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A distortion theorem for conformal mappings with an application to subharmonic functions

Dedicated to Professor T. Fuji'i'e on his 60th birthday

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1. Introduction

Let f be a conformal mapping of a domain D_0 of $\mathbf{C}$ bounded by a finitely many analytic curves. Our first purpose of this paper is to establish the following relation between $|f'(z)|$ and the Poincaré metric $\lambda_{D_0}(z)$ for D_0.

THEOREM I. *If $f(D_0)$ satisfies an exterior θ-wedge condition, then there exists $m > 0$ such that*

$$\frac{1}{|f'(z)|} \leq m\lambda_{D_0}(z)^{1-\theta}$$

for $z \in D_0$.

Here, a finitely connected domain D is said to satisfy an exterior θ-wedge condition if it is bounded and there exist $\rho > 0$ and $\theta \in (0, 1)$ such that, for every $\omega \in \partial D$, a closed sector of radius ρ and opening $\pi\theta$ with vertex at ω lies in $\mathbf{C} - D$.

Recently, N. Suzuki has obtained the following theorem:

THEOREM (Suzuki [5, Theorem 2]). *Let D be a bounded $C^{1,1}$-domain of $\mathbf{C}$, and denote by $\delta_D(z)$ the distance between $z \in D$ and ∂D. Set $\alpha(p) = 1 +$ $\max\{1 - p, 0\}$ for $0 < p < \infty$. If a nonnegative subharmonic function s on D satisfies*

$$\iint_D \delta_D(z)^{-\alpha(p)} s(z)^p \, dx \, dy < +\infty , \qquad z = x + iy ,$$

then s must vanish identically.

We apply Theorem I to generalize Suzuki's theorem:

THEOREM II. *Suppose that D satisfies an exterior θ-wedge condition. Set $\beta(p, \theta) = 2 - \min\{1, p\}/(2 - \theta)$. If a nonnegative subharmonic function s on D satisfies*

$$\iint_D \delta_D(z)^{-\beta(p,\theta)} s(z)^p \, dx \, dy < +\infty \,,$$

then $s \equiv 0$ *on* D.

After summarizing elementary properties of the Poincaré metric in §2, we will give a proof of Theorem I in §3. Theorem II will be proved in §4, where examples are also given.

2. The Poincaré metric

Let $\Delta(a; r)$ denote the open disk of radius r with center at a. We set $\Delta = \Delta(0; 1)$.

Let D be a domain of the Riemann sphere $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ whose complement $\hat{\mathbf{C}} - D$ consists of more than two points. The universal covering surface of D is conformally equivalent to Δ. Let $\pi: \Delta \to D$ be a universal covering map. The Poincaré metric $\lambda_D(z)|dz|$ for D is defined by

$$\lambda_D(\pi(\zeta))|\pi'(\zeta)| = \frac{1}{1 - |\zeta|^2}\,, \qquad \zeta \in \Delta \,.$$

Note that λ_D is a continuous function on D and does not depend on the choice of π.

The following properties of the Poincaré metric are well known (see, for example, Kra [3, Proposition 1.1 in Chapter II]).

LEMMA 1. (i) *If f is a conformal mapping, then*

$$\lambda_{f(D)}(f(z))|f'(z)| = \lambda_D(z)\,, \qquad z \in D \,.$$

(ii) *If* $D_1 \subset D_2$, *then* $\lambda_{D_2}(z) \leq \lambda_{D_1}(z)$ *for* $z \in D_1$.

(iii) *Let* $\delta_D(z) = \inf\{|z - \zeta| \,|\, \zeta \in \partial D\}$. *Then*

$$\lambda_D(z)\delta_D(z) \leq 1\,, \qquad z \in D \,.$$

(iv) *If D is simply connected and* $\infty \notin D$, *then*

$$\lambda_D(z)\delta_D(z) \geq \tfrac{1}{4}\,, \qquad z \in D \,.$$

In this paper we are concerned with finitely connected subdomains D of $\mathbf{C}$ such that every component of $\hat{\mathbf{C}} - D$ contains at least two points. We denote the class of such domains by $\mathscr{E}_0$. We need a generalization of Lemma 1 (iv) to the case $D \in \mathscr{E}_0$.

