HIROSHIMA MATH. J.
20 (1990), 399–405

Exactness and Bernoulliness of generalized random dynamical systems

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(Received August 29, 1989)

Recently the second author of the present paper proposed the idea of generalized random dynamical systems and investigated their ergodic properties in [3]. This paper is a supplementary note for [3]. We will investigate the exactness and the Bernoulliness of skew product transformations associated with generalized random dynamical systems.

§1. Preliminaries

Let $(S, \mathscr{B}, \mu)$ and $(Y, \mathscr{F}, \nu)$ be standard probability spaces. We consider $(S, \mathscr{B}, \mu)$ as a phase space and $(Y, \mathscr{F}, \nu)$ as a parameter space. Namely for each $y \in Y$ a measure-preserving transformation φ_y on $(S, \mathscr{B}, \mu)$ is given. We assume that the mapping $(s, y) \mapsto \varphi_y(s)$ is $\mathscr{B} \times \mathscr{F}/\mathscr{B}$-measurable. We are concerned with the behavior of the random orbit

$$X_n(s) = \varphi_{y_n} \circ \varphi_{y_{n-1}} \circ \cdots \circ \varphi_{y_1}(s), \qquad s \in S, \qquad y_1, \ldots, y_n \in Y, \qquad n \geq 1,$$

where $y_1, \ldots, y_n$ are taken randomly in the following manner. There are given a family of probability density functions $\{\gamma(s, y), s \in S\}$ on Y:

$$\gamma(s, y) \geq 0, \qquad \int_Y \gamma(s, y)\, d\nu(y) = 1, \qquad s \in S,$$

and a sub-σ-field $\mathscr{B}_0 \subset \mathscr{B}$ such that

(i) $\gamma(s, y)$ is $\mathscr{B}_0 \times \mathscr{F}$-measurable

and

(ii) $\varphi_y^{-1}\mathscr{B}$ and $\mathscr{B}_0$ are independent for each $y \in Y$.

Each y_k $(k \geq 1)$ is choosen according to the probability measure $\gamma(X_{k-1}(s), y)\, d\nu(y)$ where $X_0(s) = s$. Then $X = \{X_n(s), n \geq 0\}$ becomes a stationary Markov chain with the transition probability

$$P(s, B) = \int_Y 1_B(\varphi_y(s))\gamma(s, y)\, d\nu(y), \qquad B \in \mathscr{B},$$

and the stationary measure μ. Let T be the corresponding Markov operator:

$$Tf(s) = \int_S f(t)P(s, dt) = \int_Y f(\varphi_y(s))\gamma(s, y)\, d\nu(y), \qquad f \in L^1(S, \mu).$$

The quadruplet $\mathscr{D} = ((S, \mathscr{B}, \mathscr{B}_0, \mu), (Y, \mathscr{F}, \nu), \{\gamma(s, y), s \in S\}, \{\varphi_y, y \in Y\})$ is called a *generalized random dynamical system.*

Next let $(Y^*, \mathscr{F}^*) = \prod_{n=1}^{\infty} (Y_n, \mathscr{F}_n)$ be the product measurable space of the spaces $(Y_n, \mathscr{F}_n) = (Y, \mathscr{F})$, $n \geq 1$. Let ψ be the shift transformation on Y^* defined by

$$(\psi y^*)_n = y_{n+1}\,, \quad n \geq 1\,, \qquad y^* = (y_n)_{n \geq 1} \in Y^*\,.$$

Let $\Omega = S \times Y^*$, $\mathscr{M} = \mathscr{B} \times \mathscr{F}^*$ and define for $E \in \mathscr{B}$, $F = F_1 \times F_2 \times \cdots \times F_n \times \prod_{i=n+1}^{\infty} Y_i$, $F_k \in \mathscr{F}$ $(k = 1, \ldots, n)$,

$$P(E \times F) = \int_S \int_{Y^n} 1_E(s) \prod_{k=1}^{n} 1_{F_k}(y_k) \gamma(\varphi_{y_{n-1}} \circ \cdots \circ \varphi_{y_1}(s), y_n)$$

$$\ldots \gamma(s, y_1) \prod_{k=1}^{n} d\nu(y_k)\, d\mu(s)\,.$$

Then P becomes a probability measure on $(\Omega, \mathscr{M})$ by the Kolmogorov extension theorem. Define

$$\varphi^*(s, y^*) = (\varphi_{y_1}(s), \psi y^*)\,, \qquad y^* = (y_n)_{n \geq 1} \in Y^*.$$

