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Mod 3 homotopy associative H-spaces which are products of spheres

Dedicated to Professor Hirosi Toda on his 60th birthday

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§1. Introduction

For a prime p, a space X is called a mod p (*homotopy associative*) *H-space* if its localization $X_{(p)}$ at p is a (homotopy associative) H-space.

Consider the product space S of odd spheres:

$$(*) \qquad S = S^{n_1} \times \cdots \times S^{n_a} \times (S^3)^b \times (S^1)^c \quad (n_i\text{: odd integers} \geq 5).$$

Then S is a mod p H-space for any $p \geq 3$, and so is S for $p=2$ if and only if each n_i is 7, by Adams [1, 2]. Moreover S is a mod p homotopy associative H-space for any $p \geq 5$ by [2], and so is S for $p=2$ if and only if $a=0$ by Goncalves [4; Th. 1]. In case of $p=3$, the special unitary group $SU(3)$ is 3-equivalent to $S^5 \times S^3$ by Serre [9; Prop. 7]; hence we have the following typical example:

(1.1) $(S^5)^a \times (S^3)^b \times (S^1)^c$ *for* $a \leq b$ *is a* mod 3 *loop space.*

Now the main result of this paper is stated as follows:

THEOREM 1.2. *S in $(*)$ is a* mod 3 *homotopy associative H-space if and only if each n_i is 5 and $a \leq b$, i.e., S is a* mod 3 *loop space in* (1.1).

We sketch here the proof of the theorem, which is based on the methods of Zabrodsky [14], and is done by continuing to the preceding studies in [5, 6]. We assume that the localization $S_{(3)}$ of S in $(*)$ is a homotopy associative H-space. In the mod 3 Steenrod algebra, we have a decomposition

$$\mathscr{P}^n = \textstyle\sum_{j=0}^{t} \mathscr{P}^{3^j}\alpha_j \quad \text{when} \quad n_i = 2n+1,\ n = 3^t s,\ 3 \nmid s \text{ and } s \geq 2,$$

where $\mathscr{P}^m$ is the mod 3 reduced power operation. This decomposition associates an unstable secondary operation φ in the diagram

$$\begin{tikzcd}
& E_h \arrow[r, "\varphi"] \arrow[d, "r_h"] & K(Z/3, 6n-2) \\
S_{(3)} \arrow[ur, "\tilde{\xi}"] \arrow[r, "\xi"] & K(Z/3, 2n-1) \arrow[r, "h"] & \prod_{j=0}^{t} K(Z/3, 6n-1-4\cdot 3^j).
\end{tikzcd}$$

Here $h=\prod_{j=0}^{t}\alpha_j$, r_h is the homotopy fiber of h, ξ is an H-map corresponding to the factor $S^{n_i}_{(3)}$, $\tilde{\xi}$ is a lift of ξ, and $\varphi\tilde{\xi}$ is shown to be an H-map. Now, by [13; 2.5.1], we have the obstruction $\theta(\varphi\tilde{\xi})$ for $\varphi\tilde{\xi}$ to preserve the homotopies of homotopy associativity (i.e., to be an A_3-map); and we can lead a contradiction by calculating $\theta(\varphi\tilde{\xi})$ in two different ways. By this way, we have proved that

(1.3) [5; Th. A] *$n_i = 2\cdot 3^{e(i)} - 1$ $(e(i)\geq 1)$ for each i.*

On the other hand, by considering the projective 3-space of $S_{(3)}$ and by studying the Hubbuck operations S^q and Q^q on certain quotient algebra of its cohomology with coefficient in $\boldsymbol{Z}_{(3)}$, we have also proved that

(1.4) [6] *$a\leq b$ holds if each n_i is 5.*

Therefore we shall prove Theorem 1.2 by showing the following

(1.5) *If $e=e(i)=\max\{e(j)\}\geq 2$ in (1.3), then we have a contradiction.*

In this case, for $n=3^e$, we have the diagram

$$\begin{tikzcd}
 & E_h \arrow[r, "\varphi"] \arrow[d, "r_h"] & K(\boldsymbol{Z}/3,\,6n-2) \\
 & E_f \arrow[r, "h"] \arrow[d, "r_f"] & K(\boldsymbol{Z}/3,\,6n-2)\times\prod_{j=0}^{e-1}K(\boldsymbol{Z}/3,\,6n-1-4\cdot 3^j) \\
S_{(3)} \arrow[r, "\xi"] \arrow[uur, "\tilde{\xi}"] & K(\boldsymbol{Z}/3,\,2n-1) \arrow[r, "f"] & K(\boldsymbol{Z}/3,\,2n)\times\prod_{j=0}^{e-1}K(\boldsymbol{Z}/3,\,2n-1+4\cdot 3^j),
\end{tikzcd}$$

instead of the above one. Here $f=\beta\times\prod_{j=0}^{e-1}\mathscr{P}^{3^j}$ with the Bockstein operation β, and h is the secondary operation due to Shimada-Yamanoshita [10] or Liulevicius [7], which associates an unstable tertiary operation φ (see Proposition 2.4). Moreover $\tilde{\xi}$ is a suitable lift of ξ given in Proposition 3.4, which assures that $\varphi\tilde{\xi}$ is an H-map and $\theta(\varphi\tilde{\xi})$ is calculated in two ways to show (1.5). Now we prepare in §4 the ladder Toda bracket due to Zabrodsky [12], and prove Proposition 3.4 in §5.

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§2. Unstable tertiary operation

In this paper, we assume that spaces have base points $*$ which are non-degenerate, and that (continuous) maps preserve them, unless otherwise stated.

For any space X, we use the Moore path (or loop) spaces

$$PX = \{(w, r) \mid r \in [0, \infty) \text{ and } w\colon [0, \infty) \to X \text{ with } w(t) = w(r)\ (t \geq r)\},$$

$$LX = \{(w, r) \in PX \mid w(0) = *\}, \quad \Lambda X = \{(w, r) \in PX \mid w(0) = w(r)\}, \quad \text{and}$$

$$\Omega X = \{(w, r) \in PX \mid w(0) = w(r) = *\}. \quad (w \text{ in } PX \text{ is non-based.})$$

We define the maps $c\colon X \to PX$ and $e_t\colon PX \to X$ $(0 \leq t \leq \infty)$ by

$$c(x) = (\text{the constant map to } x, 0) \quad \text{and} \quad e_t(w, r) = w(\min\{t, r\}),$$

and take $* = c*$ as the non-degenerate base point for $\mathscr{L}X$ ($\mathscr{L} = P, L, \Lambda$ or Ω). Moreover we define

$$\mathscr{L}f\colon \mathscr{L}X \longrightarrow \mathscr{L}Y \quad \text{for a map} \quad f\colon X \longrightarrow Y \quad \text{by} \quad (\mathscr{L}f)(w, r) = (fw, r).$$

In PX, we define the path-multiplication $(w, r_1 + r_2) = (w_1, r_1) + (w_2, r_2)$ of $(w_i, r_i) \in PX$ with $e_\infty w_1 = e_0 w_2$ by $w(t) = w_1(t)$ for $t \leq r_1$, $= w_2(t - r_1)$ for $t \geq r_1$; and the inverse path $(w', r_1) = -(w_1, r_1)$ by $w'(t) = w_1(\max\{r_1 - t, 0\})$.

We define a *homotopy* to be a map $H\colon X \to PY$ (with $H* = *$) denoted by

$$H\colon X \longrightarrow PY; \quad f_0 \sim f_\infty, \qquad \text{for} \quad f_t = e_t H\colon X \longrightarrow Y \quad (t = 0, \infty);$$

and then we denote also by $f_0 \sim f_\infty\colon X \to Y$ or $f_0 x \sim f_\infty x$ $(x \in X)$. We note that this is the same as the usual homotopy preserving base points since they are non-degenerate. In case of

$$H\colon X \longrightarrow P^2 Y = P(PY) \quad \text{with} \quad (Pe_t)H = c(e_t^2 H) \quad (t = 0, \infty),$$

we call H a homotopy between homotopies $e_0 H$ and $e_\infty H$ *fixing the end points.*

For any spaces X and Y, we have the natural homotopy equivalence

$$\varepsilon\colon \mathscr{L}X \times \mathscr{L}Y \simeq \mathscr{L}(X \times Y) \quad (\mathscr{L} = P, L, \Lambda \text{ or } \Omega)$$

given by $\varepsilon((w, r), (v, s)) = ((w \times v)\Delta, \max\{r, s\})$ (Δ: the diagonal map).

