n)} \mathbf{x}) u_n(t, \mathbf{x}) dx_{v+1} \dots dx_n$$

= the v -component of

$$L_{t_1-t_k}^{n,p} \{ \psi_k L_{t_k-t_{k-1}}^{n,p} \{ \dots \psi_2 L_{t_2-t_1}^{n,p} \{ \psi_1 L_{t_1}^{n,p} \{ \{ f_{n|j} \}_{j=p}^n \} \} \dots \} \},$$

where ψ_i on the left-hand side is naturally regarded as a function of Ω_n^0 , i.e., identified with $\psi_i \otimes$ (the identity map on Ω_{n-p}^0). Moreover the right-hand side of (6.13) converges, as $n \rightarrow \infty$, to

$$\prod_{i=1}^p g^{u(t)}\{x_i; t, \{\phi_0^{(i)}, \dots, \phi_k^{(i)}\}\} \prod_{j=p+1}^v u(t, x_j)$$

in the same manner of convergence as in (6.4) to (6.6) of Theorem 6.1.

PROOF. We shall proceed by induction on k . The case $k=0$ is just Theorem 6.2. If we set

$$\hat{f}_n(\mathbf{x}) = u_n(t_1, \mathbf{x}) \psi_1(\mathbf{x}),$$

then the integrand on the left-hand side of (6.13) is written

$$\hat{f}_n(T_{-t+t_1} \mathbf{x}) \psi_2(T_{-t+t_2} \mathbf{x}) \dots \psi_k(T_{-t+t_k} \mathbf{x})$$

and we can apply the induction hypothesis to obtain the relation (6.13) and the required convergence, because

$$\{\hat{f}_{n|j}\}_{j=p}^n \cong \psi_1 L_{t_1}^{n,p} \{ \{ f_{n|j} \}_{j=p}^n \}$$

and in view of Theorem 6.2 the v -component of the right-hand side of the last relation converges to

$$\prod_{i=1}^p (\phi_1^{(i)} \cdot H_{t_1}^0 \{ \phi_0^{(i)} f \}) (x_i) \prod_{j=p+1}^v u(t_1, x_j)$$

in the same manner as required for $\{f_n\}$ in the assumption of Theorem 6.2.

Q. E. D.

§7. Convergence to a limiting Markov process

In this section we shall assume that $\{f_n\}$ satisfies all the assumptions of Theorem 6.1 and

$$(7.1) \quad f_n \geq 0, \quad \int_{\Omega_n^{(1/n)}} f_n d\mathbf{x} = 1.$$

For $t_0 > 0$ let $\mathbf{D}_{m,t_0} \equiv D[[0, t_0], \Omega_m^0]$ denote the set of all mappings of $[0, t_0]$ into Ω_m^0 that are left-continuous and have limits from the right. We regard it as a topological space endowed with the Skorohod topology (cf. [3]). The topological σ -field of \mathbf{D}_{m,t_0} , denoted by \mathcal{F}_m , coincides with the σ -field generated by Borel measurable cylinder sets. A generic element of \mathbf{D}_{m,t_0} will be denoted by $w = (w(t); 0 \leq t \leq t_0)$.

Let $\mu_n(d\mathbf{x}) = f_n(\mathbf{x})d\mathbf{x}$. μ_n is a probability on the σ -field \mathcal{B}_n of all Borel subsets of $\Omega_n^{(1/n)}$. For each $m \leq n$ the motion of the first m particles

$$Y^{n,m}(t) := (X_1^n(t, \mathbf{x}), \dots, X_m^n(t, \mathbf{x}))$$

is a stochastic process defined on the probability space $(\Omega_n^{(1/n)}, \mathcal{B}_n, \mu_n)$, whose sample paths restricted to $[0, t_0]$ are elements of \mathbf{D}_{m,t_0} if $\tau > t_0$. Since $\tau > t_0$ μ_n -a.e., the map $\mathbf{x} \rightarrow Y^{n,m}(\cdot)$ which is a measurable map from $(\{\tau > t_0\}, \mathcal{B}_n \cap \{\tau > t_0\})$ into $(\mathbf{D}_{m,t_0}, \mathcal{F}_m)$ induces a probability measure on \mathbf{D}_{m,t_0} from μ_n . We denote this probability measure by $P_{n|m}$. $P_{n|m}$ is simply written P_n . $P_{n|m}$ is then a marginal distribution of P_n .

THEOREM 7.1. *Let $\{f_n\}$ satisfy (7.1) as well as the hypothesis of Theorem 6.1, and f and $u(t)$ be as in Theorem 6.1. Then for each $m=1, 2, \dots$ and $0 < t_0 < (1/8M) \wedge t_+$ the family of measures $\{P_{n|m}\}_{n=m}^\infty$ on \mathbf{D}_{m,t_0} weakly converges to the m -fold product measure $P^{(f)} \otimes \dots \otimes P^{(f)}$ as $n \rightarrow \infty$ where $P^{(f)}$ is a probability measure on \mathbf{D}_{1,t_0} which solves the following martingale problem: for each $\phi \in C^1(\Omega_1^0)$*

$$(7.2) \quad \phi(w(t)) - \int_0^t \mathfrak{A}_{u(s)} \phi(w(s)) ds, \quad t \in [0, t_0], \quad \text{is a } P^{(f)}\text{-martingale,}$$

and

$$(7.3) \quad \int_{\mathbf{D}_{1,t_0}} \phi(w(0))dP^{(f)} = \int_{\Omega_1^0} \phi(x)f(x)dx.$$

($\mathfrak{A}_{u(t)}$ is defined just after (6.11).)