Then φ^* is a measure-preserving transformation on $(\Omega, \mathscr{M}, P)$ (cf. [3]). The mapping φ^* is called the *skew product transformation* associated with $\mathscr{D}$.

Set $\mathscr{B}_k = \bigvee_{y_1, \ldots, y_k} \varphi_{y_1}^{-1} \ldots \varphi_{y_k}^{-1} \mathscr{B}_0$, $k \geq 1$, and $\mathscr{B}_n^m = \bigvee_{k=n}^{m} \mathscr{B}_k$, $0 \leq n \leq m \leq \infty$. Then we have

Theorem 1 ([3] Theorems 5 and 6). (i) *T is mixing*:

$$\lim_{n \to \infty} \int_S (T^n f(s)) g(s)\, d\mu(s) = \int_S f\, d\mu \int_S g\, d\mu\,, \qquad f, g \in L^\infty(S, \mu)\,,$$

if and only if φ^* is mixing:

$$\lim_{n \to \infty} \int_\Omega F(\varphi^{*n}\omega) G(\omega)\, dP(\omega) = \int_\Omega F\, dP \int_\Omega G\, dP\,, \qquad F, G \in L^\infty(\Omega, P)\,.$$

(ii) *Assume the following conditions*:

(A) *the σ-fields $\mathscr{B}_0^{n-1}$ and $\varphi_{y_1}^{-1} \ldots \varphi_{y_n}^{-1} \mathscr{B}$ are independent for any fixed $y_1, \ldots, y_n \in Y$, $n \geq 1$, and*

(B) $\mathscr{B}_0^\infty = \mathscr{B}$.

Then φ^* is mixing.

§2. Exactness

In this section we will show that a stronger conclusion about φ^* than that of (ii) in Theorem 1 holds under the conditions (A) and (B).

Let $Y_n(y^*) = y_n$ denote the n-th coordinate function of $y^* = (y_n)_{n\geq 1} \in Y^*$, and $\mathscr{F}_n^m = \sigma(\{Y_n, \dots, Y_m\})$ the σ-field generated by $\{Y_n, \dots, Y_m\}$ for $1 \leq n \leq m \leq \infty$. Especially we see $\mathscr{F}_0^\infty = \mathscr{F}^*$.

LEMMA 1. *Under the condition* (A), *the sub-σ-fields $\mathscr{B}_0^p \times \mathscr{F}_1^q$ and $\varphi^{*-n}\mathscr{M}$ are independent for all $n \geq \max(p+1, q)$.*

PROOF. Let $F(s, y^*) = f(s, y_1, \dots, y_q)$ be a $\mathscr{B}_0^p \times \mathscr{F}_1^q$-measurable bounded function and $G(s, y^*) = g(s, y_1, \dots, y_m)$ be a $\mathscr{B} \times \mathscr{F}_1^m$-measurable bounded function where $m \geq 1$ is arbitrary. Then we have

$$(1)\quad \iint_{S\times Y^*} F(s, y^*)G(\varphi^{*n}(s, y^*))\, dP(s, y^*)$$
$$= \int_{Y^{n+m}} \Big[\int_S f(s, y_1, \dots, y_q) g(\varphi_{y_n} \circ \cdots \circ \varphi_{y_1}(s), y_{n+1}, \dots, y_{n+m})$$
$$\times\, \gamma_m(y_{n+1}, \dots, y_{n+m}; \varphi_{y_n} \circ \cdots \circ \varphi_{y_1}(s))\gamma_n(y_1, \dots, y_n; s)\, d\mu(s)\Big] \prod_{k=1}^{n+m} d\nu(y_k)$$

