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A note on the inequality $\Delta u \geqq k(x)e^u$ in R^n

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1. Introduction

This note is concerned with the problem of nonexistence of entire solutions for the differential inequality

$$(1) \qquad \Delta u \geqq k(x)e^u, \qquad x \in \boldsymbol{R}^n,$$

where $n \geqq 2$, Δ is the n-dimensional Laplacian and $k(x)$ is a nonnegative continuous function in $\boldsymbol{R}^n$. An entire solution $u(x)$ of inequality (1) is defined to be a real-valued function of class $C^2(\boldsymbol{R}^n)$ which satisfies (1) at every point of $\boldsymbol{R}^n$. The following result was established by Oleinik [5]:

THEOREM 0. *Suppose that $k(x) \geqq \theta(|x|)|x|^{-2}$ for large $|x|$, where $|\cdot|$ denotes the Euclidean length, $\theta(t) \to \infty$ as $t \to \infty$ and $\theta(t)t^{-2}$ is a nonincreasing function of t. Then inequality* (1) *has no entire solution.*

The purpose of this note is to improve and extend this result. First, we derive nonexistence criteria for (1), sharper than Oleinik's, on the basis of the consideration of certain ordinary differential inequalities. Then we attempt to obtain an extension of Theorem 0 to more general elliptic inequalities of the form (16). For other related results, we refer the reader to the papers [2, 3, 4, 6] and the references contained therein.

2. Results

First, we introduce the notation

$$k_*(r) = \min_{|x|=r} k(x) \qquad \text{for} \quad r \geqq 0,$$

and for an entire solution $u(x)$ of (1), we put

$$\bar{u}(r) = \frac{1}{\omega_n r^{n-1}} \int_{|x|=r} u(x)dS \qquad \text{for} \quad r \geqq 0,$$

where ω_n denotes the surface area of the unit sphere in $\boldsymbol{R}^n$, i.e., $\bar{u}(r)$ is the spherical mean of $u(x)$ over the sphere $|x| = r$. An improvement of Theorem 0 in the twodimensional case is given by the following theorem.

THEOREM 1. *Let $n=2$. Suppose that there exists a constant $\alpha\in(0, 1/2)$ such that*

$$\int^{\infty} r^{c+2\alpha-1}[k_*(r)]^{\alpha}dr=\infty \quad \text{for all} \quad c>0. \tag{2}$$

Then inequality (1) *has no entire solution.*

To prove this theorem, the next lemma is needed.

LEMMA 1. *Consider the ordinary differential inequality*

$$(p(t)y')'\geqq a(t)e^{y}, \qquad t\geqq t_0\geqq 0, \tag{3}$$

where $p(t)$ is a positive continuous function for $t\geqq t_0$, and $a(t)$ is a nonnegative continuous function for $t\geqq t_0$. Let $v(t)$ be a continuous function for $t\geqq t_0$. Suppose that there exists a constant $\alpha\in(0, 1/2)$ such that

$$\int^{\infty}\frac{[a(t)]^{\alpha}e^{cv(t)}}{[p(t)]^{1-\alpha}}dt=\infty \qquad \text{for all} \quad c>0. \tag{4}$$

Then inequality (2) *has no solution $y(t)$ which is defined for large t and satisfies*

$$p(t)y'(t)\geqq C_1 \quad \text{and} \quad y(t)\geqq C_2v(t) \tag{5}$$

for some positive constants C_1 and C_2 there.

PROOF OF LEMMA 1. Suppose the contrary. Let $y(t)$ be a solution of (3) satisfying (5) for $t\geqq t_1\geqq t_0$. Motivated by Wong [7], we put $w(t)=p(t)y'(t)e^{y(t)}$. Then we have

$$\begin{aligned} w'(t) &= (p(t)y'(t))'e^{y(t)}+p(t)e^{y(t)}[y'(t)]^2 \\ &\geqq a(t)e^{2y(t)}+p(t)e^{y(t)}[y'(t)]^2 \\ &= w(t)\left(\frac{a(t)e^{y(t)}}{p(t)y'(t)}+y'(t)\right), \qquad t\geqq t_1, \end{aligned}$$

which implies

$$w'(t)\geqq Cw(t)\frac{[a(t)]^{\alpha}e^{\alpha y(t)}[y'(t)]^{1-2\alpha}}{[p(t)]^{\alpha}}, \qquad t\geqq t_1,$$

