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Nonnegative entire solutions of a class of degenerate semilinear elliptic equations

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1. Introduction

This paper is concerned with the existence and qualitative behavior of nonnegative entire solutions of the degenerate elliptic equation

$$\text{(A)} \qquad \Delta(u^m) + u(1-u)(u-a) = 0\,, \qquad x \in R^n\,, \qquad n \geq 2\,,$$

where m and a are positive constants. By a radial entire solution of (A) is meant a function $u \in C(R^n)$ depending only on $|x|$ such that $u^m \in C^2(R^n)$ and that (A) is satisfied at every point of R^n.

The one-dimensional case of (A) has been studied by Aronson, Crandall and Peletier [1], who have shown, among other things, that (A) ($n = 1$) has nonnegative radial entire solutions u with compact support provided $m > 1$ and $0 < a < (m+1)/(m+3)$. Our purpose here is to extend some of the results of [1] to the higher dimensional case ($n \geq 2$) of (A) by proving the theorem below.

THEOREM. *Let $0 < a < (m+1)/(m+3)$. Then, there exists a constant $u_* \in (0, 1)$ such that* (A) *has a nonnegative radial entire solution $u(x)$ satisfying $u(0) = u_0$ if $0 < u_0 \leq u_*$, and* (A) *has no nonnegative entire solution $u(x)$ satisfying $u(0) = u_0$ if $u_* < u_0 < 1$. Furthermore, the following statements hold.*

(i) *If $0 < u_0 < u_*$, the radial entire solution $u(x)$ satisfying $u(0) = u_0$ oscillates around a and converges to a as $|x| \to \infty$.*

(ii) *The radial entire solution $u(x)$ satisfying $u(0) = u_*$ decreases monotonically to zero as $|x| \to \infty$. This solution has compact support if $m > 1$.*

The substitution $v = u^m$ reduces (A) to

$$\text{(B)} \qquad \Delta v + v^{1/m}(1 - v^{1/m})(v^{1/m} - a) = 0\,, \qquad x \in R^n\,, \qquad n \geq 2$$

which is formally a special case of the equation

$$\text{(C)} \qquad \Delta v + f(v) = 0\,, \qquad x \in R^n, \qquad n \geq 2\,.$$

Although there is a vast literature devoted to the investigation of (C) from various viewpoints (see e.g. [1–6, 13–18]), none of the existing results for (C) seems to be applicable to establish the existence of entire solutions of (B) because of lack of smoothness of the function $f(v) = v^{1/m}(1 - v^{1/m})(v^{1/m} - a)$ at $v = 0$. We therefore attempt here to develop an existence theory of nonnegative entire solutions for a class of equations of the form (C) including (B) as a particular case, so that the above-mentioned theorem immediately follows. This is accomplished in Section 3 on the basis of the asymptotic analysis, presented in Section 2, of the ordinary differential equation associated with (C). From a viewpoint of variational analysis, Yoshida [19] has recently obtained a related result to our problem.

Finally we note that equation (A) represents stationary states of phenomena described by the degenerate parabolic equation

$$\text{(D)} \qquad u_t = \Delta(u^m) + u(1-u)(u-a), \qquad (t, x) \in R_+ \times R^n.$$

Equation (D) and its variants arise in various fields of applied sciences and have been the object of numerous investigations in recent years. In the theory of population genetics, the simplest mathematical model

$$u_t = u_{xx} + u(1-u), \qquad (t, x) \in R_+ \times R,$$

was first derived by Fisher [9], and models of the form

$$\text{(E)} \qquad u_t = \Delta(u^m) + \sigma(u), \qquad (t, x) \in R_+ \times R^n,$$

were proposed by Gurtin and MacCamy [11], who studied the initial value problem for (E) in one space dimension ($n = 1$) in the following two cases:

(1) the Malthusian law: $\sigma(u) = \mu u$, $\mu > 0$;

(2) the Verhulst law: $\sigma(u) = \mu u - \lambda u^2$, $\mu, \lambda > 0$.

In the study of Fisher type equations (E), the nonlinearity

$$\sigma(u) = u(1-u)[(\tau_1 - \tau_2)(1-u) - (\tau_3 - \tau_2)u]$$

also appears (see. e.g. Aronson and Weinberger [2] or Fife [8]) and the equation (D) is regarded as a normalization of the equation (E) with this type of nonlinearity. For typical results on this subject we refer to the survey article of Fife [8] and the papers [2, 9, 11, 12].