LEMMA 2. *If* $D \in \mathscr{E}_0$, *then there exists* $m > 0$ *such that*

$$\lambda_D(z)\delta_D(z) \geq m$$

for $z \in D$.

Before proving Lemma 2, we remark the following lemma.

LEMMA 3. *If D is simply connected and contains ∞, then for each $R > 0$ there exists $m > 0$ such that*

$$\lambda_D(z)\delta_D(z) \geq m$$

for $z \in D \cap \Delta(0; R)$.

PROOF. Let $a \in \mathbf{C} - D$, and set $f(z) = 1/(z-a)$. If $z \in D - \{\infty\}$ and $w = f(z)$, then $f^{-1}(\Delta(w; r))$ contains $\Delta(z; r/|w|(|w| + r))$. In particular, letting $r = \delta_{f(D)}(w)$, we have

$$\delta_D(z) \geq \frac{\delta_{f(D)}(w)}{|w|(|w| + \delta_{f(D)}(w))}.$$

We now apply Lemma 1 (i) and (iv) to obtain

$$\lambda_D(z)\delta_D(z) = \lambda_{f(D)}(w)|w|^2\delta_D(z) \geq \frac{|w|}{4(|w| + \delta_{f(D)}(w))}.$$

Since $\delta_{f(D)}(w) = O(|w|)$ as $w \to \infty$ because of the fact that $\delta_{f(D)}(w) \leq |w - \omega|$ for $w \in f(D)$ and $\omega \in \mathbf{C} - f(D)$ and since $|w| \geq (R + |a|)^{-1}$ if $|z| \leq R$, we have the lemma.

PROOF OF LEMMA 2. Let $C_1, \ldots, C_n$ be the components of $\hat{\mathbf{C}} - D$, and set $D_j = \hat{\mathbf{C}} - C_j$ for $j = 1, \ldots, n$. We assume that $\infty \in C_1$. Since $\lambda_D(z) \geq \lambda_{D_j}(z)$ by Lemma 1 (ii), it follows from Lemma 1 (iv) and Lemma 3 that

$$\inf_{z \in D \cap \Delta(0; R)} \lambda_D(z)\delta_D(z) > 0$$

for each $R > 0$. Thus, if D is bounded, then we have done.

Assume now that D is unbounded. We can find $R_0 > 0$ such that $\delta_{D_1}(z) \leq 2\delta_{D_j}(z)$, $2 \leq j \leq n$, for $z \in D - \Delta(0; R_0)$. Then $\delta_D(z) = \min\{\delta_{D_1}(z), \ldots, \delta_{D_n}(z)\} \geq \delta_{D_1}(z)/2$, and hence by Lemma 1 (ii) and (iv)

$$\lambda_D(z)\delta_D(z) \geq \tfrac{1}{2}\lambda_{D_1}(z)\delta_{D_1}(z) \geq \tfrac{1}{8}$$

for $z \in D - \Delta(0; R_0)$. This completes the proof.

REMARK. If a is an isolated point of $\mathbf{C} - D$, then $\lambda_D(z)\delta_D(z)|\log \delta_D(z)|$ is bounded above and bounded away from zero for z sufficiently near a. For general D it is known that

$$\inf_{z \in D \cap \Delta(0;R)} \lambda_D(z)\delta_D(z)(1 + |\log \delta_D(z)|) > 0$$

for each $R > 0$ (cf. Kra [3, Lemma 2.3 in Chapter V]).