REMARK 7.1. The proof of Theorem 7.1 consists of an application of Theorem 6.3 and the proof of the relative compactness of $\{P_{n|m}\}_n$, the latter being shown under the uniform bound of $u_{n|m}(t)$ as in (6.3). In the equilibrium case Theorem 6.3 extends to the arbitrary time interval and accordingly so does Theorem 7.1 (and all the other results in this section). The convergence of finite dimensional distributions of $Y^{n,1}(t)$, which follows from Theorem 6.3, has already been observed by Spohn [14] (see also [11] and references therein), while the compactness result seems new.

LEMMA 7.1. *The martingale problem (7.2)–(7.3) has a unique solution $P^{(f)}$. The finite dimensional distribution of $P^{(f)}$ is expressed by means of $g^{u(t)}$ defined by (6.12) as follows: for $t: 0 < t_1 < \dots < t_k < t < t_0$ and $\phi = \{\phi_0, \dots, \phi_k\}$, $\phi_i \in C(\Omega_1^0)$ and for $\psi \in C(\Omega_1^0)$*

$$(7.4) \quad \int_{\mathbf{D}_{1,t_0}} \phi_0(w(0)) \dots \phi_k(w(t_k)) \psi(w(t)) dP^{(f)} = \int_{\Omega_1^0} \psi(x) g^{u(t)}\{x; t, \phi\} dx.$$

PROOF. Let P be a solution of the martingale problem (7.2) and (7.3). For $\psi \in C^1(\Omega_1^0)$

$$\langle g_t, \psi \rangle = \int \phi_0(w(0)) \psi(w(t)) dP$$

solves (6.11) with $s=0$, $\phi = \phi_0$. Hence $\langle g_t, \psi \rangle = \langle H_t^0\{\phi_0 f\}, \psi \rangle$. Repeating this we get (7.4). In particular a solution of (7.2)–(7.3) is unique. Conversely the probability measure defined by (7.4) solves the martingale problem. Q. E. D.

REMARK 7.2. By the probability law $P^{(f)}$ in Theorem 7.1 the process $w(t): 0 \leq t \leq t_0$ ($w \in \mathbf{D}_{1,t_0}$) is a time-inhomogeneous Markov process governed by the characteristic operator $\mathfrak{A}_{u(t)}$. The transition law of it is given by a Markov semi-group of linear operators $P(s, t)$, $0 \leq s \leq t \leq t_0$, acting on $C(\Omega_1^0)$ which is characterized by (a weak version of)

$$\frac{\partial}{\partial s} P(s, t) = -\mathfrak{A}_{u(s)} P(s, t) \quad \text{or} \quad \frac{\partial}{\partial t} P(s, t) = P(s, t) \mathfrak{A}_{u(t)}.$$

These equations with $P(t, t) =$ the identity operator are solved by iterating their common integrated version

$$P(s, t) \psi = U_1^0(-t+s) \psi + \int_s^t U_1^0(-r+s) K_{u(r)} P(r, t) \psi dr$$

where

$$K_{u(t)}\phi(x) = \int_{(v-v_1) \cdot t > 0} \{\phi(q, v^*) - \phi(x)\}u(t, q, v_1)(v-v_1) \cdot ldlv_1.$$

The operators H_t^s defined by (6.11) links with $P(s, t)$ by

$$\langle H_t^s\{\phi u(s)\}, \psi \rangle = \langle \phi u(s), P(s, t)\psi \rangle,$$

from which also the formula (7.4) can be derived. This relation may also be rewritten by means of the expectation $E^{(f)}$ by $P^{(f)}$ as follows:

$$H_t^s\{\phi\}(x) = E^{(f)}[(\phi/u(s))(w(s)) | w(t)=x]u(t, x).$$

PROOF OF THEOREM 7.1. Lemma 7.1 together with Theorem 6.3 proves the convergence of finite dimensional distribution of $P_{n|m}$. Therefore the proof of Theorem 7.1 is reduced to proving

LEMMA 7.2. Under the assumption of Theorem 7.1 the family $\{P_{n|m}\}_{n=m}^\infty$ is relatively compact with respect to the weak topology of probability measures on \mathbf{D}_{m,t_0} for each $m=1, 2, \dots$ and $0 < t_0 < 1/8M$.

PROOF. It suffices to show that there exists a constant C_0 such that if $0 \leq t_1 < t < t_2 \leq t_0$ and $n \geq m$, then

$$(7.5) \quad \mu_n\{V_i(t) \neq V_i(t_1) \text{ and } V_j(t_2) \neq V_j(t) \text{ for some } 1 \leq i, j \leq m\} \leq C_0(t_2 - t_1)^2.$$

In fact from (7.5) it follows that

$$\begin{aligned} P_{n|m}\{|w(t) - w(t_1)| > \lambda, |w(t_2) - w(t)| > \lambda\} \\ \leq \mu_n\{\sum_{i=1}^m |Q_i(t) - Q_i(t_1)|^2 > \lambda^2\} + \mu_n\{\sum_{i=1}^m |Q_i(t_2) - Q_i(t)|^2 > \lambda^2\} + C_0(t_2 - t_1)^2 \\ \leq (m8/\lambda^2 + C_0)(t_2 - t_1)^2, \end{aligned}$$

which together with the convergence of the finite dimensional distribution implies the weak pre-compactness of $\{P_{n|m}\}$ (cf. Billingsley [3]).

Set

$$I = \mu_n\{V_1(t) \text{ has at least two jumps in } [t_1, t_2]\},$$

$$II = \mu_n\{V_1(t) \text{ has a jump at } s \text{ and } V_2(t) \text{ has a jump in } (s, t_2] \text{ for some } s \in [t_1, t_2]\}.$$

Then the left-hand side of (7.5) is dominated by $mI + m(m-1)II$. We will show that both I and II are bounded by a constant multiple of $(t_2 - t_1)^2$.