where $C=\alpha^{-\alpha}(1-\alpha)^{\alpha-1}>0$. We rewrite this inequality as

$$w'(t)\geqq C[w(t)]^{1+\delta}\cdot\frac{[a(t)]^{\alpha}e^{(\alpha-\delta)y(t)}[p(t)y'(t)]^{1-2\alpha-\delta}}{[p(t)]^{1-\alpha}}, \tag{6}$$

where $\delta>0$ is chosen so small that

$$\delta + 2\alpha \leqq 1 \qquad \text{and} \qquad \delta < \alpha,$$

which is possible, by our assumption. From (5) and (6) it follows that

$$w'(t) \geqq \tilde{C}[w(t)]^{1+\delta} \cdot \frac{[a(t)]^\alpha e^{C_2(\alpha-\delta)v(t)}}{[p(t)]^{1-\alpha}}, \qquad t \geqq t_1, \tag{7}$$

for some $\tilde{C} > 0$. Dividing (7) by $[w(t)]^{1+\delta}$ and integrating over $[t_1, \infty)$, we have

$$\int_{t_1}^{\infty} \frac{[a(t)]^\alpha e^{C_2(\alpha-\delta)v(t)}}{[p(t)]^{1-\alpha}} dt < \infty,$$

which contradicts (4). This completes the proof of Lemma 1.

PROOF OF THEOREM 1. Let $u(x)$ be an entire solution of inequality (1). It is easily seen from Jensen's inequality that the spherical mean $\bar{u}(r)$ of $u(x)$ satisfies the following:

$$\begin{aligned} &(r\bar{u}'(r))' \geqq rk_*(r)e^{\bar{u}(r)} && \text{for} \quad r > 0, \\ &\bar{u}'(0) = 0 \quad \text{and} \quad \bar{u}'(r) \geqq 0 && \text{for} \quad r > 0. \end{aligned} \tag{8}$$

It follows that there are positive constants C_1, C_2 and R such that

$$r\bar{u}'(r) \geqq C_1 \quad \text{and} \quad \bar{u}(r) \geqq C_2 \log r \qquad \text{for} \quad r \geqq R. \tag{9}$$

However this is impossible, since applying Lemma 1 to (8), we see that condition (2) precludes solutions $\bar{u}(r)$ of (8) satisfying (9).

In the case of $n \geqq 3$, the method used in the proof of Theorem 1 does not work effectively. A slight improvement of Theorem 0 of different nature will be given below.

THEOREM 2. *Let $n \geqq 3$. Suppose that there exists an integer $m \geqq 2$ such that*

$$\liminf_{r\to\infty} r^2 \log^1 r \cdot \log^2 r \cdots \log^m r \cdot k_*(r) > 0, \tag{10}$$

where $\log^1 r = \log r$, $\log^{\nu+1} r = \log(\log^\nu r)$, $\nu = 1, 2, \ldots$. Then inequality (1) has no entire solution.

PROOF. Let $u(x)$ be an entire solution of (1). As was stated in the proof of Theorem 1, the spherical mean $\bar{u}(r)$ satisfies

$$\begin{aligned} &(r^{n-1}\bar{u}'(r))' \geqq r^{n-1}k_*(r)e^{\bar{u}(r)} && \text{for} \quad r > 0, \\ &\bar{u}'(0) = 0 \quad \text{and} \quad \bar{u}'(r) \geqq 0 && \text{for} \quad r > 0. \end{aligned} \tag{11}$$

For economy of notation we use the letter C to denote various positive constants. By (10) and (11) we have

$$(r^{n-1}\bar{u}'(r))' \geqq \frac{Cr^{n-3}}{\log^1 r\cdot\log^2 r\cdots\log^m r} \qquad \text{for large} \quad r, \tag{12}$$

say $r \geqq r_0 > 0$. Now we show that (12) also holds when m is replaced by $m-1$ in this expression. Integrating (12) on $[r_0, r]$ with use of integration by parts, we find

$$r^{n-1}\bar{u}'(r) \geqq C\left(\frac{r^{n-2}}{\log^1 r\cdots\log^m r} - C + \int_{r_0}^{r} s^{n-2}\frac{(\log^1 s\cdots\log^m s)'}{(\log^1 s\cdots\log^m s)^2}ds\right) + C,$$