2. Asymptotic analysis

This section is of preparatory nature and analysis is given of properties of nonnegative radial entire solutions of the equation (C) for which the following conditions are always assumed to hold:

(H1) $f(v)$ is continuous on $[0, 1]$ and locally Lipschitz continuous on $(0, 1]$.

(H2) $f(0) = f(1) = 0$ and there is $\alpha \in (0, 1)$ such that $f(v) < 0$ on $(0, \alpha)$, $f(\alpha) = 0$, and $f(v) > 0$ on $(\alpha, 1)$.

(H3) The integral $\int_0^1 f(v)\,dv$ exists and is positive.
(H4) $\liminf_{v\to\alpha} f(v)/(v-\alpha) > 0$.

We define

$$F(v) = \int_0^v f(s)\,ds\,. \tag{2.1}$$

From (H2) and (H3) it follows that $F(v)$ has a negative minimum at $v = \alpha$, a positive maximum at $v = 1$ and a unique zero $v = \beta$ in $(\alpha, 1)$.

A radial function $u(x) = y(|x|)$ is an entire solution of (C) if and only if $y(t) \in C^2[0, \infty)$, $t = |x|$, satisfies the initial condition $y'(0) = 0$ and the ordinary differential equation

$$y'' + \frac{n-1}{t} y' + f(y) = 0\,, \qquad t > 0\,, \tag{2.2}$$

or equivalently,

$$(t^{n-1}y')' + t^{n-1}f(y) = 0\,, \qquad t > 0\,,$$

where a prime denotes differentiation with respect to t. Thus, the solution $y \in C^2[0, \infty)$ of (2.2) subject to the initial conditions

$$y(0) = \rho\,, \qquad y'(0) = 0 \tag{2.3}$$

gives rise to an entire solution $u(x) = y(|x|)$ of (C) satisfying $u(0) = \rho$. In the present situation the solutions of (2.2)–(2.3) for various values of ρ are understood to take their values in $[0, 1]$.

Integration of (2.2) shows that

$$y'(t) = -\int_0^t \left(\frac{s}{t}\right)^{n-1} f(y(s))\,ds\,, \qquad t \geq 0 \tag{2.4}$$

and that the problem (2.2)–(2.3) is equivalent to the integral equation

$$y(t) = \rho - \frac{1}{n-2}\int_0^t s\left(1 - \left(\frac{s}{t}\right)^{n-2}\right) f(y(s))\,ds\,, \qquad t \geq 0\,, \qquad n > 2\,, \tag{2.5}$$

$$y(t) = \rho - \int_0^t s \log\left(\frac{t}{s}\right) f(y(s))\,ds\,, \qquad t \geq 0\,, \qquad n = 2\,. \tag{2.6}$$

The method of successive approximations easily establishes the existence of a local nonnegative solution $y(t)$ of (2.5) or (2.6) for every $\rho \in (0, 1)$. Let $[0, T(\rho))$ be the maximum interval of existence of $y(t)$. The problem, therefore, is to determine those values of $\rho \in (0, 1)$ for which $T(\rho) = \infty$. It can be shown that both sets $\{\rho \in (0, 1) : T(\rho) < \infty\}$ and $\{\rho \in (0, 1) : T(\rho) = \infty\}$ are non-empty.

That $T(\rho) < \infty$ for all ρ sufficiently close to 1 is a consequence of the following proposition proved in [10, Lemma 5].

PROPOSITION 1. *There exists $\rho^* \in (0, 1)$ such that if $\rho^* < \rho < 1$, the solution $y(y)$ of the initial value problem (2.2)–(2.3) has a finite zero $t = T(\rho) > 0$ at which $y'(T(\rho)) < 0$.*

The set $\{\rho \in (0, 1) : T(\rho) = \infty\}$ contains the interval $(0, \beta)$, where β is the unique zero in $(\alpha, 1)$ of the function $F(v)$ given by (2.1). In fact, let $y(t)$ be the solution of (2.2)–(2.3) defined on $[0, T(\rho))$. Multiplication of (2.2) by $y'(t)$ and integration on $[0, t]$ yield

$$\frac{1}{2}y'(t)^2 + (n-1)\int_0^t \frac{y'(s)^2}{s}\,ds + F(y(t)) = F(\rho)\,, \tag{2.7}$$

which implies that $F(y(t)) \leq F(\rho)$ for $0 \leq t < T(\rho)$. Now consider the set $\{c \in (0, 1) : F(c) \leq F(\rho)\}$. It is easy to see that if $0 < \rho < \beta$, this set forms a closed subinterval $[c_1, c_2]$ of $(0, 1)$. It follows that if $0 < \rho < \beta$, the solution $y(t)$ satisfies $c_1 \leq y(t) \leq c_2$ for $0 \leq t < T(\rho)$, which immediately implies that $T(\rho) = \infty$. Ni [13] has studied the behavior at infinity of $y(t)$ with $0 < \rho < \beta$ in the case that $f(v)$ is Lipschitz continuous on $[0, 1]$. Our aim is to show that a similar result (Proposition 2 below) holds under our weaker hypotheses on $f(v)$.

