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Three Riemannian metrics on the moduli space of 1-instantons over CP^2

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1. Introduction

The natural metric on 5-sphere of radius 1 induces the Fubini-Study metric g_{FS} on the complex projective plane CP^2. The moduli space $\mathscr{M}$ of 1-instantons over (CP^2, g_{FS}) is homeomorphic to the cone on CP^2 (Buchdahl [B] and Furuta [F]). The generic part $\mathscr{M}^*$ of the moduli space carries three natural Riemannian metrics g_J (J = I, II and I–II). We refer to Matumoto [M] for the definition of the Riemannian symmetric tensors. In this paper we will give explicit formulas of the metrics and study their basic geometric properties.

Buchdahl and Furuta defined an $SU(3)$-equivariant diffeomorphism F: $CP^2 \times (0, 1) \cong \mathscr{M}^* = \mathscr{M}$-{cone point}. We use a local coordinate system $\mathbf{C}^2 \times (0, 1) \to CP^2 \times (0, 1)$ defined by $(W_1, W_2, \lambda) \to ([1, W_1, W_2], \lambda)$ with $W_1 = X_1 + iX_2$ and $W_2 = X_3 + iX_4$. Note that $F(\mathbf{C}^2 \times (0, 1))$ is open and dense in $\mathscr{M}^*$. The metric tensors split with respect to this coordinate system as

$$F^*g_J = \varphi_J(\lambda)\, d\lambda^2 + \psi_J(\lambda) g_{FS} \quad (\mathrm{J} = \mathrm{I}, \mathrm{II} \text{ and } \mathrm{I–II}).$$

More explicitly, we can write $\varphi_J(\lambda)$ and $\psi_J(\lambda)$ by using a new paramater $Z = 1 - \lambda^2$ as follows:

$$\varphi_I(\lambda) = 8\pi^2(Z^2 \log Z + 3Z \log Z - 3Z^2 + 2Z + 1)/Z(1 - Z)^3,$$

$$\psi_I(\lambda) = 4\pi^2(-6Z^2 \log Z + Z^3 + 6Z^2 - 9Z + 2)/(Z + 2)(1 - Z)^2;$$

$$\varphi_{II}(\lambda) = 16\pi^2(Z^2 - 2Z + 6)/15Z^2, \quad \psi_{II}(\lambda) = 8\pi^2(-3Z^2 - 4Z + 12)(1 - Z)/15Z;$$

$$\varphi_{I-II}(\lambda) = \varphi_{II}(\lambda), \quad \psi_{I-II}(\lambda) = 24\pi^2(Z^4 - Z^3 + 2Z^2 + 8)(1 - Z)/5Z(Z + 2)^2.$$

In fact, $\varphi_J(\lambda)$ and $\psi_J(\lambda)$ are positive for $0 < \lambda < 1$ and g_J defines actually the positive definite Riemannian metrics for not only J = I but also J = II and I–II.

From the above formulas or their asymptotic ones given in §4, we get the following proposition, where $K_J(u, v)$ (J = I, II and I–II) denote the sectional curvatures of F^*g_J.

PROPOSITION. (a) *As $\lambda \to 1$ (near the collar) all the sectional curvatures converge to the negative constant $-5/32\pi^2$ for $(\mathscr{M}^*, g_{II})$ and $(\mathscr{M}^*, g_{I-II})$. On $(\mathscr{M}^*, g_I)$, we can induce a C^∞ metric on $\partial\bar{\mathscr{M}}$ so that $(\partial\bar{\mathscr{M}}, g_I)$ is isometric to*

$(CP^2, 4\pi^2 g_{FS})$ and $K_{\mathrm{I}}(\partial/\partial\lambda, X)$ converges to $3/8\pi^2$ as $\lambda \to 1$.