3. Distortion theorem for conformal mappings

First we introduce subclasses $\mathscr{E}_\theta$, $0 < \theta \leq 1$, of $\mathscr{E}_0$. Let $D \in \mathscr{E}_0$ be a bounded domain. If there exist $\rho > 0$ and $\theta \in (0, 1)$ such that for every $\omega \in \partial D$ a closed circular sector with vertex at ω which is congruent to

$$S(\rho, \theta) = \{w \in \mathbf{C} | 0 \leq |w| \leq \rho, 0 \leq \arg w \leq \pi\theta\}$$

lies in $\mathbf{C} - D$, then D is said to satisfy an exterior θ-wedge condition, and we denote the set of such domains D by $\mathscr{E}_\theta$. If there exists $\rho > 0$ such that for every $\omega \in \partial D$ a closed disk of radius ρ containing ω lies in $\mathbf{C} - D$, then D is said to satisfy an exterior disk condition, and we denote the set of such domains D by $\mathscr{E}_1$. If each component of ∂D is a simple analytic curve, then D is called regular.

It is trivial that $\mathscr{E}_{\theta_1} \supset \mathscr{E}_{\theta_2}$ if $0 \leq \theta_1 \leq \theta_2 \leq 1$. A bounded Lipschitz domain belongs to $\mathscr{E}_\theta$ for some $\theta \in (0, 1)$. Also, a bounded $C^{1,1}$-domain belongs to $\mathscr{E}_1$; in particular, regular domains are contained in $\mathscr{E}_1$. If a component C of the boundary of $D \in \mathscr{E}_\theta$, $\theta > 0$, is a Jordan curve, then C is rectifiable (cf. FitzGerald-Lesley [1], [2]). Note, however, that components of the boundary of $D \in \mathscr{E}_\theta$, $0 \leq \theta \leq 1$, are not necessarily Jordan curves.

Theorem I stated in the introduction is contained in the following theorem.

THEOREM 1. *Let f be a conformal mapping of a regular domain D_0 onto a domain D of* $\mathbf{C}$. *If $D \in \mathscr{E}_\theta$, $0 \leq \theta \leq 1$, then there exists $m > 0$ such that*

$$\frac{1}{|f'(z)|} \leq m\lambda_{D_0}(z)^{1-\theta}$$

for $z \in D_0$.

PROOF. The conformal mapping f induces a bijection between the set of components of ∂D_0 and the set of components of $\hat{\mathbf{C}} - D$. Let C_0 be a component of ∂D_0, and let C be the component of $\hat{\mathbf{C}} - D$ that corresponds to C_0 under the bijection. (In this case we will simply say that C_0 corresponds to C under f.) Let f_1 be a conformal mapping of the unit disk Δ onto $D_1 = \hat{\mathbf{C}} - C$. We consider three cases.

Case 1: $\theta = 0$. Assume first that $\infty \in C$. By Koebe's distortion theorem we can choose $m_1 > 0$ such that $|f_1'(\zeta)|\lambda_\Delta(\zeta) \geq m_1$ for $\zeta \in \Delta$. Set $\varphi = f_1^{-1} \circ f$

and $\varDelta_0 = f_1^{-1}(D) \subset \varDelta$. Then φ is a conformal mapping of D_0 onto $\varDelta_0$. Since φ is extended to a homeomorphism of $D_0 \cup C_0$ onto $\varDelta_0 \cup \partial\varDelta$ and both C_0 and $\partial\varDelta$ are analytic, we can continue φ conformally beyond C_0 by the reflection principle. In particular, $|\varphi'(z)| \geq m_2 > 0$ in a neighborhood of C_0. Hence, for $z \in D_0$ sufficiently near C_0, we have by Lemma 1 (i) and (ii)

$$\begin{aligned} |f'(z)|\lambda_{D_0}(z) &= |f_1'(\varphi(z))||\varphi'(z)|\lambda_{D_0}(z) \\ &= |f_1'(\varphi(z))||\varphi'(z)|^2\lambda_{\varDelta_0}(\varphi(z)) \\ &\geq m_2^2|f_1'(\varphi(z))|\lambda_{\varDelta}(\varphi(z)) \\ &\geq m_1 m_2^2 > 0\,. \end{aligned}$$