PROPOSITION 2. *Let $y \in C^2[0, \infty)$ be a solution of the problem (2.2)–(2.3) for some $\rho \in (0, 1)$, $\rho \neq \alpha$, and suppose that $\inf_{t\geq 0} y(t) > 0$. Then the following statements hold.*

(i) *The zeros of $y'(t)$ form an infinite sequence $\{t_k\}_{k=0}^{\infty}$ such that*

$$0 = t_0 < t_1 < \cdots < t_k < \cdots \quad \text{and} \quad \lim_{k\to\infty} t_k = \infty\,.$$

(ii) *$h = \sup_k |t_{k+1} - t_k|$ is finite.*

(iii) *For each k, $y(t_k) \neq \alpha$ and $y(t)$ attains a local maximum or a local minimum at $t = t_k$ according as $y(t_k) > \alpha$ or $y(t_k) < \alpha$, respectively.*

(iv) *$\lim_{t\to\infty} y(t) = \alpha$ and $\lim_{t\to\infty} y'(t) = 0$.*

PROOF. We claim that there exist constants $\eta_1, \eta_2 \in (0, 1)$ such that $\eta_1 \leq y(t) \leq \eta_2$ for $t \geq 0$. As was remarked above, this is true if the initial value ρ of $y(t)$ satisfies $0 < \rho < \beta$ (we have $c_1 \leq y(t) \leq c_2$, $t \geq 0$, in this case). If $\beta \leq \rho < 1$, then the inequality $F(y(t)) \leq F(\rho)$ implies that $y(t) \leq \rho$ for $t \geq 0$, which, combined with the assumption $\inf_{t\geq 0} y(t) > 0$, leads to the claimed conclusion. Since $f(v)$ is Lipschitz continuous on $[\eta_1, \eta_2] \subset (0, 1)$, $y(t)$ is uniquely determined by its initial value ρ, and $y'(t) \neq 0$ on the set $A = \{t \geq 0 : y(t) = \alpha\}$.

We first show that A is an infinite set, that is, $y(t)$ takes on the value α infinitely often in every neighborhood of infinity. Suppose the contrary: $A = \emptyset$ or A is a finite set. Consider the function $z_1(t) = t^{(n-1)/2}(y(t) - \alpha)$, which, by assumption, does not vanish for all sufficiently large t, say $t \geq T$. As is easily seen, $z_1(t)$ satisfies the differential equation

$$z_1'' + \left(\frac{f(y(t))}{y(t) - \alpha} - \frac{(n-1)(n-3)}{4t^2}\right) z_1 = 0\,, \qquad t \geq T\,. \tag{2.8}$$

In view of the hypotheses (H1), (H2) and (H4) a constant $\varepsilon > 0$ can be chosen so that

$$\frac{f(v)}{v - \alpha} > \varepsilon \qquad \text{for} \qquad \eta_1 \leq v \leq \eta_2\,, \qquad v \neq \alpha\,,$$

and hence we have

$$\frac{f(y(t))}{y(t) - \alpha} - \frac{(n-1)(n-3)}{4t^2} > \frac{\varepsilon}{2}$$

for $t > \max\{T, |(n-1)(n-3)/2\varepsilon|^{1/2}\}$. We are now able to apply the Sturm Comparison Theorem (see e.g. [7, Chapter 8, Theorem 1.1]) to (2.8) and the equation

$$z_2'' + \frac{\varepsilon}{2} z_2 = 0\,, \qquad t \in R\,,$$

having the solution $z_2(t) = \sin((\varepsilon/2)^{1/2} t)$ to conclude that $A \cap ((2/\varepsilon)^{1/2} k\pi, (2/\varepsilon)^{1/2}(k+1)\pi) \neq \emptyset$ for all $k > |(n-1)(n-3)|^{1/2}\pi$, which is clearly a contradiction. Therefore the set A must be an infinite set, so that $y(t)$ must have infinitely many critical points accumulating at ∞. Now the assertions (i), (ii) and (iii) are easily verified.