(b) *As $\lambda \to 0$ (near the cone point),*

$$K_{\mathrm{I}}(\partial/\partial\lambda, X) \sim -3/8\pi^2\,, \quad K_{\mathrm{I}}(X, Y) \sim (3/4\pi^2\lambda^2)(K_{FS}(X, Y) - 1) - 3/8\pi^2\,;$$
$$K_{\mathrm{II}}(\partial/\partial\lambda, X) \sim -21/16\pi^2\,, \quad K_{\mathrm{II}}(X, Y) \sim (21/16\pi^2\lambda^2)(K_{FS}(X, Y) - 1) + 3/16\pi^2\lambda^2\,;$$
$$K_{\mathrm{I-II}}(\partial/\partial\lambda, X) \sim -9/32\pi^2\,, \quad K_{\mathrm{I-II}}(X, Y) \sim (3/16\pi^2\lambda^2)(K_{FS}(X, Y) - 1) - 9/32\pi^2\,,$$

where $X, Y \in TCP^2$ and K_{FS} denotes the sectional curvature of (CP^2, g_{FS}). Note that $1 \leqq K_{FS}(X, Y) \leqq 4$.

(c) *The volume and diameter of $(\mathscr{M}^*, g_{\mathrm{I}})$ are finite and those of $(\mathscr{M}^*, g_{\mathrm{I-II}})$ and $(\mathscr{M}^*, g_{\mathrm{II}})$ are infinite.*

The computation of g_{II} is due to Hideo Doi and originally to Mikio Furuta. The author would like to thank them for permitting him to contain their results in this paper.

2. Diffeomorphism $F: CP^2 \times (0, 1) \cong \mathscr{M}^*$ due to Buchdahl and Furuta

A 1-parameter family of 1-instantons ∇_λ ($\lambda \in [0, 1)$) is defined as follows. We define a quaternion line bundle E with $c_2 = -1$ by $E = \{([X], \xi X);\ X \in \mathbf{C}^3, [X] \in CP^2,\ \xi \in \mathbf{H}\}$. We identify the Lie algebra of $SU(2)$ with $\operatorname{Im}\mathbf{H}$ of imaginary quaternions as in [M]. We fix a local frame field $u: \mathbf{C}^2 (\subset CP^2) \to E|_{\mathbf{C}^2}$ defined by $u([1, W_1, W_2]) = (1 + r^2)^{-1/2}(1, W_1, W_2)$. Then, ∇_λ is defined on $\mathbf{C}^2$ by

$$\text{(2.1)} \quad \nabla_\lambda u = uA_\lambda\,,$$
$$A_\lambda = (1 + r^2 - \lambda^2)^{-1} \operatorname{Im}\{(\overline{W_1}\, dW_1 + \overline{W_2}\, dW_2) + j\lambda(-W_2\, dW_1 + W_1\, dW_2)\}\,,$$

where A_λ is a local $\operatorname{Im}\mathbf{H}$-valued 1-form. Note that this local connection extends to a connection ∇_λ over E and ∇_0 is a reducible connection. A_λ will be called a local connection form of ∇_λ with respect to u.

Let $\mathscr{A}$ be the space of self-dual connections on E. We define $SU(3)$-action on $\mathscr{A}$ by $g \cdot \nabla = \gamma^*_{g^{-1}}(g^{-1})^*\nabla$, where $\gamma_{g^{-1}}: E \to (g^{-1})^*E$ is a $SU(3)$-bundle equivalence and $(g^{-1})^*\nabla$ is the pull back of ∇ by g^{-1}. This means in local connection forms that

$$\text{(2.2)} \qquad g \cdot \nabla u = uA'\,, \quad A' = c^{-1}\,dc + c^{-1}(g^{-1})^* Ac\,.$$

where c is determined by $g^{-1}(u(w)) = u(g^{-1}w)c$ and $(g^{-1})^*A$ is the pull back of A by g^{-1}.

We define a smooth map $\tilde{F}: SU(3) \times (0, 1) \to \mathscr{A}$ by $\tilde{F}(g, \lambda) = g \cdot \nabla_\lambda$. Note that the $SU(3)$-action on $\mathscr{A}$ has $U(2)$ as isotoropy subgroup at ∇_λ and the image of $\tilde{F}$ is transverse to the action of the gauge transformation group $\mathscr{G}$. Moreover we have

THEOREM (Buchdahl [B], Furuta [F]). *The map $\tilde{F}$ induces an $SU(3)$-equivariant diffeomorphism $F\colon SU(3)/U(2) \times (0, 1) \cong \mathscr{M}^*$.*