where

$$\gamma_k(y_1, \dots, y_k; s) = \gamma(\varphi_{y_{k-1}} \circ \cdots \circ \varphi_{y_1}(s), y_k) \dots \gamma(s, y_1)$$

which is $\mathscr{B}_0^{k-1}$-measurable for any fixed $y_1, \dots, y_k \in Y$, and it holds that

$$\gamma_{n+m}(y_1, \dots, y_{n+m}; s) = \gamma_m(y_{n+1}, \dots, y_{n+m}; \varphi_{y_n} \circ \cdots \circ \varphi_{y_1}(s))\gamma_n(y_1, \dots, y_n; s)\,.$$

Suppose $n \geq \max(p+1, q)$. Then for any fixed $y_1, \dots, y_{n+m} \in Y$, the s-function $f(s, y_1, \dots, y_q)\gamma_n(y_1, \dots, y_n; s)$ is $\mathscr{B}_0^{n-1}$-measurable and the s-function $g(\varphi_{y_n} \circ \cdots \circ \varphi_{y_1}(s), y_{n+1}, \dots, y_{n+m})\gamma_m(y_{n+1}, \dots, y_{n+m}; \varphi_{y_n} \circ \cdots \circ \varphi_{y_1}(s))$ is $\varphi_{y_1}^{-1} \dots \varphi_{y_n}^{-1}\mathscr{B}$-measurable, and hence they are independent. Therefore we obtain

$$[\dots] \text{ in the right hand side of (1)}$$
$$= \int_S f(s, y_1, \dots, y_q)\gamma_n(y_1, \dots, y_n; s)\, d\mu(s)$$
$$\times \int_S g(s, y_{n+1}, \dots, y_{n+m})\gamma_m(y_{n+1}, \dots, y_{n+m}; s)\, d\mu(s)\,,$$

and so

$$\iint_{S\times Y^*} F(s, y^*)G(\varphi^{*n}(s, y^*))dP(s, y^*)$$
$$= \int_{Y^n} \Big[\int_S f(s, y_1, \dots, y_q)\gamma_n(y_1, \dots, y_n; s)\, d\mu(s)\Big] \prod_{k=1}^{n} d\nu(y_k)$$

$$\times \int_{Y^m} \left[\int_S g(s, y_{n+1}, \dots, y_{n+m}) \gamma_m(y_{n+1}, \dots, y_{n+m}; s) \, d\mu(s) \right] \prod_{k=1}^m dv(y_{n+k})$$

$$= \int\int_{S\times Y^*} F(s, y^*) \, dP(s, y^*) \int\int_{S\times Y^*} G(\varphi^{*n}(s, y^*)) \, dP(s, y^*) \, .$$

Thus we see that $\mathscr{B}_0^p \times \mathscr{F}_1^q$ and $\varphi^{*-n}(\mathscr{B} \times \mathscr{F}_1^m)$ are independent for all $m \geq 1$, and so $\mathscr{B}_0^p \times \mathscr{F}_1^q$ and $\varphi^{*-n}\mathscr{M} = \varphi^{*-n}(\mathscr{B} \times \mathscr{F}_1^\infty)$ are independent. The proof is completed.

REMARK. Let us consider the condition

(A′) the σ-fields $\mathscr{B}_k$, $k \geq 0$, are mutually independent,

in stead of (A). Under the condition (B), (A′) implies (A). But we don't know whether (A′) is actually stronger than (A).

Assume the conditions (A) and (B) are satisfied. Then by Lemma 1, $\mathscr{B}_0^p \times \mathscr{F}_1^q$ and $\varphi^{*-n}\mathscr{M}$ are independent for all $n \geq \max(p+1, q)$. Therefore $\mathscr{B}_0^p \times \mathscr{F}_1^q$ and $\bigcap_{n\geq 0} \varphi^{*-n}\mathscr{M}$ are independent for all $p \geq 0$ and $q \geq 1$. This means by (B) that $\mathscr{M} = \mathscr{B} \times \mathscr{F}^*$ and $\bigcap_{n\geq 0} \varphi^{*-n}\mathscr{M}$ are independent. Thus we have

THEOREM 2. *Under the conditions* (A) *and* (B), $(\Omega, \mathscr{M}, P, \varphi^*)$ *is exact*: $\bigcap_{n=0}^\infty \varphi^{*-n}\mathscr{M}$ *is trivial.*

Since the exactness of a transformation implies the mixing property (cf. [2]), the second assertion (ii) of Theorem 1 follows from this theorem as a corollary.