Now we define H-spaces and the related notions (cf. [13; Ch. I–II]). An *H-space* is a pair (X, μ) of a space X and a map $\mu\colon X \times X \to X$ with $\mu \mid X \vee X = \nabla$ (the folding map). μ is called an *H-structure* or a *multiplication* for X. We also call X an H-space simply if μ is specified, and denote $\mu(x, y)$ by $x \cdot y$. If (X_i, μ_i) are H-spaces, then so are $(\mathscr{L}X_1, (\mathscr{L}\mu_1)\varepsilon)$ and $(X_1 \times X_2, (\mu_1 \times \mu_2)(1 \times T \times 1))$ (1: the identity map, T: the twisting map).

A *homotopy associative H-space*, or an *HA-space*, is a triple (X, μ, α) of an H-space (X, μ) and a homotopy $\alpha\colon X \times X \times X \to PX$; $\mu(\mu \times 1) \sim \mu(1 \times \mu)$ with $\alpha(*, x, y) = \alpha(x, *, y) = \alpha(x, y, *) = c\mu(x, y)$. α is called an *HA-structure* for X. We also call X or (X, μ) an HA-space simply if (μ, α) or α is specified. In particular, if $\mu(\mu \times 1) = \mu(1 \times \mu)$ and $\alpha = c\mu(\mu \times 1)$ hold, then (X, μ, α) (or X, (X, μ)) is called an *associative H-space*. If X_i are associative H-spaces, then so are $\mathscr{L}X_1$ and $X_1 \times X_2$.

An *H-map* between H-spaces (X_i, μ_i) $(i=1, 2)$ is a pair (f, F) of a map $f: X_1 \to X_2$ and a homotopy $F: X_1 \times X_1 \to PX_2$; $\mu_2(f\times f)\sim f\mu_1$ with $F | X_1 \vee X_1 = cf\nabla$. $F=F_f$ is called an *H-structure* for f. We call f an H-map if F_f is specified. For H-maps $(f_i, F_i): (X_i, \mu_i)\to(X_{i+1}, \mu_{i+1})$ $(i=1, 2)$, the *composition* $(f_2, F_2)\cdot(f_1, F_1)=(f_2f_1, F): (X_1, \mu_1)\to(X_3, \mu_3)$ is an H-map with the *composed H-structure* $F=F_2(f_1\times f_1)+(Pf_2)F_1: X_1\times X_1\to PX_3$.

An *HA-map* between HA-spaces (X_i, μ_i, α_i) $(i=1, 2)$ is a triple (f, F, A) of an H-map $(f, F): (X_1, \mu_1)\to(X_2, \mu_2)$ and a homotopy

$$A: X_1 \times X_1 \times X_1 \longrightarrow P^2X_2;\quad \alpha_2(f\times f\times f) \sim (Pf)\alpha_1$$

with $(Pe_0)A=(P\mu_2)(F\times cf)+F(\mu_1\times 1)$, $(Pe_\infty)A=(P\mu_2)(cf\times F)+F(1\times\mu_1)$ and $A(*, x, y)=A(x, *, y)=A(x, y, *)=(PF)(cx, cy)$. A is called an *HA-structure* for (f, F). In particular, if (X_i, μ_i, α_i) are associative H-spaces and $\mu_2(f\times f)=f\mu_1$, $F=cf\mu_1$, $\alpha_2(f\times f\times f)=(Pf)\alpha_1$ and $A=c(Pf)\alpha_1$ hold, then (f, F, A) (or f, (f, F)) is called a *homomorphism*.

Note that *the loop space ΩY of Y is an associative H-space by the path-multiplication, and Ωf: $\Omega Y_1\to\Omega Y_2$ of a map f: $Y_1\to Y_2$ is a homomorphism.*

Let (X_i, μ_i) $(i=1, 2)$ be H-spaces. Then for any map $f: X_1\to X_2$, we have

$$d(f): X_1 \wedge X_1 \longrightarrow X_2 \quad\text{with}\quad \mu_2(d(f)\,\mathrm{pr}\times f\mu_1)\Delta \sim \mu_2(f\times f)$$

(pr: $X\times\cdots\times X\to X\wedge\cdots\wedge X$ is the projection). $d(f)$ is called the *H-deviation* of f, because f is an H-map if and only if $d(f)\sim *$.

Moreover, let (X_i, μ_i, α_i) $(i=1, 2)$ be HA-spaces and $(f, F): (X_1, \mu_1)\to(X_2, \mu_2)$ be an H-map. Then we have the map $\tilde\theta$: $X_1\times X_1\times X_1\to\Lambda X_2$ defined by

$$\begin{aligned}\tilde\theta(x, y, z) &= \alpha_2(fx, fy, fz) + cfx\cdot F(y, z) + F(x, \mu_1(y, z)) - (Pf)\alpha_1(x, y, z)\\ &\quad - F(\mu_1(x, y), z) - F(x, y)\cdot cfz \quad (\cdot \text{ is induced from } \mu_2).\end{aligned}$$

Since $\tilde\theta(*, x, y)=\tilde\theta(x, *, y)=\tilde\theta(x, y, *)=F(x, y)-F(x, y)\sim *$, we get a map, which is unique up to homotopy,

$$\theta = \theta(f, F): X_1 \wedge X_1 \wedge X_1 \longrightarrow \Omega X_2 \quad \text{(due to Zabrodsky [13; 2.5])}$$

such that $\tilde\theta\sim\theta'$: $X_1\times X_1\times X_1\to\Lambda X_2$ fixing the end points, where $\theta'(x, y, z)=\theta(x, y, z)\cdot((fx\cdot fy)\cdot fz)$. We call $\theta=\theta(f, F)$ the *HA-deviation* of an H-map (f, F), because (f, F) has an HA-structure if and only if $\theta\sim *$ by definition. We denote $\theta(f, F)$ by $\theta(f)$ when F is specified.

We note that $\theta(f_0, F_0)\sim\theta(f_\infty, F_\infty)$ for two H-maps $(f_i, F_i): (X_1, \mu_1)\to(X_2, \mu_2)$, if they are homotopic as H-maps, i.e., if there are homotopies

$$H: X_1 \longrightarrow PX_2;\quad f_0 \sim f_\infty \quad\text{and}\quad G: X_1 \times X_1 \longrightarrow P^2X_2;\quad F_0 \sim F_\infty$$

with $(Pe_0)G=(P\mu_2)(H\times H)$, $(Pe_\infty)G=H\mu_1$, and $G\mid X_1\vee X_1=cH\nabla$. Moreover, we note that $\theta(\Omega g)\sim *$ for any map $g\colon Y_1\to Y_2$.

Now, for a given map $h\colon X\to Y$, let

$$\Omega Y \xrightarrow{j_h} E_h = \{(x, l)\in X\times LY \mid hx=e_\infty l\} \xrightarrow{r_h} X \xrightarrow{h} Y$$

denote the fiber sequence given by $r_h(x, l)=x$ and $j_h(l)=(*, l)$, i.e., r_h is the homotopy fiber of h and j_h is the fiber of r_h. Then for the fiber sequence

$$\Omega^2 Y \xrightarrow{j_{\Omega h}} E_{\Omega h} = \{(x', l')\in \Omega X\times L\Omega Y \mid (\Omega h)x'=e_\infty l'\} \xrightarrow{r_{\Omega h}} \Omega X \xrightarrow{\Omega h} \Omega Y$$

we see that $E_{\Omega h}$ *is an associative H-space with the multiplication induced from the ones of* ΩX *and* ΩY, *and* $j_{\Omega h}$ *and* $r_{\Omega h}$ *are homomorphisms.* Also we note that there is a natural homotopy equivalence $\varepsilon\colon E_{\Omega h}\simeq \Omega E_h$ with $(\Omega r_h)\varepsilon\sim r_{\Omega h}$.

Moreover, let $\iota_t\in H^t(K(\boldsymbol{Z}/3, t); \boldsymbol{Z}/3)$ be the fundamental class, and let $\sigma\colon H^t(X; \boldsymbol{Z}/3)\to H^{t-1}(\Omega X; \boldsymbol{Z}/3)$ be the cohomology suspension. Then:

PROPOSITION 2.1. *For given* $a\in H^{2n}(X; \boldsymbol{Z}/3)$ *and maps*

$$X \xrightarrow{h} Y \xrightarrow{g} K(\boldsymbol{Z}/3, 6n) \quad \textit{with} \quad (gh)^*\iota_{6n} = a^3,$$

there is an H-map $\varphi\colon E_{\Omega h}\to \Omega^2 K(\boldsymbol{Z}/3, 6n)$ *with* $\varphi j_{\Omega h}=\Omega^2 g$ *and*

$$\theta(\varphi)^*\iota_{6n-3} = \pm b\otimes b\otimes b \qquad \textit{for} \quad b = r_{\Omega h}^*\sigma a\in H^{2n-1}(E_{\Omega h}; \boldsymbol{Z}/3),$$

where $\theta(\varphi)\colon E_{\Omega h}\wedge E_{\Omega h}\wedge E_{\Omega h}\to K(\boldsymbol{Z}/3, 6n-3)$ *is the HA-deviation of* φ.