3. Computation of the metrics

The metrics $(\mathscr{M}^*, g_J)$ (J = I, II and I–II) are $SU(3)$-invariant and F is $SU(3)$-equivariant. So, F^*g_J splits into $F^*g_J = \varphi_J(\lambda)\, d\lambda^2 + \psi_J(\lambda) g_{FS}$, because g_{FS} is a unique $SU(3)$-invariant metric on CP^2 up to constant multiple. Define $g_t^{-1} \in SU(3)$ and $v \in T_{(0,\lambda)}CP^2$ by

$$g_t^{-1} = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad v = \frac{\partial}{\partial t} g_t^{-1}[1, 0, 0]|_0$$

so that $g_{FS}(v, v) = 1$. Then, by the definition of the metrics

$$\text{(3.1)} \qquad \varphi_J(\lambda) = F^*g_J(\partial/\partial\lambda, \partial/\partial\lambda) = g_J(\rho_* \partial_\lambda \nabla_\lambda, \rho_* \partial_\lambda \nabla_\lambda) \quad \text{and}$$

$$\text{(3.2)} \qquad \psi_J(\lambda) = F^*g_J(v, v) = g_J(\rho^* V, \rho_* V), \quad V = \frac{\partial}{\partial t}(g_t^{-1}) \cdot \nabla_\lambda|_0,$$

because $F_*(\partial/\partial\lambda) = \rho_* \partial_\lambda \nabla_\lambda$ and $F_* v = \rho_* V$. Hereafter, we fix a local coordinate system $\mathbf{C}^2 \to CP^2$ defined by $(W_1, W_2) \to [1, W_1, W_2]$ and treat an element of $\Omega^p(\mathrm{ad}\, E)$ as a local $\mathrm{Im}\,\mathbf{H}$-valued p-form. By derivating A_λ by λ and denoting $Q_\lambda = 1 + r^2 - \lambda^2$, $\beta = \overline{W_1}\, dW_1 + \overline{W_2}\, dW_2$ and $\gamma = -W_2\, dW_1 + W_1\, dW_2$, we have

$$\partial_\lambda \nabla_\lambda = 2\lambda Q_\lambda^{-2}\, \mathrm{Im}\, \beta + (2\lambda^2 Q_\lambda^{-2} + Q_\lambda^{-1}) j\gamma .$$

Let $A_{(t)}$ be the local connection form of $(g_t^{-1}) \cdot \nabla_\lambda$ with respect to u. By (2.2), $A_{(t)} = c_t^{-1} dc_t + c_t^{-1}(g_t^{-1})^* A_\lambda c_t$, where $c_t = (\cos t - W_1 \sin t)/|\cos t - W_1 \sin t|$. By derivating $A_{(t)}$ by t, we have

$$V = -2\lambda^2 Q_\lambda^{-2} X_1\, \mathrm{Im}\, (\beta + j\lambda\gamma) + \lambda Q_\lambda^{-1}\, \mathrm{Im}\, (\lambda\, dW_1 + j\, dW_2).$$

Denoting $d^4 = dX_1 \wedge dX_2 \wedge dX_3 \wedge dX_4$ and $Q = 1 + r^2$, we note also that

$$\text{(3.3)} \quad dW_i \wedge *d\overline{W_i} = 2(1 + |W_i|^2) Q^{-2} d^4 \; (i = 1, 2), \quad dW_i \wedge *dW_j = 0 \; (i, j = 1, 2),$$
$$dW_i \wedge *d\overline{W_j} = 2W_i \overline{W_j} Q^{-2} d^4 \; (i \neq j) \quad \text{and} \quad d * dW_j = 0 .$$

We will prove the formulas on $\varphi_J(\lambda)$ and $\psi_J(\lambda)$ in the introduction by the following (1)–(6).