Instead of I we estimate the quantity

$I^+[I^-] = (n-1)\mu_n$ {for some $s \in [t_1, t_2]$, the 1-st particle collides with the 2nd particle at s and one of them collides with some of the other particles in $(s, t_2]$ [resp. in $[t_1, s)]$ }.

One observes that $I \leq I^+ + I^-$. Let $\mu_n^{(s)}(d\mathbf{x}) = u_n(s, \mathbf{x})d\mathbf{x}$. For the sake of notational simplicity we have the $(n-1)$ -th and n -th particles play the role of the 1-st and 2nd particles in the definition of I^+ . Then, noting Lemma 3.3, we see that

$$\begin{aligned} I^+ &\leq (n-1)(n-2)\mu_n\{\text{for some } s \in [t_1, t_2], \text{ the } n\text{-th particle collides with the } \\ &\quad (n-1)\text{-th particle at } s \text{ and one of them collides with the} \\ &\quad \text{first particle in } (s, t_2] \text{ for the first time after } s\} \\ &\leq (n-1)(n-2) \int_{t_1}^{t_2} \mu_n^{(s)}\{\mathbf{x} = (\mathbf{x}', x_{n-1}, x_n) \in \Omega_n: \text{ there is a collision in} \\ &\quad \mathbf{F}_2[(x_{n-1}, x_n), (-ds, 0)]; \text{ and } Q_1^{(n-2)}(t, \mathbf{x}') \text{ intersects} \\ &\quad \text{with } Q^{(2)}(t, x_{n-1}, x_n) \text{ for some } t \in (0, t_2 - s)\}. \end{aligned}$$

(In the last braces ‘‘intersect’’ means that the areas occupied by particles centered at $Q_1^{(n-2)}$ and $Q^{(2)}$ overlap each other; here the shadow collisions treated in Lemma 3.3 become relevant.) The last integral equals

$$(7.6) \quad \frac{1}{n} \int_{t_1}^{t_2} ds \int d\mathbf{x} \int_{(v-v_n) \cdot l > 0} dl dv_n (v-v_n) \cdot l \hat{\mu}_{n-2}^{(s, \frac{x}{2}, l, v_n)} \{\text{the first particle intersects with } Q^{(2)}(t, x, [q+l/n, v_n]) \text{ for some } t \in (0, t_2 - s)\}$$

where $x = [q, v]$ and

$$\hat{\mu}_{n-2}^{(s, \frac{x}{2}, l, v_n)}(d\mathbf{x}') = u_n(s, \mathbf{x}', x, [q+l/n, v_n])d\mathbf{x}'.$$

Set

$$\begin{aligned} \hat{f}_{n-2}^{(s, \frac{x}{2}, l, v_n)}(\mathbf{x}') &= u_n(s, \mathbf{x}', x, [q+l/n, v_n]), \\ h^{(s, x, l, v_n)}(t, x_1) &= \int \hat{f}_{n-2}^{(s, \frac{x}{2}, l, v_n)}(T_{-t}^{(n-2)} \mathbf{x}') dx_2 \cdots dx_{n-2}. \end{aligned}$$

Though \hat{f} above is defined only via values of $u_n(s)$ on $\partial\Omega_n$, the relation $u_{n|m} = \tilde{u}_{n|m}$ a.e. on $R \times \partial\Omega_m$ enables us to make the same computation as employed to get (6.3) to see

$$\|\hat{f}_{n-2}^{(s, \frac{x}{2}, l, v_n)}\|_\infty \leq C \left(\frac{M}{1-8Ms} \right)^{m+2} \quad \text{for a.a. } (s, x, l, v_n)$$

and hence

$$\|h^{(s, x, l, v_n)}(t)\|_\infty \leq C \left(\frac{M}{1-8Ms} \right)^2 \frac{M}{1-8M(t+s)} \quad \text{for a.a. } (s, x, l, v_n).$$

Taking into account that the roles played by the n -th and $(n-1)$ -th particle are

interchangeable, one sees that (7.6) is dominated by

$$\frac{2}{n^2} \int_{t_1}^{t_2} ds \int dx \int_{(v-v_n) \cdot l > 0} dl dv_n (v-v_n) \cdot l \int_0^{t_2-s} dt \int_{(v-v_1) \cdot l' > 0} dl' dv_1 (v-v_1) \cdot l' \\ \times h^{(s, x, l, v_n)} \left(t, \left[q + tv + \frac{l}{n}, v_1 \right] \right).$$

Consequently we have

$$I^+ \leq \frac{(n-1)(n-2)}{n^2} C[M/(1-8Mt_0)]^3 \cdot 32 \int_{t_1}^{t_2} (t_2-s) ds \leq C_1(t_2-t_1)^2.$$

Similarly we have $I^- \leq C_1(t_2-t_1)^2$.

In view of the bound of I^+ and I^- above we get

$$(7.7) \quad II \leq C_2(t_2-t_1)^2$$

if we prove

$$(7.8) \quad (n-2)(n-3)\mu_n \{ \text{for some } s \in [t_1, t_2] \text{ the particles 1 and 3 collide with} \\ \text{each other at } s \text{ but each of them no more collides with any particle in} \\ \text{(} s, t_2 \text{)]; and the particles 2 and 4 collide with each other in } (s, t_2] \text{ but} \\ \text{neither of them collides with any particle in the past within this interval} \\ \leq C'(t_2-t_1)^2. \}$$

By interchanging the role of particles 2 and 3, we see that the left-hand side of (7.8) is at most

$$(n-2)(n-3) \int_{t_1}^{t_2} \mu_n^{(s)} \{ \mathbf{x} \in \Omega_2 : \text{there is a collision in } \mathbf{F}[(x_1, x_2), (-ds, 0]]; \\ \text{and also in } \mathbf{F}[(x_3, x_4), [0, t_2-s]] \} \\ = \frac{(n-2)(n-3)}{n^2} \int_{t_1}^{t_2} ds \int dx_1 \int_{(v_1-v_2) \cdot l > 0} (v_1-v_2) \cdot l dl dv_2 \\ \times \int_0^{t_2-s} dt \int dx_3 \int_{(v_3-v_4) \cdot l' > 0} (v_3-v_4) \cdot l' dl' dv_4 \\ \times u_{n|4}(s, x_1, [q_1 + \varepsilon l, v_2], T_{-t}^{(2)}(x_3, [q_3 + \varepsilon l', v_4])) \\ \leq C'(t_2-t_1)^2.$$

Thus we have (7.8) and hence (7.7). The proof of Lemma 7.2 is complete.