PROOF. Consider $K=K(\boldsymbol{Z}/3, 2n)$, $K'=K(\boldsymbol{Z}/3, 6n)$ and the maps

$$X \xrightarrow{f} K \xrightarrow{k} K' \quad \text{with } f^*\iota_{2n} = a \text{ and } k^*\iota_{6n} = \iota_{2n}^3 = \mathcal{P}^n\iota_{2n}.$$

Then $(gh)^*\iota_{6n}=(kf)^*\iota_{6n}$; hence we can take k and f to satisfy $kf=gh$. Thus we have the commutative diagram

$$(*)\qquad \begin{array}{ccccccc}
\Omega^2 Y & \xrightarrow{j_{\Omega h}} & E_{\Omega h} & \xrightarrow{r_{\Omega h}} & \Omega X & \xrightarrow{\Omega h} & \Omega Y \\
\downarrow{\scriptstyle \Omega^2 g} & & \downarrow{\scriptstyle \tilde f} & & \downarrow{\scriptstyle \Omega f} & & \downarrow{\scriptstyle \Omega g} \\
\Omega^2 K' & \xrightarrow{j} & E & \xrightarrow{r} & \Omega K & \xrightarrow{\Omega k} & \Omega K'
\end{array}$$

($E=E_{\Omega k}$, $r=r_{\Omega k}$, $j=j_{\Omega k}$, $\tilde f=\Omega f\times L\Omega g\mid E_{\Omega h}$) of the fiber sequences, consisting of the associative H-spaces and the homomorphisms. Moreover we have a homotopy

$$\eta\colon \Omega K \longrightarrow L\Omega K';\ * \sim \Omega k \quad (\Omega K=K(\boldsymbol{Z}/3, 2n-1),\ \Omega K' = K(\boldsymbol{Z}/3, 6n-1)),$$

since $(\Omega k)^*\iota_{6n-1}=\mathscr{P}^n\iota_{2n-1}=0$. This defines $w\colon \Omega K\times\Omega K\to\Omega^2K'$ by $w(x, y)=(\eta x)\cdot(\eta y)-\eta(x\cdot y)$, and $w'\colon \Omega K\wedge\Omega K\to\Omega^2K'$ with $w'\,\mathrm{pr}\sim w$ since $w\,|\,K\vee K\sim *$. Now as is shown in the proof of Zabrodsky [14; 1.3], we can take η so that $w'\sim *$, i.e., there is a homotopy

$$(\eta x)\cdot(\eta y)\sim\eta(x\cdot y)\ (x,\ y\in\Omega K)\quad\text{fixing the end points.}$$

Using these $\tilde{f}$ and η, we define

$$\varphi=\psi\tilde{f}\colon E_{\Omega h}\longrightarrow E\longrightarrow\Omega^2K'=K(Z/3,\ 6n-2)$$

by $\psi(x,\ l)=l-\eta x$ for $(x,\ l)\in E\subset\Omega K\times L\Omega K'$. Then

$$\psi j=1\quad\text{and}\quad\varphi j_{\Omega h}=\psi\tilde{f}j_{\Omega h}=\psi j\Omega^2g=\Omega^2g;$$

and $\psi\colon E\to\Omega^2K'$ is an H-map by the H-structure $F\colon E\times E\to P\Omega^2K'$, where $F((x,\ l),\ (y,\ m))$ is given by

$$(l-\eta x)\cdot(m-\eta y)\sim l\cdot m-(\eta x)\cdot(\eta y)\sim l\cdot m-\eta(x\cdot y).$$

Hence $\varphi=\psi\tilde{f}$ is an H-map, and

$$(**)\qquad \theta(\varphi)\sim\theta(\psi)(\tilde{f}\wedge\tilde{f}\wedge\tilde{f})\quad(\text{by [13; 2.5.2]}).$$

Therefore the equality $\theta(\varphi)^*\iota_{6n-3}=\pm b\otimes b\otimes b$ follows from

$$(2.2)\qquad \theta(\psi)\not\sim *\colon E\wedge E\wedge E\longrightarrow\Omega^3K'=K(Z/3,\ 6n-3).$$

In fact, by the lower fibration in $(*)$, we see that $H^{6n-3}(E\wedge E\wedge E;\ Z/3)\cong Z/3$ with a generator $c\otimes c\otimes c$ for $c=r^*\iota_{2n-1}$. Thus $\theta(\psi)^*\iota_{6n-3}=\pm c\otimes c\otimes c$ by (2.2), which implies the equality by $(**)$ and $\tilde{f}^*c=r^*_{\Omega h}(\Omega f)^*\iota_{2n-1}=b$.

To prove (2.2), suppose contrarily that $\theta(\psi)\sim *$. Then the H-map $\psi\colon E\to\Omega^2K'$ has an HA-structure, or is an A_3-map in the sence of Stasheff [11; II, Def. 4.4]. Thus, by [11; II], we have a map

$$\psi_3\colon P_3E\longrightarrow B\Omega^2K'=\Omega B\Omega K'\simeq\Omega K'\quad\text{with}\quad\psi\sim\bar{\psi}_3\colon E\longrightarrow\Omega\tilde{\Omega}K'\subset\Omega^2K'$$

for the projective t-space P_tE $(t\geqq 2)$ of the associative H-space $E=E_{\Omega k}$, where $\bar{\psi}_3$ is the adjoint of $\psi_3\varepsilon_3\colon \Sigma E\subset P_3E\to\Omega K'$ for the usual loop space $\tilde{\Omega}K'$ (which is homotopy equivalent to $\Omega K'$ by $\tilde{\Omega}K'\subset\Omega K'$).

Now ψ_3 cna be extended to $\psi_t\colon P_tE\to\Omega K'$ for all t. In fact, the obstruction for ψ_t to be extended to ψ_{t+1} is in $H^{6n-1-t}(X_t;\ Z/3)$ for $X_t=E\wedge\cdots\wedge E$ $(t+1$ copies) by [11; II, 8], which is 0 for $t\geq 3$ since E is $(2n-2)$-connected. Therefore we have a map

$$\psi_\infty=B\psi\colon P_\infty E=BE\longrightarrow\Omega K'\quad\text{with}\quad\psi\sim\bar{\psi}_\infty\sim\Omega\psi_\infty.$$

Since $\psi j=1$, this shows that $\psi_\infty Bj\sim 1$ for the fiber sequence

$$\Omega K' \xrightarrow{Bj} BE \xrightarrow{Br} B\Omega K \simeq K \xrightarrow{k} K' \simeq B\Omega K'$$

(up to homotopy equivalences) obtained from the lower one in (*). Thus $(Br\times \psi_\infty)\Delta\colon BE\simeq K\times\Omega K'$, and we have a section $s\colon K\to BE$ with $(Br)s\sim 1$. So $k\sim k(Br)s\sim *$, which contradicts $k^*\iota_{6n}=\iota_{2n}^3\neq 0$. Hence (2.2) is proved. q.e.d.

In the rest of this section, we construct a particular tertiary operation.

Let $e\geq 1$ be a fixed integer, and consider the maps in the diagram

$$(2.3)\qquad \begin{array}{l} E_f \xrightarrow{h} K_1 = \prod_{i=-1}^{e-2} K(Z/3, m_i) \xrightarrow{g} K' = K(Z/3, 6n) \\ \big\downarrow r_f \\ K=K(Z/3, 2n) \xrightarrow{f} K_0 = \prod_{i=-1}^{e-1} K(Z/3, l_i) \end{array}$$

for $n=3^e$, $l_{-1}=2n+1$, $l_i=2n+4\cdot 3^i$ $(i\geq 0)$ and $m_i=8n-l_i$, such that

$$f^*\iota_{l_i} = \mathscr{P}^{(i)}\iota_{2n},\quad g^*\iota_{6n} = \sum_{i=-1}^{e-1}\mathscr{P}^{(i)}\iota_{m_i}\quad (\mathscr{P}^{(-1)}=\beta,\ \mathscr{P}^{(i)}=\mathscr{P}^{3^i}\ (i\geq 0))$$

and $h^*\iota_{m_i}=v_i$ for some classes $v_i\in H^{m_i}(E_f;\ Z/3)$ with

$$r_f^*\iota_{2n}^3 = \mathscr{P}^n r_f^*\iota_{2n} = \sum_{i=-1}^{e-1} \mathscr{P}^{(i)}v_i\,.$$