(1) $\psi_{\mathrm{I}}(\lambda)=\langle V^h, V^h\rangle$: Recall $\delta_{\nabla_\lambda}V=-*d_{\nabla_\lambda}*V=-*\{d*V+[A_\lambda, *V]\}$. Using (3.3) we get $d*V = j\{2\lambda W_2 Q_\lambda^{-2}Q^{-1}(\lambda^2Q^{-1}-1)\}d^4$, $[A_\lambda, *V] = \lambda Q_\lambda^{-2}\{4\lambda(\mathrm{Im}\,W_1)Q^{-2}+2jW_2(\lambda^2Q^{-2}+Q^{-1})\}d^4$ and therefore

$$\delta_{\nabla_\lambda}V=-*\{4\lambda^2Q_\lambda^{-2}Q^{-2}(\mathrm{Im}\,W_1+j\lambda W_2)d^4\}\,. \tag{3.4}$$

To describe the orthogonal projection, we need the following key observation:

(3.5) LEMMA. *Let* $X = (\lambda^2-3)^{-1}Q_\lambda^{-1}\,\mathrm{Im}\,\{\lambda^2(\lambda^2+1)W_1+2j\lambda^3W_2\}\in \Omega^0(\mathrm{ad}\,E)$. *Then,* $\delta_{\nabla_\lambda}d_{\nabla_\lambda}X=-*\{4\lambda^2Q_\lambda^{-2}Q^{-2}(\mathrm{Im}\,W_1+j\lambda W_2)d^4\}$.

This lemma is verified by a direct calculation based on the definition of d_{∇_λ} and δ_{∇_λ}.

From (3.4) and (3.5) we see that V^h is given by $V^h=V-d_{\nabla_\lambda}X$. Now $\langle V^h, V^h\rangle = \langle V-d_{\nabla_\lambda}X, V^h\rangle = \langle V, V^h\rangle = \langle V, V-d_{\nabla_\lambda}X\rangle = \langle V, V\rangle - \langle V, d_{\nabla_\lambda}X\rangle$. To compute $\langle V, d_{\nabla_\lambda}X\rangle$ we first calculate the integrand and get $\mathrm{Re}\,(X\wedge *d_{\nabla_\lambda}X)=-8(\lambda^2-3)^{-1}Q_\lambda^{-3}Q^{-2}\{\lambda^4(\lambda^2+1)X_2^2+2\lambda^6|W_2|^2\}d^4$ by using (3.3). Let $A(a,b,i)$ denote $\int_{C^2}r^{2i}Q_\lambda^{-a}Q^{-b}d^4$. Then, we have

$$\langle V, d_{\nabla_\lambda}X\rangle = 2\int_{CP^2}\mathrm{Re}\,(X\wedge *d_{\nabla_\lambda}X)=-(\lambda^2-3)^{-1}(10\lambda^6+2\lambda^4)A(3,2,1)\,.$$

Similarly,

$$\begin{aligned}\langle V, V\rangle &= -\int_{CP^2}\mathrm{Re}\,(V\wedge *V)\\ &= 6\lambda^4(\lambda^2-1)A(4,1,1)+(\lambda^4+2\lambda^2)\{A(2,2,1)+2A(2,2,0)\}\,.\end{aligned}$$

Thus, using another expression $A(a,b,i)=\lambda^{2-2(a+b)}\int_{1-\lambda^2}^{1}(y-(1-\lambda^2))^{i+1}\times(1-y)^{a+b-3-i}y^{-a}\,dy$, we obtain $\psi_{\mathrm{I}}(\lambda)$ in the introduction.

(2) $\varphi_{\mathrm{I}}(\lambda)=\langle(\partial_\lambda\nabla_\lambda)^h, (\partial_\lambda\nabla_\lambda)^h\rangle$: By the definition of δ_∇ we have $\delta_{\nabla_\lambda}(\partial_\lambda\nabla_\lambda)=-*\{d*\partial_\lambda A_\lambda+[A_\lambda, *\partial_\lambda A_\lambda]\}$. By a direct computation using (3.3) we have $d*\partial_\lambda A_\lambda=0$ and $[A_\lambda, *\partial_\lambda A_\lambda]=0$. So, we have $\delta_{\nabla_\lambda}(\partial_\lambda\nabla_\lambda)=0$. In particular, $(\partial_\lambda\nabla_\lambda)^h=\partial_\lambda\nabla_\lambda$. Then, $\varphi_{\mathrm{I}}(\lambda)$ is calculated by a similar method as in (1).