Q. E. D.

The conclusion of Theorem 7.1 may be stated by means of the empirical measure α^n which is the random probability measure on \mathbf{D}_{1, t_0} defined by

$$\alpha^n = \frac{1}{n} \sum_{i=1}^n \delta_{x_i^n(\cdot)}:$$

namely, under the symmetry of f_n , the convergence of $P_{n|m}$ to $\underbrace{P^{(f)} \otimes \dots \otimes P^{(f)}}_m$ ($m=1, 2, \dots$) is equivalent to that of α^n to $P^{(f)}$ in probability as is easily shown. Since the limiting process $X(t)$ is free from any fixed points of discontinuity, the convergence of α^n is stronger than that of

$$\alpha_t^n = \frac{1}{n} \sum_{i=1}^n \delta_{x_i^n(t)}, \quad 0 \leq t \leq t_0$$

which is a random probability measure on Ω_1^0 . The next theorem which is another strengthened version of the latter convergence would not directly follow from Theorem 7.1 but is easily proved with the aid of (7.5). (The relation (7.9) in it is set forth, for a slightly different model, by Lebowitz and Spohn [11] as having “not been proved so far”.)

THEOREM 7.2. *If ϕ is a bounded Borel measurable function on Ω_1^0 such that either (i) ϕ is differentiable with respect to the space variable q with the partial derivative $(\partial/\partial q)\phi$ being bounded or (ii) ϕ is of the form $\phi(q, v) = h(v)\chi_A(q)$ where A is a domain of the torus S_2 with its boundary being smooth (χ_A is an indicator function of A), then, under the same assumption as in Theorem 7.1, for each $\delta > 0$*

$$(7.9) \quad \lim_{n \rightarrow \infty} P[\sup_{0 \leq t \leq t_0} |\langle \alpha_t^n, \phi \rangle - \langle u(t), \phi \rangle| > \delta] = 0.$$

PROOF. It suffices to show that for $0 \leq s < r < t \leq t_0$

$$(7.10) \quad \int_{\Omega_n} |\langle \alpha_s^n, \phi \rangle - \langle \alpha_r^n, \phi \rangle|^2 |\langle \alpha_t^n, \phi \rangle - \langle \alpha_r^n, \phi \rangle|^2 d\mu_n \leq C|t-s|^2,$$

where C is a constant independent of s, r, t and n . In fact this estimate proves that in the Skorohod topology of $D := D[[0, t_0], R]$ the family of processes $y_t^n = \langle \alpha_t^n, \phi \rangle$ is tight and hence, by a remark which precedes Theorem 7.2, weakly converges to a (nonradom) process $\langle u(t), \phi \rangle$; but the boundary of the set $B = \{w \in D: \sup_{0 \leq t \leq t_0} |w(t) - \langle u(t), \phi \rangle| > \delta\}$ is null with respect to the limit measure which concentrates on a single path $w_0(t) = \langle u(t), \phi \rangle$, so that the corresponding probability of B tends to zero.

One can replace (7.10) by

$$\begin{aligned} & \int_{\Omega_n} |\phi(X_i^n(r)) - \phi(X_i^n(s))| |\phi(X_j^n(r)) - \phi(X_j^n(s))| \\ & \times |\phi(X_k^n(t)) - \phi(X_k^n(r))| |\phi(X_l^n(t)) - \phi(X_l^n(r))| d\mu_n \leq C(t-s)^2 \end{aligned}$$

which should be valid for every combination of $1 \leq i, j, k, l \leq 4$. In the case (i) of ϕ this follows from (7.5) as in the proof of Lemma 7.2. As for the case (ii) of ϕ we have merely to get such an estimate as

$$\mu_n\{V_j(s) \neq V_j(r); Q_k(r) \in A \text{ but } Q_k(t) \notin A\} \leq C(t-s)^2$$

or self-evident variants of this. But these are proved by applying (7.5) in a way similar to that that proved (7.7). Q. E. D.

§ 8. The case of hard spheres

For the dynamics of hard spheres moving in R^3 the velocities after the collision are given by

$$\begin{aligned} v^* &= v - [(v - v_1) \cdot l]l \\ v_1^* &= v_1 - [(v_1 - v) \cdot l]l \end{aligned}$$

where $v, v_1 \in R^3$ and l is a unit vector in R^3 . For each l fixed the mapping $A_l: (v, v_1) \rightarrow (v^*, v_1^*)$ is a linear transformation of R^6 with $|\det A_l| = 1$, so that it preserves the volume $dv dv_1$. This proves an analogue of Lemma 1.1 which the discussions through §1 to §4 are based on. We have not to modify the expressions of $K_{m,m+1}$ in §4 and $K_{u(t)}$ in §7; with (v^*, v_1^*) defined above they are valid for the hard sphere dynamics. Though the modulus of the velocity of each particle changes through collisions, the total energy $v_1^2 + v_2^2 + \dots + v_n^2$ is kept constant. Therefore we can suppose that the velocities are uniformly bounded to derive the BBGKY hierarchy, since the number of particles are fixed in the derivation. The equations (4.4) to (4.6) are valid if ε is replaced by ε^2 (ε is the common diameter of the hard spheres). To obtain a limit theorem as $n \rightarrow \infty$, we need some upper bounds for $u_{n|m}(t)$:

LEMMA 8.1. *Let f_n be a Borel function on R^{6n} ($n=1, 2, \dots$) having the symmetry as in Theorem 4.1. Define $u_n(t)$ and $u_{n|m}(t)$ as before but by means of the dynamics of hard spheres of diameter $\varepsilon=1/\sqrt{n}$. Assume that there exist positive constants C, M and β such that*

$$(8.1) \quad |f_{n|m}(\mathbf{q}, \mathbf{v})| \leq CM^m \prod_{i=1}^m h_\beta(v_i)$$

where

$$h_\beta(v) = (\beta/2\pi)^{3/2} e^{-(\beta/2)v^2}, \quad v \in R^3.$$

Then for $0 \leq t \leq t_0$

$$(8.2) \quad |u_{n|m}(t, \mathbf{q}, \mathbf{v})| \leq eC(e^{13/10}M)^m \frac{1}{1-t/t_0} \prod_{i=1}^m h_\beta(v_i)$$

where $\beta' = e^{-1/5} \beta$ and

$$t_0 = (\sqrt{\beta/M}) [2e^{6/5}(\sqrt{8\pi} + \pi\sqrt{5/e})]^{-1} \quad (> 0.2 \times \sqrt{\beta/M}).$$

If $f_n \geq 0$, then (8.2) is valid for $0 \leq t \leq 2t_0$. (A similar bound is obtained by King [8] and Spohn [15]; cf. also [6] for a similar computation.)

Once the bound (8.2) is verified, one can proceed as before to obtain an analogue of Theorems 6.1 and 7.1 in which the limit is taken under the Grad's scaling $\varepsilon = 1/\sqrt{n}$ and the bounds i) and (6.3) in Theorem 6.1 are replaced by (8.1) and by (8.2), respectively. (The collision integral of the Boltzmann equation thus obtained differs from that of (0.1) by the factor 1/2.)

PROOF OF LEMMA 8.1. Set $v = m + k - 1$ and $\beta_j = a^{j/k} \beta$ with $0 < a < 1$ and $j = 0, 1, 2, \dots$. Then

$$\begin{aligned} & |K_{m+k-1, m+k} U(t) f_{n|m+k}| / C [(\beta/2\pi)^{3/2} M]^{m+k} \\ & \leq 4\pi \exp\left(-\frac{\beta}{2} \sum_{j=1}^v v_j^2\right) \int_{R^3} e^{-(3/2)v^2} [v|v| + \sqrt{v}(\sum_{j=1}^v v_j^2)^{1/2}] dv \\ & \quad (\text{where we applied } \sum_{j=1}^v |v_j - v| \leq v|v| + \sqrt{v}(\sum v_j^2)^{1/2} \text{ and } \int_{v \cdot l < 0} v \cdot l dl = |v|\pi) \\ & \leq 2\pi \exp\left(-\frac{\beta_1}{2} \sum_{j=1}^{m+k-1} v_j^2\right) \frac{m+k}{\beta_0^2} \{8\pi + \sqrt{2\pi^3} \sqrt{\beta/(m+k)} \sup_{r>0} r e^{-(\beta-\beta_1)r^2/2}\}. \end{aligned}$$

We apply this inequality repeatedly. In the $(p+1)$ -th step $m+k-p$ may replace $m+k$ which appears, on the last line above, in the middle (as a factor) and in the braces. We however does not make such replacement, but retain $m+k$ in these places, without violating the inequality. Now, noticing

$$\begin{aligned} \sqrt{\beta/(m+k)} \sup_{r>0} r e^{-(\beta-\beta_1)r^2/2} &= \sqrt{\beta/(m+k)} \sqrt{e \cdot (\beta-\beta_1)} \\ &\leq [ek(1-a^{1/k})]^{-1/2} =: A \end{aligned}$$

and making use of the hypothesis (8.1), we have, with $\beta' = a\beta$,

$$\begin{aligned} & |(QK)^k U(\cdot) f_{n|m+k}(t)| \\ & \leq C [(\beta/2\pi)^{3/2} M]^{m+k} \exp\left(-\frac{\beta'}{2} \sum_{j=1}^m v_j^2\right) (m+k)^k \left\{ \prod_{j=0}^{k-1} \beta_j^{-2} \right\} \\ & \quad \times [8\pi + A(2\pi)^{3/2}] \frac{(2\pi t)^k}{k!}. \end{aligned}$$

By the relations

$$\prod_{j=0}^{k-1} \beta_j^{-2} = \beta^{-2k} a^{-k+1} \leq \beta^{-2k} a^{-k}, \quad (m+k)^k \leq k^k e^m, \quad \frac{k^k}{k!} \leq e^k,$$

the last line is dominated by

$$C(a^{-3/2}eM)^m \{ \prod_{j=1}^m h_{\beta'}(v_j) \} \left\{ \frac{2\pi Me}{a\sqrt{\beta}} (2\sqrt{2/\pi} + A)t \right\}^k.$$

By elementary calculus we have $(1 - e^{-x})^{-1/2} \leq (1+x)/\sqrt{x}$ ($0 \leq x \leq 1$). Hence if $-\log a \leq 1$ ($0 < a < 1$),

$$A = [ek(1 - a^{1/k})]^{-1/2} \leq \frac{1}{\sqrt{e}} \left(\frac{1}{\sqrt{-\log a}} + \frac{1}{k} |\log a| \right) \quad (k \geq 1)$$

and by $2\sqrt{2/\pi} > 1$,

$$\begin{aligned} |u_{n|m}(t)| &\leq C(eMa^{-3/2})^m \{ \prod_{j=1}^m h_{\beta'}(v_j) \} \\ &\times \left\{ 1 + \sum_{k=1}^m \left[\frac{2\pi Met}{a\sqrt{\beta}} \left(2\sqrt{2/\pi} + \frac{1}{\sqrt{-e \log a}} \right) \left(1 + \frac{1}{k} \right) \right]^k \right\}. \end{aligned}$$

Now let us take $a = e^{-1/5}$. Then from the last inequality follows the inequality (8.2), which was to be proved. Q. E. D.