which implies

$$\bar{u}'(r) \geqq \frac{C}{r\log^1 r\cdots\log^m r} \qquad \text{for large} \quad r,$$

say $r \geqq r_1 \geqq r_0$. An integration of the above yields

$$\bar{u}(r) \geqq C + C\log^{m+1} r \geqq \log(\log^m r)^\delta \tag{13}$$

for $r \geqq r_2 \geqq r_1$, where we may assume that $\delta \in (0, 1)$ without loss of generality. Combining (13) with inequality (11) and using (10), we have

$$(r^{n-1}\bar{u}'(r))' \geqq \frac{Cr^{n-3}}{\log^1 r\cdots\log^{m-1} r\cdot(\log^m r)^{1-\delta}}$$

for $r \geqq r_2$. Integration by parts of the above gives

$$\bar{u}'(r) \geqq \frac{C}{r\log^1 r\cdots\log^{m-1} r\cdot(\log^m r)^{1-\delta}}$$

for $r \geqq r_3 \geqq r_2$, whence it follows that

$$\bar{u}(r) \geqq C(\log^m r)^\delta \tag{14}$$

for $r \geqq r_4 \geqq r_3$. From inequality (11) combined with (14), we obtain

$$(r^{n-1}\bar{u}'(r))' \geqq \frac{Cr^{n-3}}{\log^1 r\cdots\log^{m-1} r}\cdot\frac{\exp[C(\log^m r)^\delta]}{\log^m r}$$

and so

$$(r^{n-1}\bar{u}'(r))' \geqq \frac{Cr^{n-3}}{\log^1 r\cdots\log^{m-1} r}$$

for $r \geqq r_5 \geqq r_4$. Thus (12) also holds even if m is replaced by $m-1$.

Repeating the above reduction, we finally conclude that there exists an $\varepsilon \in (0, 1)$ such that

$$\bar{u}(r) \geqq C(\log r)^\varepsilon \tag{15}$$

for $r \geqq r^* > 0$. now we put $v(x) = \bar{u}(|x|)/2$. Then $v(x)$ is defined in the whole space $\boldsymbol{R}^n$, and satisfies in view of (11) and (15)

$$\Delta v(x) \geqq \frac{C\exp[C(\log|x|)^\varepsilon]}{|x|^2 \log|x|} \cdot e^{v(x)}$$

for large $|x|$. Applying Theorem 0, we are led to a contradiction immediately. This completes the proof.

EXAMPLE 1. When $n=2$, some improvements of Theorem 0 have been obtained by Ni [4]. One of them asserts that if

$$k_*(r) \geqq \frac{C}{r^2 \log r} \qquad \text{for large} \quad r$$

for some $C>0$, then inequality (1) has no entire solution. But according to our Theorem 1, the same conclusion holds under a weaker condition that

$$k_*(r) \geqq \frac{C}{r^2 (\log r)^\ell} \qquad \text{for large} \quad r$$

for some $C>0$ and $\ell \geqq 1$.

Now let us attempt to extend Theorem 0 of Oleinik for more general elliptic inequalities of the form

$$(16) \qquad Lu \equiv \textstyle\sum_{i,j=1}^n a_{ij}(x) \dfrac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \dfrac{\partial u}{\partial x_i} \geqq k(x)e^u, \qquad x \in \boldsymbol{R}^n,$$

where $n \geqq 2$, $x = (x_i)$, $a_{ij}(x)$, $b_i(x)$ are continuous for all i, j, and the symmetric matrix $(a_{ij}(x))$ is positive definite for each $x \in \boldsymbol{R}^n$. As in [5] we begin with the following lemma.

LEMMA 2. *Let* $R>0$, $x^0 = (x_i^0) \in \boldsymbol{R}^n$ *and* $k_0 = \inf_{|x-x^0| \leq R} k(x) > 0$. *Suppose that* $u(x)$ *satisfies* $Lu \geqq k(x)e^u$ *in* $|x-x^0| \leqq R$ *and that there exists a constant* $T(x^0, R)$ *such that*

$$T(x^0, R) \geqq \sup_{|x^0-y|=R} \textstyle\sum_{i=1}^n (a_{ii}(y) + b_i(y)\ (x_i^0 - y_i)\ (x_j^0 - y_j)),$$