We note that the equalities for f^* and the definition of r_f imply

$$a = r_f^*\iota_{2n}\neq 0,\quad \beta a = 0 \quad\text{and}\quad \mathscr{P}^t a = 0 \text{ for } t<n,$$

which assure the existence of such v_i by Shimada-Yamanoshita [10; Th. 5.1–2] or Liulevicius [7; Th. 4.5.1]; hence h exists. Then $(gh)^*\iota_{6n}=a^3$, and Proposition 2.1 implies the following

Proposition 2.4. (i) *$\pi_t(E_{\Omega h})=0$ for $t\geq 6n-2$.*

(ii) *There is an H-map $\varphi\colon E_{\Omega h}\to\Omega^2K'$ with $\varphi j_{\Omega h}=\Omega^2 g$ and*

$$\theta(\varphi)^*\iota_{6n-3} = \pm\, u\otimes u\otimes u \quad \textit{for}\quad u = r_{\Omega h}^*(\Omega r_f)^*\iota_{2n-1}\in H^{2n-1}(E_{\Omega h};\ Z/3).$$

§ 3. Reduction of (1.5)

Note that if a connected space X is an HA-space, then so is its universal covering space, which has the homotopy type of Y when $X=Y\times(S^1)^c$ for a simply connected space Y. Then (1.5) follows from the following

Proposition 3.1. *For the localized sphere $S^n_{(3)}$ at 3, and integers $n_i=3$ $(i>a\,(\geq 1))$ and $n_i=2\cdot 3^{e(i)}-1$ $(i\leq a)$ with $e(1)\geq e(2)\geq\cdots\geq e(a)\geq 1$, assume that $S=\prod_{i=0}^{a+b} S^{n_i}_{(3)}$ is an HA-space. Then $e(1)=1$.*

Hereafter, we study S under these assumptions.

LEMMA 3.2. (i) *$H^*(S; \boldsymbol{Z}_{(3)}) \cong \Lambda(\bar{\xi}_1, \cdots, \bar{\xi}_{a+b})$ and $H^*(S; \boldsymbol{Z}/3) \cong \Lambda(\xi_1, \cdots, \xi_{a+b})$ by primitive elements $\bar{\xi}_i$ and ξ_i such that* $\dim \bar{\xi}_i = \dim \xi_i = n_i$ *and ξ_i is the* mod 3 *reduction of $\bar{\xi}_i$.*

(ii) *Moreover, $\bar{\xi}_i$ can be chosen to be a generator of $H^{n_i}(S^{n_i}_{(3)}; \boldsymbol{Z}_{(3)})$ for any i.*

PROOF. (i) is seen in the same way as Borel [3; Th. 4.1–2, Prop. 4.3].

(ii) If $x \in H^t(S; \boldsymbol{Z}_{(3)})$ (t: odd) is a monomial of generators $\zeta_i \in H^{n_i}(S^{n_i}_{(3)}; \boldsymbol{Z}_{(3)})$, i.e., $x = c\zeta_{i(1)} \cdots \zeta_{i(l)}$ $(1 \le i(1) < \cdots < i(l) \le a+b,\ c \in \boldsymbol{Z}_{(3)})$, then

$$\psi_x\colon S \xrightarrow{\mathrm{pr}} \textstyle\prod_{j=1}^{l} S^{n_{i(j)}}_{(3)} \xrightarrow{\mathrm{pr}} \wedge_{j=1}^{l} S^{n_{i(j)}}_{(3)} = S^t_{(3)} \xrightarrow{c} S^t_{(3)}$$

(c is the map of degree c) satisfies $\psi_x^* \zeta = x$ for a generator $\zeta \in H^t(S^t_{(3)}; \boldsymbol{Z}_{(3)})$. If $x = x_1 + \cdots + x_m \in H^t(S; \boldsymbol{Z}_{(3)})$ with monomials x_j of ζ_i, then

$$\psi_x = \mu_m\left(\textstyle\prod_{j=1}^{m} \psi_{x_j}\right)\Delta\colon S \longrightarrow (S)^m \longrightarrow (S^t_{(3)})^m \longrightarrow S^t_{(3)}$$

satisfies $\psi_x^* \zeta = x$, where $\mu_m = \mu(\mu_{m-1} \times 1)$ is the iterated multiplication of $\mu = \mu_2$ of the H-space $S^t_{(3)}$. Thus we see (ii) by taking $x = \bar{\xi}_i$. q. e. d.

Let $\rho'(t)\colon S'(t) \to \prod_{i=2}^{a+b} S^{n_i}_{(3)}$ be the t-connected fibration (i.e., $S'(t)$ is t-connected and $\rho'(t)$ is a fibration inducing an isomorphism on π_n for $n > t$), and put

$$\rho(t) = 1 \times \rho'(t)\colon S(t) = S^{n_1}_{(3)} \times S'(t) \longrightarrow S = \textstyle\prod_{i=1}^{a+b} S^{n_i}_{(3)}.$$

LEMMA 3.3. *If $t \le 2n_1 - 1$, then $S(t)$ is an HA-space and $\rho(t)$ is an HA-map.*

PROOF. If $t < n_1$, then $\rho(t)$ is the t-connected fibration by definition. Thus the HA-structure for S can be lifted to that for $S(t)$, and the lemma holds.

Suppose inductively that the lemma holds for t with $n_1 - 1 \le t < 2n_1 - 1$. Let $\psi'\colon S'(t) \to K(\pi_{t+1}(S'(t)), t+1)$ be the map inducing an isomorphism on π_{t+1}. Then by the definition of $\rho'(t)$'s, the homotopy fiber of ψ' is $\rho'\colon S'(t+1) \to S'(t)$ with $\rho'(t)\rho' = \rho'(t+1)$. Thus $\rho(t+1) = \rho(t)(1 \times \rho')$, and $1 \times \rho'\colon S(t+1) \to S(t)$ is the homotopy fiber of

$$\psi = \psi' \,\mathrm{pr}_2\colon S(t) \longrightarrow S'(t) \longrightarrow K = K(\pi_{t+1}(S'(t)), t+1).$$

Therefore, if ψ is an HA-map, then the lemma holds for $t+1$ by [13; 2.5.3].

Now $d(\psi) \sim *\colon S(t) \wedge S(t) \to K$ for the H-deviation $d(\psi)$ since $n_1 - 1 \le t < 2n_1 - 1$ and $S(t) \wedge S(t)$ is $(2n_1 - 1)$-connected. Hence ψ is an H-map. Moreover $\theta(\psi) \sim *\colon S(t) \wedge S(t) \wedge S(t) \to \Omega K$ for the HA-deviation $\theta(\psi)$ since $S(t) \wedge S(t) \wedge S(t)$ is $(3n_1 - 1)$-connected. Thus ψ is an HA-map; and the lemma is proved by induction. q. e. d.

Now Proposition 3.1 follows from the following

PROPOSITION 3.4. *For S in Proposition 3.1, consider*

$$\xi_1 \in H^{2n-1}(S;\ Z/3) \quad (n=3^e,\ e=e(1),\ n_1=2n-1)$$

and the HA-map $\rho=\rho(4n-3)\colon \tilde{S}=S(4n-3)\to S$ given in Lemmas 3.2–3. Furthermore, consider $u\in H^{2n-1}(E_{\Omega h};\ Z/3)$ and the H-map

$$\varphi\colon E_{\Omega h} \longrightarrow \Omega^2 K' = K(Z/3,\ 6n-2) \quad \text{with} \quad \theta(\varphi)^*\iota_{6n-3} = \pm\, u\otimes u\otimes u$$

given in Proposition 2.4 (ii). If $e\geq 2$, then there are a space X and maps

$$S\wedge S \xrightarrow{\lambda_1} X \xrightarrow{\lambda_2} E_{\Omega h} \quad \text{and} \quad s\colon S \longrightarrow E_{\Omega h}$$

such that $\lambda_1(\rho\wedge\rho)\sim *$, $\varphi\lambda_2\sim *$, $d(s)\sim\lambda_2\lambda_1$ and $s^*u=\xi_1$.

COROLLARY 3.5. *In Proposition 3.4, the compositions*

$$\tilde{\varphi} = \varphi s\colon S \longrightarrow \Omega^2 K' \quad \text{and} \quad \tilde{\rho} = s\rho\colon \tilde{S} \longrightarrow E_{\Omega h}$$

are H-maps so that the composed H-maps $\tilde{\varphi}\rho$, $\varphi\tilde{\rho}\colon \tilde{S}\to\Omega^2 K'$ are mutually homotopic as H-maps (hence $\theta(\tilde{\varphi}\rho)\sim\theta(\varphi\tilde{\rho})$ as is noted in §2).