Appendix I. Measurability of T_t

We shall prove Borel measurability of sets and functions which appeared in §3. The parameter ε will be fixed and suppressed from the notation.

Set for $t > 0$

$$B_k(t) = \{ \mathbf{x} \in \Omega_n : \tau > t \text{ and the number of collisions in } \mathbf{F}_n[\mathbf{x}; [jt/k, (j+1)t/k]] \text{ is at most one for each } j=0, 1, \dots, k-1 \}.$$

$B_k(t)$ for $t < 0$ is analogously defined (“ $\tau > t$ ” is replaced by “ $\tau^{(-)} < t$ ” and $[jt/k, (j+1)t/k]$ by $[(j+1)t/k, jt/k]$).

LEMMA 1. *Let $t > 0$ and B be a Borel subset of Ω_n . If $B \subset B_1(-t)$, then $T_t^{-1}B$ and $T_{-t}B$ are Borel measurable. The analogous statement holds for $t \leq 0$.*

PROOF. Let $B' = B \cap \partial\Omega_n$ and $B'' = B' \setminus \partial\Omega_n$. Since T_t is a continuous map from $G_t = \{ \mathbf{x} : (t, \mathbf{x}) \in G \}$ into $\Omega_n \setminus \partial\Omega_n$ (see Lemma 1.2), $T_t^{-1}B''$ is Borel. B' is decomposed into a disjoint union of two Borel sets $B'_\pm = B' \cap (\partial\Omega_n \setminus \Sigma_n)_\pm$ where $(\partial\Omega_n \setminus \Sigma_n)_{+(-)}$ is the set of all $\mathbf{x} \in \partial\Omega_n \setminus \Sigma_n$ s.t. a pair of \mathbf{x} is in in-coming (out-going) collision. Since $B \subset B_1(-t)$, $T_t^{-1}B'_+ = T_{-t}^0 B'_+$ and $T_t^{-1}B'_- = T_{-t}^0 (B'_-)^*$; hence these are all Borel. ($A^* := \{ \mathbf{x}^* : \mathbf{x} \in A \}$.) Consequently $T_t^{-1}B$ is Borel. Since $T_{-t}B = T_t^{-1}B \setminus (\partial\Omega_n \setminus \Sigma_n)_-$, $T_{-t}B$ also is Borel. Q. E. D.

LEMMA 2. *For every $k=1, 2, \dots$ $B_k(t)$ is a Borel set.*

PROOF. The induction with the help of Lemma 1 and the relation

$$B_k(t) = T_{t/k}^{-1}(B_1(-t/k) \cap B_{k-1}(t-t/k))$$

proves the lemma, if one shows that $B_1(t)$ is Borel for every t . Let $t > 0$ and set

$$B^{(i,j)}(t) = \{ \mathbf{x} \in B_1(t) : \text{a collision in } F[\mathbf{x}, [0, t]] \text{ if any may occur only} \\ \text{between the } i\text{-th and } j\text{-th particle} \}.$$

Then $B_1(t) = \cup_{i < j} B^{(i,j)}(t)$ and $B^{(i,j)}(t)$ equals

$$\bigcup_{N=1}^{\infty} \bigcap_{\substack{0 \leq r \leq t \\ r \text{ is rational}}} \{ \mathbf{x} \in \Omega_n : \text{there is no grazing collision in } F_2[(x_i, x_j), [0, t]], \\ \rho(Q^{(2)}(r, (x_i, x_j)), q_k + rv_k) \geq \varepsilon + 1/N \text{ for } k \neq i, j, \text{ and} \\ \rho(q_k + rv_k, q_v + rv_v) \geq \varepsilon + 1/N \text{ for } k, v \neq i, j, k \neq v \}.$$

It would be clear that $B^{(i,j)}(t)$ is Borel. The case $t < 0$ is similarly treated.

Q. E. D.

LEMMA 3. *If $t \geq 0$ [$t > 0$] and B is a Borel set included in $\{\tau > t\}$ [resp. $\{\tau^{(-)} < t\}$], then $T_t B$ and $T_t^{-1} B$ are Borel measurable.*

PROOF. Let $t > 0$. Since $\{\tau > t\} = \cup_{k=1}^{\infty} B_k(t)$, it suffices to show that $T_t A$ and $T_t^{-1} A$ are Borel if A is a Borel subset of $B_k(t)$. As for $T_t A$ this is inductively deduced from

$$T_t A = T_{t-t/k} T_{t/k} A,$$

because Lemma 1 together with $B_k(t) \subset B_1(t/k)$ shows that $T_{t/k} A$ is Borel and, on the other hand, $T_{t/k} A \subset T_{t/k} B_k(t) \subset B_{k-1}(t-t/k)$. The proof for $T_t^{-1} A$ is similar.

Q. E. D.

The measurability of $A^{i,j}$ in the proof of Lemma 3.2 is shown in that of Lemma 2 above. The measurability of $T_t A$ and A_m in the proofs of Lemmas 3.2 and 3.1 follows from Lemmas 1 and 2. From Lemma 3 it follows that $\phi(T_t \mathbf{x})$ is Borel measurable on $\{\tau > t\}$ [resp. $\{\tau^{(-)} < t\}$] if $t \geq 0$ [resp. $t < 0$] and τ is a Borel function on Ω_n . All the other measurability questions may be answered by the next theorem. Recall that $\Sigma_n = \Sigma_n^{(e)}$ denotes the set of configurations of multiple touch, twin collision or grazing collision.