Next assume that C is bounded. Let $a \in C$, and set $h(w) = 1/(w-a)$ and $D_2 = h(D)$. Then $f_2 = h \circ f : D_0 \to D_2$ is conformal, and C_0 corresponds to the unbounded component of $\hat{\mathbf{C}} - D_2$ under f_2. Thus, as was shown in the preceding paragraph, there exists $m_3 > 0$ such that $|f_2'(z)|\lambda_{D_0}(z) \geq m_3$ for $z \in D$ near C_0, and hence

$$|f'(z)|\lambda_{D_0}(z) \geq m_3|f(z) - a|^2\,.$$

Taking distinct points $a, b \in C$, we see that

$$|f'(z)|\lambda_{D_0}(z) \geq m_4 \max\{|f(z)-a|^2, |f(z)-b|^2\} \geq m_4\frac{|a-b|^2}{4} > 0$$

near C_0.

We have shown that $|f'(z)|\lambda_{D_0}(z)$ is bounded away from 0 near ∂D_0. Since $|f'(z)|\lambda_{D_0}(z)$ is positive and continuous in D_0, we obtain the desired estimate.

Case 2: $0 < \theta < 1$. We suppose that for each $\omega \in \partial D$ a closed circular sector S_ω with vertex at ω which is congruent to $S(\rho, \theta)$ lies in $\mathbf{C} - D$. Fix a conformal mapping $g : \varDelta \to \hat{\mathbf{C}} - S(\rho, \theta)$, which is extended homeomorphically to the closed disk, such that $g(1) = 0$. Since there exists a conformal mapping $\tilde{g}$ of a neighborhood of 1 such that $g(\zeta) = \tilde{g}(\zeta)^{2-\theta}$, we see that for some $m_5 > 0$

$$g(\varDelta(0; r)) \cap \varDelta(0; m_5(1-r)^{2-\theta}) = \varnothing\,, \qquad 0 \leq r < 1\,. \tag{1}$$

Now take $z \in D_0$ sufficiently near C_0 so that a point $\omega \in C$ lies on the circle $\partial\varDelta(w; \delta_D(w))$, where $w = f(z)$. For an appropriate $\eta \in \partial\varDelta$ the function $g_\omega = \eta g + \omega$ maps $\varDelta$ conformally onto $\hat{\mathbf{C}} - S_\omega$, and $g_\omega(1) = \omega$. Since $g_\omega^{-1} \circ f_1(\varDelta) \subset \varDelta$, it follows from Schwarz's lemma that

$$f_1(\varDelta(0; r)) \subset g_\omega \circ h_\omega(\varDelta(0; r))\,,$$

where h_ω is a Möbius transformation of $\varDelta$ which maps 0 to $c_\omega = g_\omega^{-1} \circ f_1(0) = g^{-1}(\bar{\eta}(f_1(0) - \omega))$. Observe that c_ω, $\omega \in C$, stay in a compact subset of $\varDelta$.

Therefore, in view of (1) we have for some $m_6 > 0$

$$f_1(\Delta(0; r)) \cap \Delta(\omega; m_6(1-r)^{2-\theta}) = \varnothing\,, \qquad 0 \le r < 1\,,$$

for all $\omega \in C$. In particular, letting $r = |\zeta|$, $\zeta = f_1^{-1}(w)$, we obtain

$$\delta_{D_1}(w) = \delta_D(w) \ge m_6(1-r)^{2-\theta} \ge m_7 \lambda_\Delta(\zeta)^{-(2-\theta)}$$

for all w near C. Consequently, by Lemma 1 (i) and (iii)

$$\frac{1}{|f_1'(\zeta)|} = \frac{\lambda_{D_1}(w)}{\lambda_\Delta(\zeta)} \le \frac{1}{\lambda_\Delta(\zeta)\delta_{D_1}(w)} \le m_8 \lambda_\Delta(\zeta)^{1-\theta}$$