PROOF OF PROPOSITION 3.1 FROM PROPOSITION 3.4 AND COROLLARY 3.5. Suppose $e=e(1)\geq 2$. Then, by these results, the HA-deviation

$$\theta(\tilde{\varphi})\colon S\wedge S\wedge S \longrightarrow \Omega^3 K' = K(Z/3,\ 6n-3)$$

of the H-map $\tilde{\varphi}$ is calculated as follows:

$\theta(\rho)\sim *$ since ρ is an HA-map; and $\theta(\tilde{\rho})\sim *\colon \tilde{S}\wedge\tilde{S}\wedge\tilde{S}\to\Omega E_{\Omega h}$ by Proposition 2.4 (i) since $\tilde{S}$ is $(2n-2)$-connected. Thus

$$\theta(\tilde{\varphi})(\rho\wedge\rho\wedge\rho) \sim \theta(\tilde{\varphi}\rho) \sim \theta(\varphi\tilde{\rho}) \sim \theta(\varphi)(\tilde{\rho}\wedge\tilde{\rho}\wedge\tilde{\rho})$$

by [13; 2.5.2]. Hence it follows from Proposition 3.4 that

$$\begin{aligned}\theta(\tilde{\varphi})^*\iota &\equiv (s^*\otimes s^*\otimes s^*)\theta(\varphi)^*\iota \quad \text{mod } \mathrm{Ker}(\rho^*\otimes\rho^*\otimes\rho^*) \quad (\iota=\iota_{6n-3})\\ &= \pm\, s^*u\otimes s^*u\otimes s^*u = \pm\,\xi_1\otimes\xi_1\otimes\xi_1.\end{aligned}$$

Also by Lemma 3.2 (ii) and the definition of ρ, there is a homology class $t\in H_{2n-1}(S;\ Z/3)$ with $\langle t,\ \xi_1\rangle=1$ and $\langle t,\ \mathrm{Ker}\,\rho^*\rangle=0$; hence

(3.6) $$\langle t\otimes t\otimes t,\ \theta(\tilde{\varphi})^*\iota\rangle = \pm\,\langle t,\ \xi_1\rangle^3 = \pm\, 1.$$

On the other hand, $\tilde{\varphi}^*\iota_{6n-2}\in H^{6n-2}(S;\ Z/3)$ is primitive since $\tilde{\varphi}$ is an H-map; and $H^*(S;\ Z/3)$ has no even dimensional primitive classes by Lemma 3.2 (i).

Hence $\tilde{\varphi}^*\iota_{6n-2}=0$ and $\tilde{\varphi}\sim *$. This implies by Zabrodsky [14; 1.2.1] that

$$\theta(\tilde{\varphi})^*\iota = (1\otimes\bar{\mu}-\bar{\mu}\otimes 1)z \quad \text{for some} \quad z\in H^*(S\wedge S;\ \boldsymbol{Z}/3),$$

where $\bar{\mu}\alpha=\mu^*\alpha-1\otimes\alpha-\alpha\otimes 1$ for the multiplication of μ of S. Thus

$$(3.7) \qquad \begin{aligned}\langle t\otimes t\otimes t,\ \theta(\tilde{\varphi})^*\iota\rangle &= \langle t\otimes t\otimes t,\ (1\otimes\bar{\mu}-\bar{\mu}\otimes 1)z\rangle \\ &= \langle t\otimes t\otimes t,\ (1\otimes\mu^*-\mu^*\otimes 1)z\rangle = \langle t\otimes t^2-t^2\otimes t,\ z\rangle.\end{aligned}$$

Here $t^2=tt$ is the Pontrjagin product in $H_*(S;\ \boldsymbol{Z}/3)$ given by μ, which is commutative by Milnor-Moore [8; 4.20] since $H^*(S;\ \boldsymbol{Z}/3)$ is primitively generated by Lemma 3.2 (i). Therefore $t^2=0$ since $\dim t$ is odd; and the last in (3.7) is 0, which contradicts (3.6). q.e.d.

Proof of Corollary 3.5 from Proposition 3.4. Let

$$\nu_1: \tilde{S}\wedge\tilde{S} \longrightarrow LX;\ *\sim\lambda_1(\rho\wedge\rho), \quad \nu_2: X\longrightarrow L\Omega^2K';\ *\sim\varphi\lambda_2 \quad \text{and}$$
$$\omega: S\wedge S\longrightarrow PE_{\Omega h};\ d(s)\sim\lambda_2\lambda_1$$

be homotopies given by Proposition 3.4. Then $\tilde{\varphi}$ is an H-map with the H-structure $F_{\tilde{\varphi}}: S\times S\to P\Omega^2K'$ given by

$$\begin{aligned}F_{\tilde{\varphi}}(x,\ y) = F_{\varphi}(sx,\ sy) &+ (P\varphi)(\zeta(sx,\ sy)+\omega(x,\ y)\cdot cs(x\cdot y)) \\ &- F_{\varphi}(\lambda_2\lambda_1(x,\ y)\cdot s(x\cdot y)) - \nu_2\lambda_1(x,\ y)\cdot c\varphi s(x\cdot y),\end{aligned}$$

and so is $\tilde{\rho}$ with $F_{\tilde{\rho}}: \tilde{S}\times\tilde{S}\to PE_{\Omega h}$ given by

$$F_{\tilde{\rho}}(\tilde{x},\ \tilde{y}) = \zeta(\rho\tilde{x},\ \rho\tilde{y}) + (\omega(\rho\tilde{x},\ \rho\tilde{y})-(L\lambda_2)\nu_1(\tilde{x},\ \tilde{y}))\cdot(Ps)F_{\rho}(\tilde{x},\ \tilde{y}),$$

where $F_{\varphi}: E_{\Omega h}\times E_{\Omega h}\to P\Omega^2K'$ and $F_{\rho}: \tilde{S}\times\tilde{S}\to PS$ are those of φ and ρ, and $\zeta: S\times S\to PE_{\Omega h}$ is a homotopy $sx\cdot sy\sim d(s)(x,\ y)\cdot s(x\cdot y)$.

Now the homotopy $(L\nu_2)\nu_1: \tilde{S}\wedge\tilde{S}\to L^2\Omega^2K'$ gives us a homotopy

$$\nu_2\lambda_1(\rho\wedge\rho)\sim(L(\varphi\lambda_2))\nu_1 \quad \text{fixing the end points.}$$

Also the one $\tilde{S}\times\tilde{S}\to P^2S$, defined by $(\tilde{x},\ \tilde{y})\to(PF_{\varphi})((L\lambda_2)\nu_1(\tilde{x},\ \tilde{y}),\ cs(\rho\tilde{x}\cdot\rho\tilde{y}))$, gives us a homotopy

$$\begin{aligned}&-F_{\varphi}(\lambda_2\lambda_1(\rho\tilde{x},\ \rho\tilde{y}),\ s(\rho\tilde{x}\cdot\rho\tilde{y}))-(L(\varphi\lambda_2))\nu_1(\tilde{x},\ \tilde{y})\cdot c\varphi s(\rho\tilde{x}\cdot\rho\tilde{y}) \\ &\quad\sim -(P\varphi)((L\lambda_2)\nu_1(\tilde{x},\ \tilde{y})\cdot cs(\rho\tilde{x}\cdot\rho\tilde{y})) \quad \text{fixing the end points.}\end{aligned}$$

By these homotopies, we can define the homotopy $F_{\tilde{\varphi}}(\rho\times\rho)+(P\tilde{\varphi})F_{\rho}\sim F_{\varphi}(\tilde{\rho}\times\tilde{\rho})+(P\varphi)F_{\tilde{\rho}}: \tilde{S}\times\tilde{S}\to P\Omega^2K'$ between the composed H-structures of $\tilde{\varphi}\rho$ and $\varphi\tilde{\rho}$, so that this and the stationary homotopy $H=c\tilde{\varphi}\rho: \tilde{S}\to P\Omega^2K'$; $\tilde{\varphi}\rho=\varphi\tilde{\rho}$ show $\tilde{\varphi}\rho\sim\varphi\tilde{\rho}$ as H-maps. q.e.d.

Therefore we have proved that Proposition 3.4 implies Proposition 3.1, which implies (1.5) and Theorem 1.2.

§ 4. Ladder Toda Bracket

In this section we discuss a simple case of the ladder Toda bracket introduced by Zabrodsky [12].

