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Linearized oscillations for equations with positive and negative coefficients

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1. Introduction

Recently a linearized oscillation theory has been developed in [6]–[9] for nonlinear delay differential equations which in some sense parallels the so called linearized stability theory for differential equations. Roughly speaking, it has been shown that, under appropriate hypotheses, certain nonlinear differential equations have the same oscillatory behavior as an associated linear equation with constant coefficients.

Our aim in this paper is to present a linearized oscillation result for the neutral differential equation with positive and negative coefficients

$$(1)\quad \frac{d}{dt}[x(t) - P(t)G(x(t-\tau))] + Q_1(t)H_1(x(t-\sigma_1)) - Q_2(t)H_2(x(t-\sigma_2)) = 0$$

where we will assume that there are constants P_0, p_0, q_1, q_2 and M such that the following hypotheses are satisfied:

$$(2)\quad P, Q_1, Q_2 \in C[[t_0, \infty), R^+], \qquad G, H_1, H_2 \in C[R, R],$$

$$(3)\quad \tau \in (0, \infty), \quad \sigma_1, \sigma_2 \in [0, \infty), \qquad \sigma_1 \geqq \sigma_2,$$

$$(4)\quad \limsup_{t\to\infty} P(t) = P_0 \in (0, 1), \qquad \liminf_{t\to\infty} P(t) = p_0 \in (0, 1),$$

$$(5)\quad \liminf_{t\to\infty} Q_1(t) = q_1, \qquad \limsup_{t\to\infty} Q_2(t) = q_2,$$

$$(6)\quad 0 \leqq \frac{G(u)}{u} \leqq 1 \quad \text{for } u \neq 0, \qquad \lim_{u\to 0} \frac{G(u)}{u} = 1,$$

$$(7)\quad \frac{H_1(u)}{H_2(u)} \geqq 1 \quad \text{and} \quad 0 < \frac{H_2(u)}{u} \leqq M \quad \text{for } u \neq 0, \quad \lim_{u\to 0} \frac{H_2(u)}{u} = 1$$

and

$$(8)\quad 1 - P_0 - Mq_2(\sigma_1 - \sigma_2) > 0.$$

With Eq.(1) we associate the linear equation with constant coefficients

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(9) $$\frac{d}{dt}[y(t) - p_0 y(t-\tau)] + q_1 y(t-\sigma_1) - q_2 y(t-\sigma_2) = 0.$$

In Section 2 we establish some basic lemmas which we will use in Sections 3 and 4 and which are interesting in their own right. In Section 3 we establish sufficient conditions for the oscillation of all solutions of Eq.(1) in terms of the oscillation of all solutions of the linear Eq.(9). Finally, in Section 4 we show that, under appropriate hypotheses, if the linear Eq.(9) has a positive solution, so does Eq.(1).

Let $m = \max\{\tau, \sigma_1, \sigma_2\}$. By a *solution* of Eq.(1) we mean a function $x \in C[[t_1 - m, \infty), R]$, for some $t_1 \geqq t_0$, such that $[x(t) - P(t)G(x(t-\tau))]$ is continuously differentiable on $[t_1, \infty)$ and such that Eq.(1) is satisfied for $t \geqq t_1$.

As is customary, a solution of Eq.(1) is said to *oscillate* if it has arbitrarily large zeros. Otherwise the solution is called *nonoscillatory.*

In the sequel, unless otherwise specified, when we write a functional inequality we will assume that it holds for all sufficiently large values of t.

2. Some basic lemmas

In this section we will present some basic lemmas which are needed for our proofs of the main theorems in Sections 3 and 4.

The first lemma is extracted from [2].

LEMMA 1([2]). *Consider the neutral delay differential equation*

(10) $$\frac{d}{dt}[y(t) - py(t-\tau)] + q_1 y(t-\sigma_1) - q_2 y(t-\sigma_2) = 0$$

where p, q_1, q_2, τ, σ_1 *and* σ_2 *are real numbers. Then every solution of* (10) *oscillates if and only if its characteristic equation*

(11) $$\lambda - p\lambda e^{-\lambda\tau} + q_1 e^{-\lambda\sigma_1} - q_2 e^{-\lambda\sigma_2} = 0$$

has no real roots.