and

$$T(x^0, R) \geqq \sup_{|x^0-y|=R} \frac{2}{|x^0-y|^2} \textstyle\sum_{i,j=1}^n a_{ij}(y)\ (x_i^0 - y_i)\ (x_j^0 - y_j),$$

where $y = (y_i)$. *Then, we have*

$$e^{u(x^0)} \leqq 4T(x^0, R)/(k_0 R^2).$$

PROOF. We adapt the argument due to Oleinik [5]. Put

$$a=4T(x^0, R)R^2/k_0$$

and define

$$V(x)=a/(R^2-r^2)^2, \qquad \text{where} \quad r=|x-x^0|.$$

Then $V(x)$ satisfies

$$L[\log V(x)] \leqq k(x)V(x), \qquad |x-x^0|<R. \tag{17}$$

In fact,

$$\begin{aligned} L[\log V(x)] &= \frac{LV(x)}{V(x)} - \frac{1}{V^2(x)}\sum_{i,j=1}^n a_{ij}(x)\frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_j} \\ &= 4(R^2-r^2)^{-2}(\textstyle\sum_{i=1}^n (a_{ii}(x)+b_i(x)\,(x_i-x_i^0))\,(R^2-r^2) \\ &\quad + 2\textstyle\sum_{i,j=1}^n a_{ij}(x)\,(x_i-x_i^0)\,(x_j-x_j^0)) \\ &\leqq 4(R^2-r^2)^{-2}(T(x^0, R)\,(R^2-r^2)+2\textstyle\sum_{i,j=1}^n a_{ij}(x)\,(x_i-x_i^0)\,(x_j-x_j^0)) \\ &\leqq 4(R^2-r^2)^{-2}(T(x^0, R)\,(R^2-r^2)+T(x^0, R)r^2) \\ &= 4(R^2-r^2)^{-2}T(x^0, R)R^2 \leqq k(x)V(x), \quad |x-x^0|<R. \end{aligned}$$

Next we put $v(x)=e^{u(x)}$ and assert that

$$v(x) \leqq V(x), \quad |x-x^0|<R. \tag{18}$$

Suppose the contrary. Since $\log v(x)-\log V(x) \to -\infty$ as $|x-x^0| \to R$, $\log v(x) -\log V(x)$ takes a positive maximum in $|x-x^0|<R$ at some point x'. Clearly $v(x') > V(x')$. Noting that $L[\log v(x)] \geqq k(x)v(x)$ in $|x-x^0| \leqq R$ and using (17), we find

$$L[\log v-\log V]\,(x') \geqq k(x')\,[v(x')-V(x')]>0.$$

But this contradicts the fact that x' is a point of maximum of $\log v(x)-\log V(x)$. Thus (18) holds. By putting $x=x^0$ in (18), we have the desired conclusion.

THEOREM 3. *Suppose that there exist functions $T(r)$ and $m(r)$ such that*

$$T(r) \geqq \sup_{\substack{|x|=r \\ |x-y|=r/2}} \sum_{i=1}^n (a_{ii}(y)+b_i(y)\,(x_i-y_i)), \tag{19}$$

$$T(r) \geqq \sup_{\substack{|x|=r \\ |x-y|=r/2,\, x\neq y}} \frac{2}{|x-y|^2}\sum_{i,j=1}^n a_{ij}(y)\,(x_i-y_i)\,(x_j-y_j), \tag{20}$$

$$\inf_{r/2 \leqq |x| \leqq 3r/2} k(x) \geqq m(r)>0 \tag{21}$$

for large r, say $r \geqq R_0$, and

$$T(r)/(m(r)r^2) \to 0 \quad \textit{as} \quad r \to \infty. \tag{22}$$

Then inequality (16) *has no entire solution.*

PROOF. Let $u(x)$ be an entire solution of (16). Consider a point x such that $|x| \geqq R_0$. Applying Lemma 2 to the ball $\{y: |y-x| \leqq |x|/2\}$ and taking account of the fact that $|y-x| \leqq |x|/2$ implies $|x|/2 \leqq |y| \leqq 3|x|/2$, we find

$$e^{u(x)} \leqq \frac{16T(|x|)}{(\inf_{|x|/2 \leqq |y| \leqq 3|x|/2} k(y))\ |x|^2},$$

and hence

$$e^{u(x)} \leqq 16T(|x|)/(m(|x|)|x|^2)$$

This shows that $e^{u(x)} \to 0$ as $|x| \to \infty$. On the other hand, it is easy to see that

$$L[e^{u(x)}] = e^{u(x)}\left(Lu(x) + \sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i}\right) \geqq 0, \quad x \in R^n.$$

Hence by the maximum principle $e^{u(x)} \equiv 0$ in R^n, and this contradiction proves our assertion.