§3. Factor transformations

Now we consider a factor transformation of $(\Omega, \mathscr{M}, P, \varphi^*)$. Let $(S^*, \mathscr{B}^*) = (S, \mathscr{B})^N$ be the product measurable space of $(S, \mathscr{B})$, where $N = \{0, 1, 2, \dots\}$. Let θ be the shift transformation on S^*: $(\theta s^*)_n = s_{n+1}$, $n \geq 0$, $s^* = (s_n)_{n\geq 0} \in S^*$. The transition probability $P(s, B)$ and the stationary measure μ given in §1 induce a Markov measure Q on $(S^*, \mathscr{B}^*)$: for $B = B_0 \times \cdots \times B_n \times \Pi_{i=n+1}^\infty S_i$, $B_k \in \mathscr{B}$, $0 \leq k \leq n$, $S_i = S$, $i \geq n+1$,

$$Q(B) = \int_{B_0} d\mu(s_0) \int_{B_1} P(s_0, ds_1) \dots \int_{B_n} P(s_{n-1}, ds_n) \tag{2}$$

$$= \int_S \int_{Y^n} 1_{B_0}(s) \prod_{k=1}^n 1_{B_k}(\varphi_{y_k} \circ \cdots \circ \varphi_{y_1}(s))$$

$$\times \gamma_n(y_1, \dots, y_n; s) \prod_{k=1}^n dv(y_k) \, d\mu(s) \, .$$

The measure Q on $(S^*, \mathscr{B}^*)$ is nothing else but the probability law of the Markov chain $\{X_n(s)\}_{n\geq 0}$. Clearly θ is a measure-preserving transformation on $(S^*, \mathscr{B}^*, Q)$, which is called a Markov shift.

Let $\pi: \Omega \to S^*$ be a mapping defined by $\pi(s, y^*) = (s, X_1(s), \dots, X_n(s), \dots) = (s, \varphi_{y_1}(s), \dots, \varphi_{y_n} \circ \cdots \circ \varphi_{y_1}(s), \dots)$, $y^* = (y_n)_{n\geq 1} \in Y^*$. It is easy to see that $\pi \circ \varphi^* = \theta \circ \pi$ and $\pi^{-1}(\mathscr{B}^*) \subset \mathscr{M}$. We see also $P \circ \pi^{-1} = Q$. Indeed, for any bounded measurable function $H(s^*) = h(s_0, s_1, \dots, s_n)$, $s^* = (s_k)_{k\geq 0} \in S^*$, we have

$$\begin{aligned}
&\int_\Omega H(\pi(s, y^*))\, dP(s, y^*) \\
&= \int_S \int_{Y^n} h(s, \varphi_{y_1}(s), \dots, \varphi_{y_n} \circ \cdots \circ \varphi_{y_1}(s))\gamma_n(y_1, \dots, y_n; s) \textstyle\prod_{k=1}^n d\nu(y_k)\, d\mu(s) \\
&= \int_{S^*} H(s^*)\, dQ(s^*)\,.
\end{aligned}$$

Hence $(S^*, \mathscr{B}^*, Q, \theta)$ is a factor transformation of $(\Omega, \mathscr{M}, P, \varphi^*)$, namely the former is an image of the latter, under the mapping π (cf. [2]). Therefore by Theorem 2 we have

THEOREM 3. *Under the conditions* (A) *and* (B), *the Markov shift* $(S^*, \mathscr{B}^*, Q, \theta)$ *is exact*: $\bigcap_{n=0}^\infty \theta^{-n}\mathscr{B}^* =$ *trivial.*

Next we consider a factor transformation of $(S^*, \mathscr{B}^*, Q, \theta)$. Here we don't assume the conditions (A) and (B). Let $\mathscr{B}_0^* = \mathscr{B}_0^N$ be the product σ-field of $\mathscr{B}_0$ which is given in §1. Then $(S^*, \mathscr{B}_0^*, Q, \theta)$ is a factor transformation of $(S^*, \mathscr{B}^*, Q, \theta)$, and we have

THEOREM 4. *The factor transformation* $(S^*, \mathscr{B}_0^*, Q, \theta)$ *is a Bernoulli transformation* (cf. [1]).