The next lemma is interesting in its own right. Roughly speaking, it states that if Eq.(10) oscillates at the point $(p, q_1, q_2) \in (0, 1] \times R^+ \times R^+$ then it also oscillates in some neighborhood of the same point.

LEMMA 2. *Assume that*

$$p \in (0, 1],\ q_1, q_2 \in (0, \infty),\ \tau \in (0, \infty),\ \sigma_1, \sigma_2 \in [0, \infty)$$

and suppose that every solution of Eq.(10) *oscillates. Then there exists an*

$\varepsilon_0 > 0$ such that for any ε, ε_1, $\varepsilon_2 \in [0, \varepsilon_0]$ every solution of the differential equation

$$(12) \quad \frac{d}{dt}[y(t) - (p - \varepsilon)y(t - \tau)] + (q_1 - \varepsilon_1)y(t - \sigma_1) - (q_2 + \varepsilon_2)y(t - \sigma_2)$$

also oscillates.

PROOF. By Lemma 1 it suffices to show that the characteristic equation of Eq.(12) has no real roots. The hypothesis that every solution of Eq.(10) oscillates implies that the characteristic equation

$$f(\lambda) = \lambda - \lambda p e^{-\lambda\tau} + q_1 e^{-\lambda\sigma_1} - q_2 e^{-\lambda\sigma_2} = 0$$

of Eq.(10) has no real roots. As $f(\infty) = \infty$, it follows that

$$(13) \qquad f(\lambda) > 0 \qquad \text{for all} \quad \lambda \in R .$$

By (13) we see that $f(0) = q_1 - q_2 > 0$ and either $\sigma_1 \geqq \sigma_2$ or $\sigma_1 < \sigma_2 \leqq \tau$. Then it follows that $f(-\infty) = \infty$ and so $m = \min_{\lambda \in R} f(\lambda)$ exists and is positive. Therefore

$$(14) \qquad \lambda - \lambda p e^{-\lambda\tau} + q_1 e^{-\lambda\sigma_1} - q_2 e^{-\lambda\sigma_2} \geqq m , \qquad \lambda \in R .$$

Set

$$\delta = \tfrac{1}{3} \min \{p, q_1 - q_2\} \qquad \text{and} \qquad g(\lambda) = \delta(|\lambda| e^{-\lambda\tau} + e^{-\lambda\sigma_1} + e^{-\lambda\sigma_2}) .$$

Observe that

$$f(\lambda) - g(\lambda) = \lambda - (p\lambda + \delta|\lambda|)e^{-\lambda\tau} + (q_1 - \delta)e^{-\lambda\sigma_1} - (q_2 + \delta)e^{-\lambda\sigma_2}$$

and so

$$\lim_{|\lambda| \to \infty} [f(\lambda) - g(\lambda)] = \infty .$$

In particular, there exists a $\lambda_0 > 0$ such that

$$f(\lambda) - g(\lambda) > \frac{m}{2} \qquad \text{for} \qquad |\lambda| \geqq \lambda_0 .$$

Let $\eta = \lambda_0 e^{\lambda_0\tau} + e^{\lambda_0\sigma_1} + e^{\lambda_0\sigma_2}$ and set $\varepsilon_0 = \min \{\delta, m/2\eta\}$. To complete the proof, it suffices to show that for every ε, ε_1, $\varepsilon_2 \in [0, \varepsilon_0]$ the characteristic equation

$$\lambda - \lambda(p - \varepsilon)e^{-\lambda\tau} + (q_1 - \varepsilon_1)e^{-\lambda\sigma_1} - (q_2 + \varepsilon_2)e^{-\lambda\sigma_2} = 0$$

of Eq.(12) has no real roots. Indeed for $|\lambda| \geqq \lambda_0$,

$$\begin{aligned}
&\lambda - \lambda(p-\varepsilon)e^{-\lambda\tau} + (q_1 - \varepsilon_1)e^{-\lambda\sigma_1} - (q_2 + \varepsilon_2)e^{-\lambda\sigma_2} \\
&= f(\lambda) - [-\varepsilon\lambda e^{-\lambda\tau} + \varepsilon_1 e^{-\lambda\sigma_1} + \varepsilon_2 e^{-\lambda\sigma_2}] \\
&\geqq f(\lambda) - g(\lambda) > m/2 > 0\,.
\end{aligned}$$