(3) (H. Doi) $\varphi_{\mathrm{II}}(\lambda)$: Let $F(\nabla)$ be a curvature form of a connection ∇, and let us denote $F(\nabla_\lambda)$ by F_λ. Since $d_{\nabla_\lambda}\partial_\lambda\nabla_\lambda=\partial_\lambda F_\lambda$, we have $\varphi_{\mathrm{II}}(\lambda)=\langle(\partial_\lambda F_\lambda)^h, (\partial_\lambda F_\lambda)^h\rangle$ by (3.1) and the definition of the metric of type II. By a direct computation we have

$$F_\lambda=(1-\lambda^2)Q_\lambda^{-2}\{K+2j\lambda\,dW_1\wedge dW_2\}\quad\text{and}$$
$$\partial_\lambda F_\lambda=2\lambda(1-\lambda^2-r^2)Q_\lambda^{-3}K+2\{4\lambda^2(1-\lambda^2)+(1-3\lambda^2)Q_\lambda\}Q_\lambda^{-3}j\,dW_1\wedge dW_2\,,$$

where K is $(1+r^2)^2$ times the Kähler form of g_{FS}, more explicitly, $K = \{(1+|W_2|^2)\,d\overline{W_1} \wedge dW_1 + (1+|W_1|^2)\,d\overline{W_2} \wedge dW_2 - \overline{W_1}W_2\,d\overline{W_2} \wedge dW_1 - \overline{W_2}W_1\,d\overline{W_1} \wedge dW_2\}$. Since $*\partial_\lambda F_\lambda = \partial_\lambda F_\lambda$, we have $\delta_{\nabla_\lambda}\delta_{\nabla_\lambda}\partial_\lambda F_\lambda = *d_{\nabla_\lambda}d_{\nabla_\lambda}\partial_\lambda F_\lambda = *[F_\lambda, \partial_\lambda F_\lambda]$ which vanishes because $[K, \boldsymbol{j}\,dW_1 \wedge dW_2] = 0$. This means that $(\partial_\lambda F_\lambda)^h = \partial_\lambda F_\lambda$. Using $A(a, b, i)$, we compute the L^2-norm of $\partial_\lambda F_\lambda$ and we obtain $\varphi_{\mathrm{II}}(\lambda)$.

(4) (H. Doi) $\psi_{\mathrm{II}}(\lambda) = \langle (XF)^h, (XF)^h \rangle$, where $XF = (\partial/\partial t)F(g_t^{-1}\nabla_\lambda)|_0$: Since $F(g_t^{-1}\nabla_\lambda) = \mathrm{Ad}\,(c_t^{-1})(g_t^{-1})^*F_\lambda$, where $(g_t^{-1})^*F_\lambda$ is a pull back of F_λ by g_t^{-1}. Let $\alpha = -4Q_\lambda^{-3}X_1\lambda^2K - 2\boldsymbol{j}\lambda Q_\lambda^{-3}X_1(Q_\lambda + 4\lambda^2)\,dW_1 \wedge dW_2$. Since α is self-dual, $\delta_{\nabla_\lambda}\delta_{\nabla_\lambda}\alpha = *[F_\lambda, \alpha]$. We note $[K, \boldsymbol{j}\,dW_1 \wedge dW_2] = 0$ again and get $[F_\lambda, \alpha] = 0$. Hence, $\alpha \in \mathrm{Ker}\,\delta_{\nabla_\lambda}\delta_{\nabla_\lambda}$. By a direct computation, we obtain

$$XF = \frac{\partial}{\partial t}c_t^{-1}|_0 F_\lambda + F_\lambda \frac{\partial}{\partial t}c_t|_0 + \frac{\partial}{\partial t}(g_t^{-1})^*F_\lambda|_0 = \frac{\partial}{\partial t}(g_t^{-1})^*F_\lambda|_0 - [F_\lambda, \mathrm{Im}\,W_1] \quad \text{and}$$

$$\frac{\partial}{\partial t}(g_t^{-1})^*F_\lambda|_0 = (1-\lambda^2)\alpha + (1-\lambda^2)\boldsymbol{k}(-6Q_\lambda^{-2}\lambda X_2\,dW_1 \wedge dW_2)\,.$$

Since $d_{\nabla_\lambda}d_{\nabla_\lambda}\boldsymbol{i} = [F_\lambda, \boldsymbol{i}] = -4(1-\lambda^2)Q_\lambda^{-2}\lambda\boldsymbol{k}\,dW_1 \wedge dW_2$, we can write $XF = (1-\lambda^2)\alpha + d_{\nabla_\lambda}d_{\nabla_\lambda}\beta$, for some $\beta \in \Omega^0(\mathrm{ad}\,E)$. This implies that $(XF)^h = (1-\lambda^2)\alpha$. By computing the L^2-norm of $(1-\lambda^2)\alpha$, we obtain $\psi_{\mathrm{II}}(\lambda)$.