PROOF. In the equation (2), take $B_k \in \mathscr{B}_0$, $0 \leq k \leq n$, and change the order of integrations. In the integrand of the obtained integral we see that $\gamma_n(y_1, \dots, y_n; s) = \gamma_{n-1}(y_2, \dots, y_n; \varphi_{y_1}(s))\gamma(s, y_1)$, the s-functions $1_{B_0}(s)\gamma(s, y_1)$ and the remainder are independent (because the former is $\mathscr{B}_0$-measurable and the latter is $\varphi_{y_1}^{-1}\mathscr{B}$-measurable) and φ_{y_1} is μ-preserving. Hence we get

$$\begin{aligned}
Q(B) = \mu(B_0) \int_S \int_{Y^{n-1}} &1_{B_1}(s) \textstyle\prod_{k=2}^n 1_{B_k}(\varphi_{y_k} \circ \cdots \circ \varphi_{y_2}(s)) \\
&\times \gamma_{n-1}(y_2, \dots, y_n; s) \textstyle\prod_{k=2}^n d\nu(y_k)\, d\mu(s)\,.
\end{aligned}$$

Repeating the same arguments, we obtain finally

$$Q(B) = \textstyle\prod_{k=0}^n \mu(B_k)\,.$$

Since n is arbitrary, this completes the proof.

§4. Bernoulliness

Let us consider $(\Omega, \mathscr{M}, P, \varphi^*)$. Define $\mathscr{G} = \mathscr{B}_0 \times \mathscr{F}_1^1$. Let $F(s, y^*) = f(s, y_1)$ be $\mathscr{G}$-measurable. Then $F(\varphi^{*k}(s, y^*)) = f(\varphi_{y_k} \circ \cdots \circ \varphi_{y_1}(s), y_{k+1})$ is $\mathscr{B}_0^k \times \mathscr{F}_1^{k+1}$-measurable. Hence $\varphi^{*-k}\mathscr{G} \subset \mathscr{B}_0^k \times \mathscr{F}_1^{k+1}$ and so $\bigvee_{k=0}^{n-1} \varphi^{*-k}\mathscr{G} \subset \mathscr{B}_0^{n-1} \times \mathscr{F}_1^n$. On the other hand we have $\bigvee_{k=n}^{\infty} \varphi^{*-k}\mathscr{G} \subset \varphi^{*-n}\mathscr{M}$. By Lemma 1 we obtain

LEMMA 2. *Under the condition* (A), *the sub-σ-fields* $\varphi^{*-n}\mathscr{G}$, $n \geq 0$, *are mutually independent.*

If $\bigvee_{n=0}^{\infty} \varphi^{*-n}\mathscr{G} = \mathscr{M}$ holds, then $\mathscr{G}$ is called a generator for φ^*. In this case φ^* becomes a Bernoulli transformation by Lemma 2. In the following theorem, we give some sufficient conditions for $\mathscr{G}$ to be a generator for φ^*.

THEOREM 5. *In addition to the conditions* (A) *and* (B), *we assume the following conditions*:

(C) *Y is a countable set, and*

(D) *for any $n \geq 1$, $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in Y$ and $B \in \mathscr{B}_0$, there exists $B' \in \mathscr{B}_0$ such that $\varphi_{b_1}^{-1} \ldots \varphi_{b_n}^{-1}(B') = \varphi_{a_1}^{-1} \ldots \varphi_{a_n}^{-1}(B)$.*

Then $(\Omega, \mathscr{M}, P, \varphi^*)$ is a Bernoulli transformation with the generator $\mathscr{G}$.