THEOREM. *If B is a Borel subset of Ω_n such that either $B \cap \Sigma_n = \emptyset$ or $\Sigma_n \subset B$, then $\{(t, \mathbf{x}) \in R \times \Omega_n : T_t \mathbf{x} \in B\}$ is a Borel set. In other words the mapping $(t, \mathbf{x}) \rightarrow T_t \mathbf{x}$ is Borel measurable if all the points of Σ_n are identified.*

PROOF. First we show that $T_t^{-1} B$ is a Borel set of Ω_n . Let $t < 0$ and decompose B into three disjoint sets $B_1 = B \cap \{\tau^{(-)} < t\}$, $B_2 = B \cap \{\tau^{(-)} \geq t\} \setminus \Sigma_n$

and $B_3 = B \cap \Sigma_n$. Since $\{\tau^{(-)} < t\}$ is Borel (in fact open), B_1 and B_2 are Borel. By Lemma 3 $T_\tau^{-1}B_1$ is a Borel set. Also we have $T_\tau^{-1}B_2 = \emptyset$ and $B_3 = \emptyset$ or Σ . Therefore $T_\tau^{-1}B$ is a Borel set. The case where $t \geq 0$ is similar. Thus we have shown that if all the points of Σ_n are identified, the mapping $\mathbf{x} \rightarrow T_t \mathbf{x}$ is Borel measurable. But, under this identification, $t \rightarrow T_t \mathbf{x}$ is left-continuous up to τ for all \mathbf{x} and τ is a Borel function; hence $T_t \mathbf{x}$ is jointly Borel measurable. Q. E. D.

Appendix II. The uniform convergence of $u_{n|m}(t)$

The lemma given below is a strong version of Lemma 6.1 and proves the uniform convergence of $u_{n|m}(t, \mathbf{x})$ on each compact set of $J_m^-(t_- - t)$ in Theorem 6.1, provided that the convergence of $f_{n|m}$ to $f^{m\otimes}$ also is uniform on each compact set of $J_m^-(t_-)$. This formulation of the main theorem is found in [8], [10] and [14].

LEMMA. Let A be a compact set of $J_m^-(t_- - t)$ ($t_- \leq 0, t > 0$). Then for any $\eta > 0$ there exists a Borel set B of $\Gamma_k \times S^{2k} \times A$ such that

$$|\Gamma_k \times S^{2k} \setminus B(\mathbf{x})| < \eta \quad \text{for all } \mathbf{x} \in A$$

($B(\mathbf{x})$ is the \mathbf{x} -section of B ; $\Gamma_k = \Gamma_k(t)$ is the same as in Lemma 6.1), for all σ, j

$$T_{-t+s_k}^{(\varepsilon)} M_{k,\Delta}^{(\varepsilon)} \mathbf{x} \longrightarrow T_{-t+s_k}^0 M_{k,\Delta}^0 \mathbf{x} \quad \text{as } \varepsilon \downarrow 0 \quad (\Delta = (s, \mathbf{l}, \mathbf{v}, \sigma, \mathbf{j}))$$

uniformly in $(s, \mathbf{l}, \mathbf{v}, \mathbf{x}) \in B$, and $\{T_{-t+s_k}^0 M_{k,\Delta}^0 \mathbf{x} : (s, \mathbf{l}, \mathbf{v}, \mathbf{x}) \in B\}$ is relatively compact in $J_{m+k}^-(t_-)$.

PROOF. For $v = 1, 2, \dots, k$ we write

$$\Delta_v = (s_v, \dots, \sigma_v), \quad s_v = (s_{1v}, \dots, s_{kv}) \quad \text{etc.}$$

and

$[M_{v,\Delta_v}^0 \mathbf{x}]_1$ = the element of Ω_{m+v-1}^0 obtained from $M_{v,\Delta_v}^0 \mathbf{x}$ by discarding the lastly added (i.e., the $(v+m)$ -th) particle.

$[M_{v,\Delta_v}^0 \mathbf{x}]_2$ is defined as above but by discarding the j_v -th particle instead of the added particle. Set for $\mathbf{x} \in A$

$$h^d(\mathbf{x}) = \min_{\substack{v=1, \dots, k \\ i=1, 2}} \inf_{s_v \leq s \leq t-t_-} \text{dist} \{T_{-s+s_v}^0 [M_{v,\Delta_v}^0 \mathbf{x}]_i, d_{m+v-1}\}$$

where $\text{dist} \{\mathbf{x}, d_{m+v-1}\}$ is the distance of \mathbf{x} from d_{m+v-1} if $\mathbf{x} \in \Omega_{m+v-1}$ and $\text{dist} \{\partial, d_{m+v-1}\} = \infty$. It suffices to prove that for any $\eta > 0$ there exists a constant $\alpha_0 > 0$ and a Borel set $B \subset \Gamma_k \times S^{2k} \times A$ such that

$$(II.1) \quad |\Gamma_k \times S^{2k} \setminus B(\mathbf{x})| < \eta \quad \text{for } \mathbf{x} \in A$$

and

$$(II.2) \quad h^A(\mathbf{x}) > \alpha_0 \quad \text{if } (s, \mathbf{l}, \mathbf{v}, \mathbf{x}) \in B.$$