(5) $\varphi_{\mathrm{I-II}}(\lambda)$: Since $(\partial_\lambda\nabla_\lambda)^h = \partial_\lambda\nabla_\lambda$, we have $\varphi_{\mathrm{I-II}}(\lambda) = \varphi_{\mathrm{II}}(\lambda)$.

(6) $\psi_{\mathrm{I-II}}(\lambda)$: Since $*d_{\nabla_\lambda}V^h = d_{\nabla_\lambda}V^h$, we have

$$\psi_{\mathrm{I-II}}(\lambda) = \langle d_{\nabla_\lambda}V^h, d_{\nabla_\lambda}V^h \rangle = -2\int_{CP^2} \mathrm{Re}\,(d_{\nabla_\lambda}V^h \wedge d_{\nabla_\lambda}V^h)\,.$$

The computation of this integral is complicated and we used the formula processing software REDUCE 3.2 to complete it.

4. Asymptotic behavior of the metrics

We will give asymptotic formulas of the metrics near the cone point and the collar. We study their sectional curvatures, too.

As $\lambda \to 0$ (near the cone point) the metrics are asymptotically

$$\begin{aligned} g_{\mathrm{I}} &\sim 2\pi^2(27\lambda^4 + 20\lambda^2 + 10)\,d\lambda^2/15 + 2\pi^2(5\lambda^4 + 6\lambda^2)g_{\mathrm{FS}}/9, \\ g_{\mathrm{II}} &\sim 16\pi^2(16\lambda^4 + 10\lambda^2 + 5)\,d\lambda^2/15 + \pi^2(24\lambda^4 + 8\lambda^2)g_{\mathrm{FS}}/3 \quad \text{and} \\ g_{\mathrm{I-II}} &\sim 16\pi^2(16\lambda^4 + 10\lambda^2 + 5)\,d\lambda^2/15 + \pi^2(56\lambda^4 + 48\lambda^2)g_{\mathrm{FS}}/9\,. \end{aligned}$$

We take a new parameter Y defined by $Y^2 = 1 - \lambda^2$, that is, $Y = Z^{1/2}$. Then, as $Y \to 0$ (near the collar) the metrics are asymptotically

$$g_{\mathrm{I}} \sim 8\pi^2((26Y^4 + 6Y^2)\log Y + 15Y^4 + 6Y^2 + 1)\,dY^2$$
$$+ 2\pi^2(-12Y^4 \log Y - 3Y^4 - 6Y^2 + 2)g_{\mathrm{FS}}\,,$$
$$g_{\mathrm{II}} \sim 16\pi^2(5Y^4 + 4Y^2 + 6)\,dY^2/15Y^2 + 8\pi^2(Y^4 - 16Y^2 + 12)g_{\mathrm{FS}}/15Y^2 \quad \text{and}$$
$$g_{\mathrm{I-II}} \sim 16\pi^2(5Y^4 + 4Y^2 + 6)\,dY^2/15Y^2 + 48\pi^2(2Y^4 - 2Y^2 + 1)g_{\mathrm{FS}}/5Y^2\,.$$

This implies that g_{I} is C^1-asymptotic to the product metric near the collar and extends to the boundary of collar in C^1 sense.

Applying the well-known lemma below to the (asymptotic) formulas of the metrics, we can easily get Proposition in the introduction.