for all ζ near $\partial\Delta$. Since $\varphi(D_0) \subset \Delta$ and $\varphi = f_1^{-1} \circ f$ is continued conformally across C_0, we have by Lemma 1

$$\begin{aligned}\frac{1}{|f'(z)|} &= \frac{1}{|f_1'(\zeta)||\varphi'(z)|} \le m_8 \frac{\lambda_\Delta(\zeta)^{1-\theta}}{|\varphi'(z)|} \\ &\le m_8 \frac{\lambda_{\varphi(D_0)}(\zeta)^{1-\theta}}{|\varphi'(z)|} = m_8 \frac{\lambda_{D_0}(z)^{1-\theta}}{|\varphi'(z)|^{2-\theta}} \le m_9 \lambda_{D_0}(z)^{1-\theta}\end{aligned}$$

for all $z \in D_0$ near C_0. Now our assertion easily follows.

Case 3: $\theta = 1$. The proof is quite similar to the preceding case, and may be omitted.

REMARK. Let D be a Jordan domain satisfying an exterior θ-wedge condition, and let $f: \Delta \to D$ be conformal. Lesley [4] proved that the inverse mapping f^{-1} is then Hölder continuous on $\bar{D}$ with exponent $1/(2-\theta)$. Also, FitzGerald-Lesley [2] showed that $1/f'$ belongs to the Hardy space H^p for all $p < 1/2(1-\theta)$.

4. Application

Let a set $X(D)$ of measurable functions on D be assigned to each subdomain D of $\mathbf{C}$, and assume that X is conformally invariant; that is, if $f: D \to D'$ is conformal, then $X(D) = \{s' \circ f \mid s' \in X(D')\}$. Our problem is to find an exponent γ for which

$$\iint_D \delta_D(z)^{-\gamma}|s(z)|\,dx\,dy = +\infty \qquad \text{for all} \quad s \in X(D) - \{0\}\,. \tag{2}$$

THEOREM 2. *Suppose that, for all regular domains D, condition* (2) *holds for $\gamma = \gamma_0 \le 2$, where γ_0 does not depend on D. Then, for all domains D in $\mathscr{E}_\theta$, $0 \le \theta \le 1$, condition* (2) *holds for $\gamma = (\gamma_0 - 2\theta + 2)/(2-\theta)$.*

PROOF. Let $D \in \mathscr{E}_\theta$. Note that, by Lemma 1 (iii) and Lemma 2,

$$\iint_D \delta_D(z)^{-\gamma}|s(z)|\,dx\,dy = +\infty$$

if and only if

$$\iint_D \lambda_D(z)^{\gamma}|s(z)|\,dx\,dy = +\infty\,.$$

Now D is conformally equivalent to some regular region D_0; let $f: D_0 \to D$ be conformal. Then Theorem 1 together with Lemma 1 (i) implies that, for each $\gamma \leq 2$,

$$\begin{aligned}
&\iint_D \lambda_D(z)^{\gamma}|s(z)|\,dx\,dy \\
&= \iint_{D_0} \lambda_{D_0}(z)^{\gamma}|s(f(z))|\,|f'(z)|^{2-\gamma}\,dx\,dy \\
&\geq m_\gamma \iint_{D_0} \lambda_{D_0}(z)^{\gamma+(2-\gamma)(\theta-1)}|s(f(z))|\,dx\,dy
\end{aligned}$$

for some $m_\gamma > 0$. Thus, if $\gamma = (\gamma_0 - 2\theta + 2)/(2-\theta)(\leq 2)$, then the last integral diverges for $s \in X(D) - \{0\}$, and we obtain the theorem.