It remains to prove (iv). Suppose that

$$\eta = \inf_k |y(t_k) - \alpha| > 0 \tag{2.9}$$

Define

$$M = \max\{|f(v)| : 0 \leq v \leq 1\}\,, \tag{2.10}$$

$$\mu = \min\{|f(v)| : |v - \alpha| \geq \eta/2, \eta_1 \leq v \leq \eta_2\}\,. \tag{2.11}$$

Obviously, $M \geq \mu > 0$. Since $y'(t_k) = 0$, it follows from (2.4) and (2.10) that

$$|y'(t)| \leq \int_{t_k}^{t} \left(\frac{s}{t}\right)^{n-1} |f(y(s))| ds \leq M(t - t_k)\,, \qquad t \geq t_k\,,$$

and

$$|y(t) - y(t_k)| \leq \frac{M}{2}(t - t_k)^2, \qquad t \geq t_k. \tag{2.12}$$

From (2.9) and (2.12) we see that

$$|y(t) - \alpha| > \eta/2 \qquad \text{for} \qquad 0 < t - t_k < \delta = (\eta/M)^{1/2},$$

from which it follows that $|f(y(t))| \geq \mu$ for $t \in [t_k, t_k + \delta]$, and accordingly

$$\begin{aligned}|y'(t)| &= \int_{t_k}^{t} \left(\frac{s}{t}\right)^{n-1} |f(y(s))| ds \geq \mu \int_{t_k}^{t} \left(\frac{s}{t}\right)^{n-1} ds \\ &= \frac{\mu}{n}\left(t - \left(\frac{t_k}{t}\right)^{n-1} t_k\right) \geq \frac{\mu}{n}(t - t_k), \qquad t_k \leq t \leq t_k + \delta.\end{aligned}$$

Using the last inequality and noting that $t_k + \delta \leq t_{k+1} \leq (k+1)h$ by the assertion (ii) of this proposition, we obtain

$$\begin{aligned}\int_{t_k}^{t_k+\delta} \frac{y'(s)^2}{s} ds &\geq \frac{1}{(k+1)h} \int_{t_k}^{t_k+\delta} y'(s)^2 ds \\ &\geq \frac{1}{(k+1)h} \int_{t_k}^{t_k+\delta} \left(\frac{\mu}{n}\right)^2 (s - t_k)^2 ds \\ &= \frac{1}{(k+1)h}\left(\frac{\mu}{n}\right)^2 \frac{\delta^3}{3}, \qquad k = 0, 1, 2, \ldots,\end{aligned}$$

which implies that $\int_0^\infty y'(s)^2/s\, ds = \infty$. On the other hand, from (2.7) we have

$$(n-1) \int_0^t \frac{y'(s)^2}{s} ds \leq F(\rho) - F(y(t)) \leq F(\rho) - F(\alpha)$$

for all $t \geq 0$, which gives a contradiction as $t \to \infty$. Thus we conclude that (2.9) is impossible, that is, $\eta = \inf_k |y(t_k) - \alpha| = 0$.

Since $F(y(t_k)) \geq F(\alpha)$ for all k, we have $\inf_k F(y(t_k)) = F(\alpha)$, and since, by (2.7),

$$F(y(t_k)) = F(\rho) - (n-1) \int_0^{t_k} \frac{y'(s)^2}{s} ds, \tag{2.13}$$

the sequence $\{F(y(t_k))\}$ is decreasing and $\lim_{k\to\infty} F(y(t_k)) = F(\alpha)$. Substitution of (2.7) into (2.13) gives

$$F(y(t_k)) = \frac{1}{2} y'(t)^2 + F(y(t)) + (n-1) \int_{t_k}^{t} \frac{y'(s)^2}{s} ds, \tag{2.14}$$

which shows that

$$F(y(t_k)) \geq F(y(t)) \geq F(\alpha)\,, \qquad t \geq t_k\,. \tag{2.15}$$

It follows therefore that $\lim_{t\to\infty} F(y(t)) = F(\alpha)$, and consequently $\lim_{t\to\infty} y(t) = \alpha$. That $\lim_{t\to\infty} y'(t) = 0$ follows from the relation

$$F(y(t_k)) \geq \tfrac{1}{2}y'(t)^2 + F(\alpha)\,, \qquad t \geq t_k\,,$$

which is a consequence of (2.14). This completes the proof.