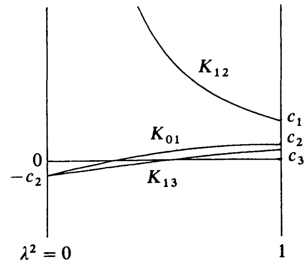
(4.1) LEMMA. *$K_{\mathrm{J}}(\partial/\partial\lambda, X) = \varphi_{\mathrm{J}}^{-1}\psi_{\mathrm{J}}^{-1}\{-\psi_{\mathrm{J}}''/2 + \psi_{\mathrm{J}}'^2\psi_{\mathrm{J}}^{-1}/4 + \varphi_{\mathrm{J}}'\psi_{\mathrm{J}}'\varphi_{\mathrm{J}}^{-1}/4\}$ and $K_{\mathrm{J}}(X, Y) = \psi_{\mathrm{J}}^{-1}\{K_{\mathrm{FS}}(X, Y) - \psi_{\mathrm{J}}'^2\varphi_{\mathrm{J}}^{-1}\psi_{\mathrm{J}}^{-1}/4\}$, where $X, Y \in TCP^2$.*

We recall the results of Groisser-Parker [GP2] on the metric of type I which can be applied also to a general metric on CP^2: There are a number r_0, a neighborhood U of the cone point $[V_0]$ in $\mathscr{M}$ and a diffeomorphism F_0: $CP^2 \times (0, r_0) \to U - \{[V_0]\}$ so that g_{I} satisfies $F_0^* g_{\mathrm{I}} = dr^2 + r^2(g_{\mathrm{FS}} + O(r^2))$ and the sectional curvature K_0 of $F_0^* g_{\mathrm{I}}$ satisfies $K_0(\partial/\partial r, X) = O(1)$ and $K_0(X, Y) = (K_{\mathrm{FS}}(X, Y) - 1)/r^2 + O(1)$ for $X, Y \in TCP^2$ as $r \to 0$ in this coordinate system. Near the collar g_{I} is C^0-asymptotic to the product metric $4\pi^2(2dt^2 + g_{\mathrm{FS}})$ for some coordinate system. We find for example that the constants $O(1)$ in the curvature K_0 are equal to $-3/8\pi^2$ in our standard metric case.

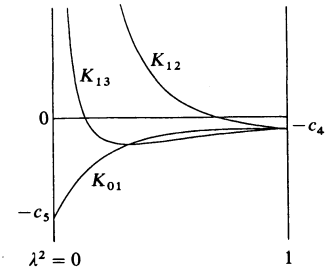
Followings are the graphs showing the behavior of the sectional curvatures. Let $X_0 = \lambda^2$. In each case $\mathrm{J} = \mathrm{I}$, II and I–II, let K_{ij} denote $K_{\mathrm{J}}(\partial/\partial X_i, \partial/\partial X_j)$ and put $c_1 = 1/\pi^2$, $c_2 = 3/8\pi^2$, $c_3 = 1/4\pi^2$, $c_4 = 5/32\pi^2$, $c_5 = 21/16\pi^2$ and $c_6 = 9/32\pi^2$. Suppose the metric is given by $g_{\mathrm{J}} = \tilde{\varphi}_{\mathrm{J}}(X_0)\,dX_0^2 + \psi_{\mathrm{J}}(X_0)g_{\mathrm{FS}}$ and let $e_0 = \tilde{\varphi}_{\mathrm{J}}^{-1/2}\partial/\partial X_0$ and $e_i = \psi_{\mathrm{J}}^{-1/2}\partial/\partial X_i$ $(1 \le i \le 4)$. Then, K_{J} is calculated by

$$K_{\mathrm{J}}(a_0e_0 + a_1e_1, b_0e_0 + b_1e_1 + b_2e_2 + b_3e_3) = (a_0^2 + b_0^2)K_{01} + a_1^2b_2^2K_{12} + a_1^2b_3^2K_{13}\,,$$

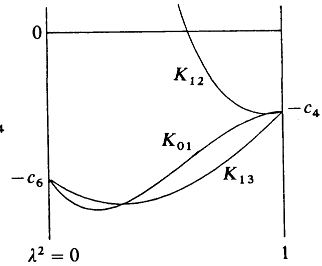
where $a_0^2 + a_1^2 = 1$, $b_0^2 + b_1^2 + b_2^2 + b_3^2 = 1$ and $a_0b_0 + a_1b_1 = 0$. Note that $\lambda^2 = 0$ corresponds to the cone point in these graphs.



Case of g_{I}



Case of g_{II}



Case of $g_{\mathrm{I-II}}$

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