Also for $|\lambda| \leqq \lambda_0$,

$$\begin{aligned}
&\lambda - \lambda(p-\varepsilon)e^{-\lambda\tau} + (q_1 - \varepsilon_1)e^{-\lambda\sigma_1} - (q_2 + \varepsilon_2)e^{-\lambda\sigma_2} \\
&\geqq f(\lambda) - \varepsilon_0[\lambda_0 e^{\lambda_0\tau} + e^{\lambda_0\sigma_1} + e^{\lambda_0\sigma_2}] \geqq m - m/2 > 0\,.
\end{aligned}$$

and the proof is complete. ■

The following lemma about integral inequalities is interesting in its own right.

LEMMA 3. *Assume that* $F \in C[[T, \infty), (0, \infty)]$, $H \in C[R^+, R^+]$, $\tau \in (0, \infty)$, σ_1, $\sigma_2 \in [0, \infty)$, c_1, $c_2 \in [0, \infty)$, *$H(u)$ is nondecreasing in a neighborhood of the origin and* $\sigma_1 \geqq \sigma_2$. *Let* $m = \max\{\tau, \sigma_1\}$ *and suppose that the integral inequality*

$$(15)\quad F(t)z(t-\tau) + c_1 \int_{t-\sigma_1}^{t-\sigma_2} H(z(s))\,ds + c_2 \int_{t-\sigma_1}^{\infty} H(z(s))\,ds \leqq z(t)\,, \qquad t \geqq T$$

has a continuous positive solution $z\colon [T-m, \infty) \to (0, \infty)$ *such that*

$$(16)\qquad \lim_{t\to\infty} z(t) = 0\,.$$

Then there exists a positive solution $x\colon [T-m, \infty) \to (0, \infty)$ *of the corresponding integral equation*

$$(17)\quad F(t)x(t-\tau) + c_1 \int_{t-\sigma_1}^{t-\sigma_2} H(x(s))\,ds + c_2 \int_{t-\sigma_1}^{\infty} H(x(s))\,ds = x(t)\,, \qquad t \geqq T.$$

PROOF. Choose a $T' \geqq T$ and a $\delta > 0$ such that $z(t) > z(T')$ for $T - m \leqq t < T'$, $0 < z(t) < \delta$ for $t \geqq T' - m$ and $H(u)$ is nondecreasing in $[0, \delta]$. Define the set of functions

$$W = \{w \in C[[T-m, \infty), R^+]\colon 0 \leqq w(t) \leqq z(t) \text{ for } t \geqq T - m\}$$

and define the mapping S on W as follows:

$$(Sw)(t) = \begin{cases} F(t)w(t-\tau) + c_1 \displaystyle\int_{t-\sigma_1}^{t-\sigma_2} H(w(s))\,ds + c_2 \int_{t-\sigma_1}^{\infty} H(w(s))\,ds\,, & t \geqq T' \\ (Sw)(T') + z(t) - z(T')\,, & T - m \leqq t < T'\,. \end{cases}$$

Clearly, the function $Sw\colon [T-m, \infty) \to R^+$ is continuous. Also w_1, $w_2 \in W$

with $w_1 \leqq w_2$ implies $Sw_1 \leqq Sw_2$. Clearly, $Sz \leqq z$ and so $w \in W$ implies $Sw \leqq Sz \leqq z$. Thus, $S: W \to W$. We now define the following sequence on W:

$$z_0 = z \quad \text{and} \quad z_n = Sz_{n-1} \quad \text{for} \quad n = 1, 2, \ldots .$$

It is clear by induction that for $n = 1, 2, \ldots,$

$$0 \leqq z_n(t) \leqq z_{n-1}(t) \leqq z(t) \quad \text{for} \quad t \geqq T - m .$$

Set $x(t) = \lim_{t\to\infty} z_n(t)$ for $t \geqq T - m$. It follows by Lebesgue's dominated convergence theorem that $x(t)$ satisfies (17). Then by an argument similar to that in Lemma 2 in [1] (see also [4]) we can easily prove that $x(t)$ is continuous and positive. The proof is complete. ■

3. Linearized oscillations

The main result in this section is the following linearized oscillation result.