3. Main results

The purpose of this section is to prove the existence theorem for equation (A) stated in the Introduction. The theorem is easily seen to be equivalent to the following theorem regarding the initial value problem

$$y'' + \frac{n-1}{t}y' + y^{1/m}(1 - y^{1/m})(y^{1/m} - a) = 0\,, \qquad t > 0\,, \tag{3.1}$$

$$y(0) = \rho\,, \qquad y'(0) = 0\,. \tag{3.2}$$

Theorem 1. *Suppose that $m > 0$ and $0 < a < (m+1)/(m+3)$. Then there exists a constant $\rho_* \in (0, 1)$ such that the problem* (3.1)–(3.2) *has a nonnegative solution $y \in C^2[0, \infty)$ if $0 < \rho \leq \rho_*$, and the problem* (3.1)–(3.2) *has no nonnegative solution $y \in C^2[0, \infty)$ if $\rho_* < \rho < 1$. Furthermore, the following statements hold.*

(i) *If $0 < \rho < \rho_*$, the solution $y(t)$ of* (3.1)–(3.2) *oscillates around a^m and converges to a^m as $t \to \infty$.*

(ii) *If $\rho = \rho_*$, there exists a solution $y_*(t)$ of* (3.1)–(3.2) *which decreases monotonically to zero as $t \to \infty$. This solution has compact support if $m > 1$.*

This theorem is essentially a corollary of the following theorem for equation (C) which is formulated in terms of the initial value problem (2.2)–(2.3). The hypotheses (H1)–(H4) are assumed to hold without further mention.

Theorem 2. *There exists a constant $\rho_* \in (0, 1)$ such that the problem* (2.2)–(2.3) *has a nonnegative solution $y \in C^2[0, \infty)$ if $0 < \rho \leq \rho_*$. Furthermore the following statements hold.*

(i) *If $0 < \rho < \rho_*$, the solution $y(t)$ of* (2.2)–(2.3) *has the property $\inf_{t\geq 0} y(t) > 0$, oscillates around α and converges to α as $t \to \infty$.*

(ii) *If $\rho = \rho_*$, there exists a solution $y_*(t)$* of (2.2)–(2.3) *which decreases monotonically to zero as $t \to \infty$.*

PROOF OF THEOREM 2. Denote by P the set of all $\rho \in (0, 1)$ such that the problem (2.2)–(2.3) has a solution $y \in C^2[0, \infty)$ with $\inf_{t\geq 0} y(t) > 0$. As was noted before Proposition 2, the set P contains the interval $(0, \beta)$. Let ρ_* denote the least upper bound of $\rho \in (0, 1)$ such that $(0, \rho) \subset P$. Proposition 1 implies that $\rho_* < 1$, and Proposition 2 guarantees the truth of the statement (i).

To prove (ii) take a sequence $\{\rho_k\}$ of positive numbers such that $\rho_k < \rho_*$ and $\lim_{k\to\infty} \rho_k = \rho_*$, and let $y_k \in C^2[0, \infty)$ be the nonnegative solution of (2.2)–(2.3) for $\rho = \rho_k$, $k = 1, 2, \ldots$. Since the sequence $\{y_k(t)\}$ is uniformly bounded on $[0, \infty)$ and since, by (2.4) and (2.10),

$$|y_k'(t)| \leq \int_0^t \left(\frac{s}{t}\right)^{n-1} |f(y_k(s))| ds \leq Mt \,, \qquad t \geq 0 \,, \qquad k = 1, 2, \ldots,$$

the Ascoli-Arzela theorem shows that the sequence $\{y_k(t))\}$ contains a sequence converging to a function $y_\infty(t)$ uniformly on any compact subinterval of $[0, \infty)$. It is clear that $y_\infty(t)$ is continuous and satisfies the integral equation (2.5) or (2.6), so that $y_\infty(t)$ is a nonnegative C^2-solution of (2.2)–(2.3) on $[0, \infty)$. We will show that $\inf_{t\geq 0} y_\infty(t) = 0$. Suppose the contrary $\inf_{t\geq 0} y_\infty(t) > 0$. Let $\{t_k\}$ be the sequence of zeros of $y_\infty'(t)$ as described in Proposition 2. Note that $y_\infty(t)$ has a local maximum at $t = t_{2j}$ and a local minimum at $t = t_{2j-1}$. In view of (2.15) with $y = y_\infty$ we have $y_\infty(t_1) \leq y_\infty(t) \leq \rho_*$ for $t \geq 0$, and hence there exists $\rho_0 > \rho_*$ such that for any $\rho \in (\rho_*, \rho_0)$ the solution $y(t)$ of (2.2)–(2.3) exists on $[0, t_2]$ and satisfies

$$|y(t) - y_\infty(t)| < (y_\infty(t_2) - y_\infty(t_1))/2 \,, \qquad 0 \leq t \leq t_2 \,.$$