Consider the following diagram of spaces and maps:

$$\begin{tikzcd} & & Y \arrow[r, "g"] & Y_0 \arrow[r, "g_0"] \arrow[d, "h_0"] & Y_1 \arrow[d, "h_1"] & \\ \Omega X_2 & E \arrow[l, "\psi"'] \arrow[rr, "r"] & & X_0 \arrow[r, "f_0"] & X_1 \arrow[r, "f_1"] & X_2. \end{tikzcd}$$

Here f_0, f_1, g, g_0, h_0 and h_1 are given maps with

$$f_1f_0 \sim *, \quad g_0g \sim *, \quad h_1g_0 \sim f_0h_0 \quad \text{and} \quad f_1h_1 \sim *,$$

r is the homotopy fiber of f_0, i.e., $E=E_{f_0}=\{(x, l)\in X_0\times LX_1 \,|\, f_0x=e_\infty l\}$ and $r(x, l)=x$, and ψ is the map defined by

$$\psi(x, l) = (Lf_1)l - vx \quad \text{by a fixing homotopy } v\colon X_0 \longrightarrow LX_2;\ * \sim f_1f_0.$$

Then we prove the following

PROPOSITION 4.1. *There are maps $h\colon Y\to E$ and $h'\colon Y_0\to \Omega X_2$ with*

$$rh = h_0g \quad \textit{and} \quad h'g \sim \psi h.$$

PROOF. By using homotopies $\eta\colon Y\to LY_1;\ *\sim g_0g$, $\omega\colon Y_0\to PX_1;\ h_1g_0\sim f_0h_0$ and $\zeta\colon Y_1\to LX_2;\ *\sim f_1h_1$, we define h and h' by

$$h = \{h_0g\times((Lh_1)\eta+\omega g)\}\Delta \quad \text{and} \quad h' = \zeta g_0 + (Pf_1)\omega - vh_0.$$

Then $rh=h_0g$. Moreover $L(f_1h_1)\eta\sim\zeta g_0g\colon Y\to LX_2$ fixing the end points by $(L\zeta)\eta\colon Y\to L^2X_2$. Therefore

$$h'g = \zeta g_0g + (Pf_1)\omega g - vh_0g \sim L(f_1h_1)\eta + (Pf_1)\omega g - vh_0g = \psi h. \quad \text{q.e.d.}$$

§ 5. Proof of Proposition 3.4

By the consturction given in §2, we have the diagram

$$
\begin{array}{ll}
& \Omega^2K_1 \xrightarrow{i_1=j_{\Omega h}} E_2=E_{\Omega h} \xrightarrow{\varphi} \Omega^2K'=K(\boldsymbol{Z}/3,\,6n-2) \\
& \qquad\qquad\qquad \big\downarrow{\scriptstyle r_2=r_{\Omega h}} \\
(5.1) & \Omega^2K_0 \xrightarrow{j_0=\Omega j_f} E_1=\Omega E_f \xrightarrow{\Omega h} \Omega K_1=\prod_{i=-1}^{e-1} K(\boldsymbol{Z}/3,\,m_i-1) \xrightarrow{\Omega g} \Omega K' \\
& \qquad\qquad\qquad \big\downarrow{\scriptstyle r_1=\Omega r_f} \\
& \qquad\quad S \xrightarrow{s_0} \Omega K=K(\boldsymbol{Z}/3,\,2n-1) \xrightarrow{\Omega f} \Omega K_0=\prod_{i=-1}^{e-1} K(\boldsymbol{Z}/3,\,l_i-1)
\end{array}
$$

for f, g, h and φ in (2.3) and Proposition 2.4 (ii) and s_0 with $s_0^*\iota_{2n-1}=\xi_1$ in Lemma 3.2.

Hereafter assume that $n=3^e$ and $e=e(1)\geq 2$. Then:

LEMMA 5.2. *s_0 is an H-map, and there are maps*

$$s_1\colon S \longrightarrow E_1 \quad \textit{and} \quad d_0\colon S \wedge S \longrightarrow \Omega^2K_0 = \prod_{i=-1}^{e-1} K(\boldsymbol{Z}/3,\,l_i-2)$$

such that $r_1s_1\sim s_0$, $d(s_1)\sim j_0d_0$, $d_0^*\iota_t=0$ for $t=l_i-2$ $(i\neq 0)$, and

$$d_0^*\iota_{2n+2}\in PH^{2n-1}(S;\,\boldsymbol{Z}/3)\otimes PH^3(S;\,\boldsymbol{Z}/3) \quad (l_1-2=2n+2),$$

where PH^* denotes the primitive module of H^*.

PROOF. s_0 is an H-map since ξ_1 is primitive; and we fix an H-structure F: $S\times S\to P\Omega K$ for s_0.

The mod 3 Steenrod algebra $\mathscr{A}$ acts on $H^*(S;\,\boldsymbol{Z}/3)$ trivially. Hence $s_0^*(\Omega f)^*\iota_t=0$ for all t by the definition of f in (2.3). Thus $(\Omega f)s_0\sim *$. By choosing a homotopy v: $S\to L\Omega K_0$; $*\sim(\Omega f)s_0$, we define

$$d\colon S \wedge S \longrightarrow \Omega^2K_0 \quad \text{by} \quad d(x,\,y)\sim vx\cdot vy + (P\Omega f)F(x,\,y) - v(x\cdot y).$$

Then by [14; 1.2.1] and [13; 2.5.2], we see that

$$(1\otimes\bar{\mu}-\bar{\mu}\otimes 1)d^*\iota_t = \theta((\Omega f)s_0)^*\iota_t = \theta(s_0)^*(\Omega^2 f)^*\iota_t = 0$$

for any t ($\bar{\mu}$ is the one in (3.7)). Thus $d^*\iota_t$ represents some element in $\Gamma=\mathrm{Ext}^{2,*}_{H_*}(\boldsymbol{Z}/3,\,\boldsymbol{Z}/3)$ for $H_*=H_*(S;\,\boldsymbol{Z}/3)$. Here Γ is isomorphic to $\oplus\{\boldsymbol{Z}/3$ generated by $\xi_i\otimes\xi_j|1\leq i\leq j\leq a+b\}$, since $H^*=H^*(S;\,\boldsymbol{Z}/3)$ is given in Lemma 3.2 (ii). Therefore, by dimensional reason, $d^*\iota_t=0$ in Γ for $t\neq 2n+2$, and $d^*\iota_{2n+2}$ in Γ is represented by a class in $PH^{2n-1}\otimes PH^3$. Thus there are $a_t\in H^t$ for $t=l_i-2$ $(-1\leq i<e)$ such that

$$d^*\iota_t = \bar{\mu}a_t \text{ if } t\neq 2n+2, \text{ and } d^*\iota_t - \bar{\mu}a_t\in PH^{2n-1}\otimes PH^3 \text{ if } t=2n+2.$$

Now we take a map ω: $S\to\Omega^2K_0$ with $\omega^*\iota_t=a_t$, and define d_0: $S\wedge S\to\Omega^2K_0$ by

$$d_0(x,\,y)\sim(\omega+v)x\cdot(\omega+v)y + (P\Omega f)F(x,\,y) - (\omega+v)(x\cdot y).$$

Then $d_0(x, y) \sim d(x, y) + \omega x \cdot \omega y - \omega(x \cdot y)$ since $+$ is homotopy commutative in $\Omega^2 K_0$. Hence $d_0^* \iota_t = d^* \iota_t - \bar{\mu}\omega^* \iota_t$, and so d_0 satisfies the last conditions in the lemma by the definition of a_t and ω. Moreover, by using the natural homotopy equivalence $\varepsilon: E_{\Omega f} \to \Omega E_f$ (see §2), we put

$$s_1 = \varepsilon(s_0 \times (\omega + \nu))\Delta : S \longrightarrow E_{\Omega f} \longrightarrow \Omega E_f = E_1 .$$

Then $r_1 s_1 \sim s_0$ and we see that $d(s_1) \sim j_0 d_0$ (cf. [13; 2.2.1 (b)]). q. e. d.

The above lemma implies that

(5.3) $\quad d_0^* \iota_{2n+2} = \sum_{i=1}^{b} \zeta_i \otimes \xi_{a+i}$ for some $\zeta_i \in PH^{2n-1}(S; \boldsymbol{Z}/3)$.

Therefore by the proof of Lemma 3.2 (ii), there are maps

(5.4) $\quad \psi_i : S \longrightarrow \Sigma = S^{2n-1}_{(3)}$ with $\psi_i^* \zeta = \zeta_i$ for $1 \leq i \leq b$,

where $\zeta \in H^{2n-1}(\Sigma; \boldsymbol{Z}/3)$ is a generator. Consider the maps

(5.5) $\quad \sigma_i : S \longrightarrow K(\boldsymbol{Z}_{(3)}, 3)$ with $\sigma_i^* \bar{\iota}_3 = \bar{\xi}_{a+i}$ for $1 \leq i \leq b$, and

$\tau : \Sigma \wedge K(\boldsymbol{Z}_{(3)}, 3) \longrightarrow \Omega^2 K_0$ with $\tau^* \iota_t = 0 \ (t \neq 2n+2)$, $\tau^* \iota_{2n+2} = \zeta \otimes \tilde{\iota}_3$,

for the fundamental class $\bar{\iota}_3$ and its mod 3 reduction $\tilde{\iota}_3$. Then Lemma 5.2 together with (5.3–5) implies the following

LEMMA 5.6. $d_0 \sim \tau(\psi_1 \wedge \sigma_1) \cdot \tau(\psi_2 \wedge \sigma_2) \cdot \cdots \cdot \tau(\psi_b \wedge \sigma_b)$.