We note that $h^A(\mathbf{x})$ is continuous in \mathbf{x} for a.a. $(s, \mathbf{l}, \mathbf{v})$. Set for $\alpha > 0$

$$B_\alpha = \{(\mathbf{x}, s, \mathbf{l}, \mathbf{v}) : \mathbf{x} \in A, h^A(\mathbf{x}) > \alpha\}$$

and

$$f_\alpha(\mathbf{x}) = |\Gamma_k \times S^{2k} \setminus B_\alpha(\mathbf{x})|.$$

Then on the one hand

$$(II.3) \quad f_\alpha(\mathbf{x}) \downarrow 0 \quad \text{as } \alpha \downarrow 0 \quad \text{for all } \mathbf{x} \in A$$

and on the other hand, by Fatou's lemma, $f_\alpha(\mathbf{x})$ is upper semi-continuous in \mathbf{x} for each α . Therefore the convergence in (II.3) is uniform. Now take $\alpha_0 > 0$ so that $f_{\alpha_0}(\mathbf{x}) < \eta$ for all $\mathbf{x} \in A$ and set $B = B_{\alpha_0}$. Then (II.1) and (II.2) are satisfied as desired. Q. E. D.

Appendix III. An example of $\{f_n\}$

Let A be a bounded measurable set of R^d ($d=2, 3, \dots$) with the symmetry $A = -A$, and define for $\varepsilon > 0$ and $n=1, 2, \dots$ a function

$$H_n^{(\varepsilon)}(\mathbf{q}), \quad \mathbf{q} = (q_1, \dots, q_n) \in R^d \times \dots \times R^d \quad (n\text{-fold product}),$$

by

$$H_n^{(\varepsilon)}(\mathbf{q}) = \begin{cases} 1 & \text{if } q_i - q_j \notin \varepsilon A \quad \text{for all pairs } 1 \leq i < j \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Let (V, \mathcal{F}, ν) be a measure space and $f=f(q, v)$ be a probability density on the product space $R^d \times V$. Set $\rho(q) = \int_V f(q, v) d\nu(v)$,

$$c_k^{(\varepsilon)} = \int_{R^{kd}} \rho^{k \otimes} H_k^{(\varepsilon)} d\mathbf{q}$$

and

$$f_n^{(\varepsilon)}(\mathbf{q}, \mathbf{v}) = f^{n \otimes}(\mathbf{q}, \mathbf{v}) H_n^{(\varepsilon)}(\mathbf{q}) / c_n^{(\varepsilon)}.$$

LEMMA. Let f be bounded. Then for each $m=1, 2, \dots$ there exists a constant A_m such that if $n\varepsilon^d |A| \|\rho\|_\infty \leq 1/2$, then

$$(III.1) \quad \|f_n^{(\varepsilon)} - f^{m \otimes} H_m^{(\varepsilon)}\|_\infty \leq A_m n \varepsilon^d.$$

For the Grad scaling $n\epsilon^{d-1} = \text{const}$ the right-hand side of (III.1) vanishes in the limit so that the assumption ii) on $\{f_n\}$ in Theorem 6.1 are satisfied; the other assumption i) in it is ready from (III.4) below and $H_n^{(\epsilon)} \leq H_m^{(\epsilon)} \otimes H_{n-m}^{(\epsilon)}$.

PROOF OF LEMMA. Fixing ϵ and n , we suppress the superscript ϵ from the notations. Set $g_m = f_m^{\otimes} H_m$. Let us divide the difference which is to be estimated into two parts:

$$(III.2) \quad f_{n|m} - g_m = (f_{n|m} - (c_{n-m}/c_n)g_m) - (1 - c_{n-m}/c_n)g_m.$$

Noticing the relation

$$H_n(\mathbf{q}, \mathbf{q}') = H_m(\mathbf{q}) \cdot H_{n-m}(\mathbf{q}') \cdot \prod_{k=m+1}^n H_{m+1}(\mathbf{q}, q_k)$$

where $\mathbf{q} = (q_1, \dots, q_m)$ and $\mathbf{q}' = (q_{m+1}, \dots, q_n)$, and applying the identity

$$1 - \prod_{k=1}^p a_k = (1 - a_1) + (1 - a_2)a_1 + (1 - a_3)a_2a_1 + \dots + (1 - a_p)\prod_{k=1}^{p-1} a_k,$$

we see that

$$(III.3) \quad |f_{n|m}(\mathbf{q}) - (c_{n-m}/c_n)g_m(\mathbf{q})| \\ = \frac{1}{c_n} g_m(\mathbf{q}) \int_{R^{(n-m)d}} [1 - \prod_{k=m+1}^n H_{m+1}(\mathbf{q}, q_k)] H_{n-m}(\mathbf{q}') \rho^{(n-m)\otimes}(\mathbf{q}') d\mathbf{q}' \\ \leq \frac{1}{c_n} g_m(\mathbf{q}) \int_{R^{(n-m)d}} \sum_{k=m+1}^n (1 - H_{m+1}(\mathbf{q}, q_k)) H_{n-m}(\mathbf{q}') \rho^{(n-m)\otimes}(\mathbf{q}') d\mathbf{q}' \\ = \frac{1}{c_n} g_m(\mathbf{q}) (n-m)c_{n-m-1} \int_{R_d} (1 - H_{m+1}(\mathbf{q}, q_{m+1})) \rho(q_{m+1}) dq_{m+1} \\ \leq (c_{n-m-1}/c_n)(n-m)m \|\rho\|_{\infty} |A| \epsilon^d g_m(\mathbf{q}).$$

Similarly we have $0 \leq c_{k+1} - c_k \leq k\epsilon^d |A| \|\rho\|_{\infty} c_k$, or, what is the same,

$$(III.4) \quad 1 \geq c_{k+1}/c_k \geq 1 - k\epsilon^d |A| \|\rho\|_{\infty}.$$

The required relation (III.1) is now deduced by applying the inequalities (III.3) and (III.4) to the right-hand side of (III.2). Q. E. D.

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