This inequality (with $t = t_1$ and $t = t_2$) implies in particular that

$$y(t_1) < (y_\infty(t_1) + y_\infty(t_2))/2 < y(t_2) \,,$$

and so $y(t)$ has a positive local minimum at some point in $(0, t_2)$ and $y(t)$ can be continued to the entire interval $[0, \infty)$ in such a way that $\inf_{t\geq 0} y(t) > 0$. This, however, contradicts the definition of ρ_* and proves that $\inf_{t\geq 0} y_\infty(t) = 0$.

Since $y_\infty(t) < y_\infty(0)$ for $t > 0$, there are two possible cases: Either $y_\infty'(t) < 0$ on $(0, \infty)$ or there is $t_1 > 0$ such that $y_\infty'(t) < 0$ on $(0, t_1)$ and $y_\infty'(t_1) = 0$. In the first case, it is enough to put $y_* = y_\infty$. If the second case holds, then we have $y_\infty(t_1) \leq y_\infty(t) \leq \rho_*$ for $t \geq 0$ by (2.15) and conclude that $y_\infty(t_1) = 0$. In this case the function y_* defined by $y_*(t) = y_\infty(t)$ for $0 \leq t \leq t_1$ and $y_*(t) = 0$ for $t \geq t_1$ is easily seen to be of class $C^2[0, \infty)$ and satisfy (2.2)–(2.3). In either case the solution $y_*(t)$ is decreasing to zero as $t \to \infty$. This completes the proof of Theorem 2.

REMARK 1. According to Peletier and Serrin [15, Theorem 5] the above solution $y_*(t)$ has compact support if and only if

$$\int_0^\alpha |F(v)|^{-1/2}\,dv < \infty\,. \tag{3.3}$$

Proof of Theorem 1. Notice that (3.1) is a special case of (2.2) with $f(y) = y^{1/m}(1 - y^{1/m})(y^{1/m} - a)$, for which the hypotheses (H1), (H2) and (H4) are clearly satisfied. The hypothesis (H3) is also satisfied for this $f(y)$, since

$$\int_0^1 f(v)\,dv = \frac{-m}{(m+1)(m+2)}\left(a - \frac{m+1}{m+3}\right). \tag{3.4}$$

Theorem 2 then ensures the existence of a number $\rho_* \in (0, 1)$ for which the statements (i) and (ii) hold true. If $m > 1$, then (3.3) holds, so that, by Remark 1, the solution $y_*(t)$ in (ii) has compact support.

The proof will be complete if it is shown that there is no nonnegative solution $y \in C^2[0, \infty)$ of (3.1)–(3.2) for any $\rho \in (\rho_*, 1)$. Suppose to the contrary that (3.1)–(3.2) has a nonnegative solution $y = y_\# \in C^2[0, \infty)$ for some $\rho = \rho_\# > \rho_*$. The solution $y_\#(t)$ satisfies $\inf_{t\geq 0} y_\#(t) > 0$, since otherwise there exist two distinct solutions $y_*(t)$ and $y_\#(t)$ of (3.1) having the property $\inf_{t\geq 0} y_*(t) = \inf_{t\geq 0} y_\#(t) = 0$, which contradicts the uniqueness of such *decaying* solutions of (3.1) (see Peletier and Serrin [15, Theorem 3]). Let ρ^* $(\geq \rho_\#)$ denote the least upper bound of the set P of all $\rho \in (0, 1)$ such that the problem (3.1)–(3.2) possesses a solution $y(t)$ satisfying $\inf_{t\geq 0} y(t) > 0$. Then we have $\rho^* \notin P$. Using a sequence $\{\rho_k\}$ such that $\rho_k \in P$ and $\lim_{k\to\infty} \rho_k = \rho^*$ and arguing as in the proof of Theorem 1, we can construct a nonnegative solution $y^* \in C^2[0, \infty)$ of (3.1) such that $\inf_{t\geq 0} y^*(t) = 0$. But this again contradicts the uniqueness result of Peletier and Serrin [15]. The proof of Theorem 1 is thus complete.

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