Now we consider the special maps

$$K_3 = K(\boldsymbol{Z}_{(3)}, 3) \xrightarrow{\eta_1} K_7 = K(\boldsymbol{Z}/3, 7) \xrightarrow{\eta_2} K_{12} = K(\boldsymbol{Z}/3, 12)$$

with $\eta_1^* \iota_7 = \mathscr{P}^1 \tilde{\iota}_3$ and $\eta_2^* \iota_{12} = \mathscr{P}^1 \beta \iota_7$. Then $(\eta_2 \eta_1)^* \iota_{12} = (\mathscr{P}^2 \beta + \beta \mathscr{P}^2) \tilde{\iota}_3 = 0$. Thus we have the maps

$$F_1 \xrightarrow{p_1} K_3 \xrightarrow{\eta} F_2 \xrightarrow{p_2} K_7 \quad \text{with} \quad p_2 \eta = \eta_1,$$

where p_2 is the homotopy fiber of η_2 and p_1 is that of η. Then:

LEMMA 5.7. *$\pi_t(F_1) = 0$ for $t \geq 11$, $p_1^* : \tilde{H}^*(K_3; \boldsymbol{Z}/3) \to \tilde{H}^*(F_1; \boldsymbol{Z}/3)$ is 0 for $* \neq 3$, and there are maps $\tilde{\sigma}_i : S \to F_1$ with $p_1 \tilde{\sigma}_i \sim \sigma_i$ for σ_i in* (5.5) $(1 \leq i \leq b)$.

PROOF. By definition, $\pi_t(F_2) = 0$ for $t \geq 12$, and $\pi_t(F_1) = 0$ for $t \geq 11$. Moreover we see the second assertion since $p_1^* \mathscr{P}^1 \tilde{\iota}_3 = 0$.

Fix i with $1 \leq i \leq b$. Then by the proof of Lemma 3.2 (ii), σ_i is factored through as $S \xrightarrow{\sigma'} S^3_{(3)} \xrightarrow{\sigma} K_3$, $\sigma_i \sim \sigma\sigma'$. $\eta\sigma \sim *$ since F_2 is 6-connected; hence $\sigma \sim p_1 \tilde{\sigma}$ for some $\tilde{\sigma} : S^3_{(3)} \to F_1$. Thus $p_1 \tilde{\sigma}_i \sim \sigma_i$ for $\tilde{\sigma}_i = \tilde{\sigma}\sigma'$. q. e. d.

LEMMA 5.8. *For the diagram* (5.1), *there is a map*

$$\alpha_2\colon \Sigma \wedge F_1 \longrightarrow E_2 \quad (\Sigma = S^{2n-1}_{(3)})$$

with $r_2\alpha_2(\psi_i \wedge \tilde{\sigma}_i) \sim j_0\tau(\psi_i \wedge \sigma_i)\colon S \wedge S \to E_1$ *and* $\varphi\alpha_2 \sim *\colon \Sigma \wedge F_1 \to \Omega^2 K'$.

PROOF. $(\Omega h)j_0 = \Omega(hj_f)\colon \Omega^2 K_0 \to \Omega K_1 = \prod_{i=-1}^{e-1} K(\boldsymbol{Z}/3, m_i - 1)$, and so

$$((\Omega h)j_0\tau)^*\iota_t \in PH^{2n-1}(\Sigma;\ \boldsymbol{Z}/3) \otimes PH^*(K_3;\ \boldsymbol{Z}/3) \qquad \text{for } t = m_i - 1$$

by (5.5). On the other hand, it is well known that

$$H^*(K_3;\ \boldsymbol{Z}/3) = \Lambda(\tilde{\iota}_3,\ \mathscr{P}^{(i)}\cdots\mathscr{P}^{(0)}\tilde{\iota}_3 | i \geq 0) \otimes \boldsymbol{Z}/3[\beta\mathscr{P}^{(i)}\cdots\mathscr{P}^{(0)}\tilde{\iota}_3 | i \geq 0],$$

$(\mathscr{P}^{(i)} = \mathscr{P}^{3^i})$. Thus, by dimensional reason, we see that

$$((\Omega h)j_0\tau)^*\iota_t = c\zeta \otimes (\beta\mathscr{P}^1\tilde{\iota}_3)^{n'} \quad (c \in \boldsymbol{Z}/3,\ n' = 3^{e-1}) \qquad \text{for } t = m_{e-1} - 1,$$

and $((\Omega h)j_0\tau)^*\iota_t = 0$ otherwise. Define a map $\tilde{\tau}\colon \Sigma \wedge K_7 \to \Omega K_1$ by

$$\tilde{\tau}^*\iota_t = c\zeta \otimes (\beta\iota_7)^{n'} \text{ for } t = m_{e-1} - 1 \quad \text{and} \quad \tilde{\tau}^*\iota_t = 0 \text{ otherwise.}$$

Since $\eta_1^*\iota_7 = \mathscr{P}^1\tilde{\iota}_3$ and $p_2\eta = \eta_1$ by definition, these imply that

$$(\Omega h)j_0\tau \sim \tilde{\tau}(1 \wedge \eta_1) = \tilde{\tau}(1 \wedge p_2)(1 \wedge \eta).$$

Therefore we have the homotopy commutative diagram

$$\begin{array}{ccccccccc}
 & & & & \Sigma \wedge F_1 & \xrightarrow{1\wedge p_1} & \Sigma \wedge K_3 & \xrightarrow{1\wedge\eta} & \Sigma \wedge F_2 \\
 & & & & & & \downarrow{\scriptstyle j_0\tau} & & \downarrow{\scriptstyle \tilde{\tau}(1\wedge p_2)} \\
\Omega^2 K' & \xleftarrow{\varphi} & E_2 & \xrightarrow{r_2} & & & E_1 & \xrightarrow{\Omega h} & \Omega K_1 & \xrightarrow{\Omega g} & \Omega K'.
\end{array}$$

Here $(1 \wedge \eta)(1 \wedge p_1) \sim *$ by definition. Moreover, by the definitions of g and h in (2.3), $(gh)^*\iota_{6n} = r_f^*\iota_{2n}^3$ and so $\Omega(gh) \sim *$. Also

$$\tilde{\tau}^*(\Omega g)^*\iota_{6n-1} = \mathscr{P}^{n'}(c\zeta \otimes (\beta\iota_7)^{n'}) = c\zeta \otimes (\mathscr{P}^1\beta\iota_7)^{n'} = (1 \wedge \eta_2)^*(c\zeta \otimes \iota_{12}^{n'});$$

hence $((\Omega g)\tilde{\tau}(1 \wedge p_2))^*\iota_{6n-1} = 0$. Thus $(\Omega g)\tilde{\tau}(1 \wedge p_2) \sim *$. Therefore we can apply Proposition 4.1 to get two maps

$$\alpha_2\colon \Sigma \wedge F_1 \longrightarrow E_2 \text{ with } r_2\alpha_2 = j_0\tau(1 \wedge p_1) \quad \text{and} \quad \tilde{\alpha}_2\colon \Sigma \wedge K_3 \longrightarrow \Omega^2 K'$$

with $\tilde{\alpha}_2(1 \wedge p_1) \sim \varphi\alpha_2$, because ψ in Proposition 4.1 for the above diagram is equal to φ by a suitable homotopy $\Omega(gh) \sim *$ (see the proof of Proposition 2.1).

Now we have the first homotopy for α_2 using Lemma 5.7; and the second one because $p_1^* = 0$ in dimension $\neq 3$ by Lemma 5.7, $\Omega^2 K' = K(\boldsymbol{Z}/3, 6n-2)$ and $6n-2 \neq 2n+2$. q.e.d.