PROOF. It remains to prove that $\mathscr{G}$ is a generator for φ^*. To do this it suffices to show that $\mathscr{B}_n \times \mathscr{F}_1^{n+1} \subset \bigvee_{k=0}^{n} \varphi^{*-k}\mathscr{G}$ for all $n \geq 0$. Consider a typical element $A = \varphi_{a_1}^{-1} \ldots \varphi_{a_n}^{-1}(B) \times \{y^* = (y_k)_{k \geq 1} \in Y^*; y_1 = b_1, \ldots, y_{n+1} = b_{n+1}\} \in \mathscr{B}_n \times \mathscr{F}_1^{n+1}$ where $B \in \mathscr{B}_0$. By the condition (D) there is $B' \in \mathscr{B}_0$ such that $\varphi_{b_1}^{-1} \ldots \varphi_{b_n}^{-1}(B') = \varphi_{a_1}^{-1} \ldots \varphi_{a_n}^{-1}(B)$. Hence we have

$$\begin{aligned} 1_A(s, y^*) &= 1_{\varphi_{b_1}^{-1} \ldots \varphi_{b_n}^{-1}(B')}(s) \prod_{k=1}^{n+1} 1_{\{y_y = b_k\}}(y^*) \\ &= 1_{\varphi_{y_1}^{-1} \ldots \varphi_{y_n}^{-1}(B')}(s) \prod_{k=1}^{n+1} 1_{\{y_k = b_k\}}(y^*) \\ &= 1_{B'}(\varphi_{y_n} \circ \cdots \circ \varphi_{y_1}(s)) \prod_{k=1}^{n+1} 1_{\{y_1 = b_k\}}(\psi^{k-1} y^*) . \end{aligned}$$

Put $F(s, y^*) = 1_{B'}(s) 1_{\{y_1 = b_{n+1}\}}(y^*)$ and $G_k(s, y^*) = 1_{\{y_1 = b_k\}}(y^*)$, $1 \leq k \leq n$. Then $F(s, y^*)$ and $G_k(s, y^*)$, $1 \leq k \leq n$, are $\mathscr{G}$-measurable, and hence

$$1_A(s, y^*) = F(\varphi^{*n}(s, y^*)) \prod_{k=1}^{n} G_k(\varphi^{*(k-1)}(s, y^*))$$

is $\bigvee_{k=0}^{n} \varphi^{*-k}\mathscr{G}$-measurable. Thus $A \in \bigvee_{k=0}^{n} \varphi^{*-k}\mathscr{G}$. Noting the condition (C) we see $\mathscr{B}_n \times \mathscr{F}_1^{n+1} \subset \bigvee_{k=0}^{n} \varphi^{*-k}\mathscr{G}$ by a routine argument. We have proved the theorem.

REMARK. If every φ_y $(y \in Y)$ is invertible then the condition (D) is satisfied.

EXAMPLE. Let us consider the following generalized random dynamical system:

(i) $S = [0, 1)$, $\mathscr{B}$ = the Borel field of $[0, 1)$ and μ = the Lebesgue measure on $[0, 1)$.

(ii) $Y = \{0, 1\}$, $\mathscr{F}$ = the all subsets of Y and $\nu(\{0\}) = \nu(\{1\}) = 1/2$.

(iii) $\varphi_0(s) = 3s$ (mod 1) and $\varphi_1(s) = 3(1 - s)$ (mod 1).

(iv) $\mathscr{B}_0$ = the field generated by $[0, 1/3)$, $[1/3, 2/3)$ and $[2/3, 1)$.

(v) $\gamma(s, 0) = 2/3$ $(0 \leq s < 1/3)$, $= 1$ $(1/3 \leq s < 2/3)$, $= 4/3$ $(2/3 \leq s < 1)$, and $\gamma(s, 1) = 2 - \gamma(s, 0)$.

This system satisfies clearly all the conditions (A) $\sim$ (D), and so the associated skew product transformation φ^* is a Bernoulli transformation.

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