Let $S(D)$ denote the set of subharmonic functions on D, and set $S^+(D) = \{s \in S(D) | s \geq 0\}$. Theorem 2 and Suzuki's theorem stated in the introduction yield the following corollary, which contains Theorem II. Recall that $\beta(p, \theta) = 2 - \min\{1, p\}/(2-\theta)$.

COROLLARY 1. *Let $D \in \mathscr{E}_\theta$, $0 \leq \theta \leq 1$. If $s \in S^+(D)$ satisfies*

$$\iint_D \delta_D(z)^{-\beta(p,\theta)} s(z)^p\,dx\,dy < +\infty$$

for some $p \in (0, \infty)$, then $s \equiv 0$.

Also, we obtain the following result (cf. Suzuki [5, Theorem 2]).

COROLLARY 2. *Let $D \in \mathscr{E}_\theta$, $0 \leq \theta \leq 1$. If $s \in S(D)$ satisfies*

$$\iint_D \delta_D(z)^{-\beta(p,\theta)}|s(z)|^p\,dx\,dy < +\infty$$

for some $p \in (0, 1/2)$, then $s \equiv 0$.

EXAMPLE 1. Let $D = \Delta(0; 2) - \{0\}$ and $s(z) = \max\{-\log|z|, 0\}$. Then

$s \in S^+(D) - \{0\}$ and

$$\iint_D \delta_D(z)^{-\gamma} s(z)^p \, dx \, dy < +\infty$$

for $\gamma < 2$. This example shows that in the above corollaries the assumption that every component of $\hat{\mathbf{C}} - D$ is a continuum is essential.

REMARK. In [6] Suzuki has shown that for *any* proper subdomain D of $\mathbf{C}$

$$\iint_D \delta_D(z)^{-2} |s(z)|^p \, dx \, dy = +\infty$$

for all $s \in S(D) - \{0\}$ and $0 < p \leq 1$.

EXAMPLE 2. For each $\theta \in [0, 1]$ we construct a Jordan domain D_θ as follows. We set $D_1 = \Delta$. For $\theta \in [0, 1)$, the exponential function maps the circular arc $|z + \pi \tan(\pi\theta/2) - \pi i| = \pi \sec(\pi\theta/2)$, $\operatorname{Re} z \geq 0$, onto a Jordan curve through 1. We let D_θ be the domain bounded by the Jordan curve. Clearly, D_θ belongs to $\mathscr{E}_\theta$. Let $f_\theta : \Delta \to D_\theta$ be a conformal mapping such that $f_\theta(1) = 1$. It is easy to see that there exists $m_1 = m_1(\theta) > 0$ such that

$$|f_\theta'(\zeta)| \leq m_1 |\zeta - 1|^{1-\theta} \tag{3}$$

for $\zeta \in \Delta$.

Consider the domains $U_\psi = \Delta(c_\psi; r_\psi) \cap \Delta(\overline{c_\psi}; r_\psi)$, where $c_\psi = (1 + i \cot \pi\psi)/2$, $r_\psi = (\csc \pi\psi)/2$ and $0 < \psi < 1/2$. The inner angle of U_ψ at 1 is $2\pi\psi$. The function $g_\psi(\zeta) = (1 - \omega)/(1 + \omega)$, $\omega^{2\psi} = (1 - \zeta)/\zeta$, maps U_ψ conformally onto Δ. Let $P_\psi = P \circ g_\psi$, where $P(w) = (1 - |w|^2)/|w - 1|^2$. Then P_ψ is a positive harmonic function of U_ψ vanishing continuously on $\partial U_\psi - \{1\}$. We set $P_\psi(\zeta) = 0$ for $\zeta \in \Delta - U_\psi$ to obtain a nonnegative subharmonic function on Δ. Thus $P_\psi \circ f_\theta^{-1} \in S^+(D_\theta)$. Note that

$$P_\psi(\zeta) \leq m_2 |\zeta - 1|^{-1/2\psi} \tag{4}$$

for $\zeta \in \Delta$.