PROOF OF PROPOSITION 3.4. By Lemma 5.2, we see that

$$d((\Omega h)s_1) \sim (\Omega h)d(s_1) \sim (\Omega h)j_0 d_0 = \Omega(hj_f)d_0 \sim *,$$

because Ωh and $\Omega(hj_f)$ are given by some mod 3 Steenrod operations of positive degree which are trivial on $H^*(S \wedge S; \boldsymbol{Z}/3)$. Therefore $(\Omega h)s_1: S \to \Omega K_1 = \prod_{i=-1}^{e-1} K(\boldsymbol{Z}/3, m_i - 1)$ is an H-map. Thus for any $t = m_i - 1$, $((\Omega h)s_1)^* \iota_t$ is primitive, and so it is 0 by dimensional reason. Hence $(\Omega h)s_1 \sim *$, and we have a lift

$$s: S \longrightarrow E_2 = E_{\Omega h} \quad \text{with } r_2 s = s_1, \text{ i.e., } r_1 r_2 s \sim s_0 \text{ and } s^* u = \xi_1.$$

Now we consider the maps

(5.9) $$S \wedge S \xrightarrow{\delta_1} Y = (\Sigma \wedge F_1)^b \xrightarrow{\delta_2} E_2 = E_{\Omega h} \quad (\Sigma = S_{(3)}^{2n-1})$$

given by $\delta_1 = (\prod_{i=1}^{b} (\psi_i \wedge \tilde{\sigma}_i))\Delta$ and $\delta_2 = \alpha_2 \cdot \cdots \cdot \alpha_2$ (b times), where ψ_i, $\tilde{\sigma}_i$ and α_2 are the maps in (5.4) and Lemmas 5.7–8. Then

$$\begin{aligned} r_2\delta_2\delta_1 &= r_2(\alpha_2(\psi_1 \wedge \tilde{\sigma}_1) \cdot \cdots \cdot \alpha_2(\psi_b \wedge \tilde{\sigma}_b)) \\ &\sim j_0(\tau(\psi_1 \wedge \sigma_1) \cdot \cdots \cdot \tau(\psi_b \wedge \sigma_b)) \sim j_0 d_0 \sim d(s_1) \sim r_2 d(s) \end{aligned}$$

by Lemmas 5.8, 5.6 and 5.2, since $r_2 = \Omega r_h$ and $j_0 = \Omega j_f$ are H-maps. Thus

(5.10) $$j_1 d + \delta_2\delta_1 \sim d(s): S \wedge S \longrightarrow E_2 \quad \text{for some} \quad d: S \wedge S \longrightarrow \Omega^2 K_1.$$

Here, by dimensional reason, we see that

(5.11) $$d^* \iota_{t-1} \in DH^* \otimes \tilde{H}^* + \tilde{H}^* \otimes DH^* \quad \text{for any} \quad t = m_i - 1,$$

where $H^* = H^*(S; \boldsymbol{Z}/3)$, and DH^* is the decomposable module of H^*. Consider the maps

$$S \xrightarrow{\gamma_1} S' = \prod_{i=2}^{a+b} S_{(3)}^{n_i} \xrightarrow{\gamma_2} \tilde{K} = \prod_{i=2}^{a+b} K(\boldsymbol{Z}/3, n_i),$$

where γ_1 is the projection and $\gamma_2^* \iota_{n_i} = \xi_i$ (in Lemma 3.2). Then we see that

(5.12) $$\operatorname{Im}[\gamma^*: \tilde{H}^*(\tilde{K} \wedge S; \boldsymbol{Z}/3) \to \tilde{H}^*] = DH^* \quad \text{for} \quad \gamma = (\gamma_2\gamma_1 \wedge 1)\Delta: S \longrightarrow \tilde{K} \wedge S.$$

Thus by (5.10–12), there are two maps

(5.13) $$d_1: \tilde{K} \wedge S \wedge S \longrightarrow \Omega^2 K_1 \quad \text{and} \quad d_2: S \wedge \tilde{K} \wedge S \longrightarrow \Omega^2 K_1 \quad \text{with}$$
$$d \sim d_1(\gamma \wedge 1) + d_2(1 \wedge \gamma): S \wedge S \longrightarrow \Omega^2 K_1.$$

Furthermore, by putting $e(i) = 1$ for $i > a$, define

$$f_i: K(\boldsymbol{Z}/3, n_i) \longrightarrow \tilde{K}_i = K(\boldsymbol{Z}/3, n_i + 1) \times \prod_{j=0}^{e(i)-1} K(\boldsymbol{Z}/3, n_i + 4 \cdot 3^j)$$

by $f_i^*\iota_t=\beta\iota_{n_i}$ for $t=n_i+1$ and $f_i^*\iota_t=\mathscr{P}^{(j)}\iota_{n_i}$ for $t=n_i+4\cdot 3^j$; and consider the homotopy fiber

$$\tilde{r}\colon \tilde{E} \longrightarrow \tilde{K} \quad \text{of} \quad \textstyle\prod_{i=2}^{a+b} f_i\colon \tilde{K} \longrightarrow \prod_{i=2}^{a+b} \tilde{K}_i.$$

Then

$$(5.14) \qquad \pi_t(\tilde{E}) = 0 \text{ for } t \geq n_1 + 4\cdot 3^{e-1} = 10\cdot 3^{e-1} \quad (n_1=2n-1,\ n=3^e).$$

Furthermore $(\prod_i f_i)\gamma_2 \sim *$ since $\gamma_2^*(\prod_i f_i)^*\iota_t=0$ for any t, and so we see that

$$(5.15) \qquad \gamma_2 = \tilde{r}\tilde{\gamma}_2 \quad \text{for some} \quad \tilde{\gamma}_2\colon S' \longrightarrow \tilde{E}.$$

Moreover the mod 3 Steenrod algebra $\mathscr{A}$ acts trivially on Im $[\tilde{r}^*\colon H^*(\tilde{K};\, \boldsymbol{Z}/3) \to H^*(\tilde{E};\, \boldsymbol{Z}/3)]$ by definition, and $\varphi j_1=\Omega^2 g$. Thus

$$(5.16) \qquad \varphi j_1 d_1(\tilde{r}\wedge 1\wedge 1) \sim * \quad \text{and} \quad \varphi j_1 d_2(1\wedge \tilde{r}\wedge 1) \sim *.$$

On the other hand, $\rho\colon \tilde{S}\to S$ is defined by

$$\rho = \rho(4n-3) = 1 \times \rho'(4n-3)\colon \tilde{S} = \Sigma \times S'(4n-3) \longrightarrow S = \Sigma \times S',$$

and $S'(4n-3)$ is $(4n-3)$-connected. Therefore,

$$\tilde{\gamma}_2\gamma_1\rho = \tilde{\gamma}_2\rho'(4n-3)\,\mathrm{pr}_2\colon \tilde{S} \longrightarrow S'(4n-3) \longrightarrow S' \longrightarrow \tilde{E}$$

is homotopic to $*$ by (5.14) and $4n-3 \geq 10\cdot 3^{e-1}$. Thus

$$(5.17) \qquad (\tilde{r}\wedge 1)\tilde{\gamma} = \gamma \quad \text{and} \quad \tilde{\gamma}\rho \sim * \qquad \text{for} \quad \tilde{\gamma} = (\tilde{\gamma}_2\gamma_1\wedge 1)\Delta\colon S \longrightarrow \tilde{E}\wedge S.$$

Now using Y, δ_1, δ_2 in (5.9) and the above maps, we define

$$S\wedge S \xrightarrow{\lambda_1} X=(\tilde{E}\wedge S\wedge S)\times(S\wedge \tilde{E}\wedge S)\times Y \xrightarrow{\lambda_2} E_2=E_{\Omega h}$$

by $\lambda_1=((\tilde{\gamma}\wedge 1)\times(1\wedge\tilde{\gamma})\times\delta_1)\Delta$ and $\lambda_2=j_1 d_1(\tilde{r}\wedge 1\wedge 1)\,\mathrm{pr}_1+j_1 d_2(1\wedge\tilde{r}\wedge 1)\,\mathrm{pr}_2+\delta_2\,\mathrm{pr}_3$. Then, noticing that j_1 and φ are H-maps, we see that $d(s)\sim\lambda_2\lambda_1$ by (5.10), (5.13) and (5.17), and $\varphi\lambda_2\sim\varphi\delta_2\,\mathrm{pr}_3\sim(\varphi\alpha_2\cdots\varphi\alpha_2)\,\mathrm{pr}_3\sim *$ by (5.16) and Lemma 5.8. Moreover, $\pi_t(F_1)=0$ for $t\geq 11$ by Lemma 5.7, and $\tilde{S}=\Sigma\times S'(4n-3)$ is $(2n-2)$-connected. Thus $\tilde{\sigma}_i\rho\sim *\colon \tilde{S}\to F_1$ because $2n-2=2\cdot 3^e-2\geq 11$ by the assumption $e\geq 2$. Therefore

$$\lambda_1(\rho\wedge\rho) = ((\tilde{\gamma}\rho\wedge\rho)\times(\rho\wedge\tilde{\gamma}\rho)\times\delta_1(\rho\wedge\rho))\Delta \sim (\textstyle\prod_{i=1}^{b}(\psi_i\rho\wedge\tilde{\sigma}_i\rho))\Delta \sim *$$

by (5.17) and (5.9). This completes the proof of Proposition 3.4. q.e.d.

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