COROLLARY. *Suppose that there exist constants a, b, $c > 0$ and α, β, $\sigma \in R$ such that $\sigma > \max\{\alpha, \beta+1\}$ and*

$$a_{ij}(x) \leqq a|x|^\alpha, \quad |b_i(x)| \leqq b|x|^\beta, \quad 1 \leqq i, \quad j \leqq n;$$
$$k(x) \geqq c|x|^{\sigma-2}$$

for sufficiently large $|x|$. Then inequality (16) *has no entire solution.*

PROOF. It is easily seen by our assumption that the function $T(r) = C_1(r^\alpha + r^{\beta+1})$ satisfies (19) and (20) provided $C_1 > 0$ is large enough, and that the function $m(r) = C_2 r^{\sigma-2}$ satisfies (21) and (22) provided $C_2 > 0$ is small enough. Thus according to Theorem 3, inequality (16) has no entire solution.

EXAMPLE 2. Consider the equation

$$(23) \qquad Lu = f(x)e^u, \qquad x \in R^n, \qquad n \geqq 3,$$

where L is the same operator as in (16). Suppose that $a_{ij}(x)$, $b_i(x)$ and $f(x)$ are locally Hölder continuous in R^n. Suppose moreover that the limits $\bar{a}_{ij} = \lim_{|x| \to \infty} a_{ij}(x)$ exist and the matrix $(\bar{a}_{ij})$ has at least three positive eigenvalues, that $b_i(x) = o(|x|^{-1})$ as $|x| \to \infty$, and that

$$(24) \qquad |f(x)| \leqq C|x|^{-2-\mu} \qquad \text{for large} \quad |x|$$

for some C, $\mu > 0$. Then by applying Friedman's existence theorem [1, Corollary 2], equation (23) is shown to have a bounded entire solution. Actually there exists a bounded function $w(x)$ such that

$$Lw(x) = -(1+|x|)^{-2-\mu}, \qquad x \in \boldsymbol{R}^n,$$

and it is easily verified that the functions $u_1(x) = w(x) - C_1$ and $u_2(x) = -w(x) - C_2$, respectively, become a supersolution and a subsolution of (23) satisfying $u_1(x) \geqq u_2(x)$ in $\boldsymbol{R}^n$ provided C_1, $C_2 > 0$ are sufficiently large. Therefore the well-known supersolution and subsolution method ensures the existence of an entire soslution $u(x)$ of (23) such that $u_2(x) \leqq u(x) \leqq u_1(x)$ in $\boldsymbol{R}^n$.

On the other hand, if (24) is replaced by the condition that

$$f(x) \geqq C|x|^{-2+\mu} \qquad \text{for large} \quad |x|$$

for some C, $\mu > 0$, with the other conditions being kept to hold, then by Corollary, equation (23) admits no entire solution.

References

[1] A. Friedman, Bounded entire solutions of elliptic equations, Pacific J. Math., **44** (1973), 497–507.

[2] N. Kawano, On bounded entire solutions of semilinear elliptic equations, Hiroshima Math. J., **14** (1984), 125–158.

[3] R. C. McOwen, On the equation $\Delta u + Ke^{2u} = f$ and prescribed negative curvature in $\boldsymbol{R}^2$, J. Math. Anal. Appl., **103** (1984), 365–370.

[4] W.-M. Ni, On the elliptic equation $\Delta u + K(x)e^{2u} = 0$ and conformal metrics with prescribed Gaussian curvatures, Invent. Math., **66** (1982), 343–352.

[5] O. A. Oleinik, On the equation $\Delta u + k(x)e^u = 0$, Russian Math. Surveys, **33**–2 (1978), 243–244.

[6] D. H. Sattinger, Conformal metrics in $\boldsymbol{R}^2$ with prescribed curvature, Indiana Univ. Math. J., **22** (1972), 1–4.

[7] P.-K. Wong, Existence and asymptotic behavior of proper solutions of a class of second-order nonlinear differential equaitons, Pacific J. Math., **13** (1963), 737–760.

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