Let $0 < p \leq 1$ and $\gamma = 2 - (1 - \psi)p/\psi(2 - \theta)$. Then $\gamma < \beta(p, \theta)$ and $\lim_{\psi \to 1/2} \gamma = \beta(p, \theta)$. Making use of (3) and (4), we have

$$\begin{aligned} &\iint_{D_\theta} \lambda_{D_\theta}(z)^\gamma [(P_\psi \circ f_\theta^{-1})(z)]^p \, dx \, dy \\ &= \iint_\Delta \lambda_\Delta(\zeta)^\gamma P_\psi(\zeta)^p |f_\theta'(\zeta)|^{2-\gamma} \, d\xi \, d\eta \\ &\leq m_3 \iint_{U_\psi} \lambda_\Delta(\zeta)^\gamma |\zeta - 1|^{(2-\gamma)(1-\theta) - p/2\psi} \, d\xi \, d\eta \end{aligned}$$

$$\leq m_4 \iint_{\Delta} |\zeta - 1|^{-\gamma+(2-\gamma)(1-\theta)-p/2\psi} \, d\xi \, d\eta$$

$$= m_4 \iint_{\Delta} |\zeta - 1|^{-2+(1-2\psi)p/2\psi} \, d\xi \, d\eta < +\infty .$$

This example shows that in Corollary 1 we cannot replace $\beta(p, \theta)$ by any smaller number when $0 < p \leq 1$.

Finally, we consider nonnegative harmonic functions. Let $H^+(D)$ denote the set of nonnegative harmonic functions on D. The following result is a corollary to Theorem 2 and Suzuki [5, Remark 2].

COROLLARY 3. *Let $D \in \mathscr{E}_\theta$, $0 \leq \theta \leq 1$, and set*

$$\gamma(p, \theta) = 2 - \frac{\min\{1, \max\{p, 1-p\}\}}{2-\theta} .$$

If $s \in H^+(D)$ satisfies

$$\iint_D \delta_D(z)^{-\gamma(p,\theta)} s(z)^p \, dx \, dy < +\infty$$

for some $p \in (0, \infty)$, then $s \equiv 0$.

The next example shows that in the above corollary we cannot replace $\gamma(p, \theta)$ by any smaller number when $(2-\theta)/(3-\theta) < p \leq 1$.

EXAMPLE 3. Let D_θ, f_θ and P be as in Example 2. Then $P \circ f_\theta^{-1} \in H^+(D_\theta)$, and

$$\iint_{D_\theta} \lambda_{D_\theta}(z)^\gamma [(P \circ f_\theta^{-1})(z)]^p \, dx \, dy$$

$$= \iint_{\Delta} \lambda_\Delta(\zeta)^\gamma P(\zeta)^p |f_\theta'(\zeta)|^{2-\gamma} \, d\xi \, d\eta$$

$$\leq m_1 \iint_{\Delta} \lambda_\Delta(\zeta)^{\gamma-p} |\zeta - 1|^{-2p+(2-\gamma)(1-\theta)} \, d\xi \, d\eta .$$

Now it is easy to verify that, for each $q < -1$,

$$\int_0^{2\pi} |re^{it} - 1|^q \, dt = O((1-r)^{q+1})$$

as $r \uparrow 1$. If $(2-\theta)/(3-\theta) < p \leq 1$ and $2 - (2p-1)/(1-\theta) < \gamma < \gamma(p, \theta) = 2 - p/(2-\theta)$, then $-2p + (2-\gamma)(1-\theta) < -1$ so that

$$\iint_{D_\theta} \lambda_{D_\theta}(z)^\gamma [(P \circ f_\theta^{-1})(z)]^p \, dx \, dy$$
$$\leq m_2 \int_0^1 (1-r)^{p-\gamma-2p+(2-\gamma)(1-\theta)+1} \, dr$$
$$= m_2 \int_0^1 (1-r)^{(2-\theta)(\gamma(p,\theta)-\gamma)-1} \, dr < +\infty \, .$$

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