THEOREM 1. *Assume that* (2)–(8) *are satisfied and that every solution of Eq.*(9) *oscillates. Then every solution of Eq.* (1) *also oscillates.*

PROOF. Assume, for the sake of contradiction, that Eq.(1) has a nonoscillatory solution $x(t)$. We will assume that $x(t)$ is eventually positive. The case where $x(t)$ is eventually negative is similar and will be omitted. By Lemma 1 the characteristic equation

$$f(\lambda) = \lambda - p_0 \lambda e^{-\lambda\tau} + q_1 e^{-\lambda\sigma_1} - q_2 e^{-\lambda\sigma_2} = 0$$

of Eq.(9) has no real roots. As $f(0) = q_1 - q_2$ and $f(\infty) = \infty$ it follows that

(18) $$q_1 > q_2 .$$

Choose $\eta > 0$ and so small that

(19) $$q_1 - q_2 - 2\eta > 0$$

and

(20) $$l - P_0 - M(q_2 + \eta)(\sigma_1 - \sigma_2) > 0 .$$

Then by conditions (5) and (7) for $t \geqq t_1$, where t_1 is sufficiently large, we have

(21)
$$\frac{d}{dt}[x(t) - P(t)G(x(t-\tau))] + (q_1 - \eta)H_1(x(t-\sigma_1)) - (q_2 + \eta)H_2(x(t-\sigma_2)) \leqq 0 .$$

Set

$$z(t) = x(t) - P(t)G(x(t-\tau)) - (q_2+\eta)\int_{t-\sigma_1}^{t-\sigma_2} H_2(x(s))\,ds. \tag{22}$$

Then by (21), (7) and (19)

$$\begin{aligned} z'(t) &= [x(t) - P(t)G(x(t-\tau))]' - (q_2+\eta)[H_2(x(t-\sigma_2)) - H_2(x(t-\sigma_1))] \\ &\leqq -(q_1-\eta)H_1(x(t-\sigma_1)) + (q_2+\eta)H_2(x(t-\sigma_1)) \\ &\leqq -(q_1-q_2-2\eta)H_2(x(t-\sigma_1)) \leqq 0. \end{aligned} \tag{23}$$

Hence, $z(t)$ is decreasing. Now we claim that $x(t)$ is bounded. Otherwise there exists a sequence of points $\{t_n\}$ such that

$$\lim_{t\to\infty} t_n = \infty, \qquad \lim_{t\to\infty} x(t_n) = \infty \qquad \text{and} \qquad x(t_n) = \max_{s\leqq t_n} x(s).$$

Then by (4), (6), (7) and (20)

$$\begin{aligned} z(t_n) &= x(t_n) - P(t_n)G(x(t_n-\tau)) - (q_2+\eta)\int_{t_n-\sigma_1}^{t_n-\sigma_2} H_2(x(s))\,ds \\ &= x(t_n) - P(t_n)\frac{G(x(t_n-\tau))}{x(t_n-\tau)}x(t_n-\tau) - (q_2+\eta)\int_{t_n-\sigma_1}^{t_n-\sigma_2}\frac{H_2(x(s))}{x(s)}x(s)\,ds \\ &\geqq x(t_n)[1-P(t_n) - M(q_2+\eta)(\sigma_1-\sigma_2)] \to \infty \qquad \text{as} \qquad n\to\infty \end{aligned}$$

which contradicts (23). Then by (22) and (23) we see that $z(t)$ is also bounded and $\lim_{t\to\infty} z(t) = l \in R$. By integrating both sides of (23) from t_1 to ∞ we obtain

$$l - z(t_1) \leqq -(q_1-q_2-2\eta)\int_{t_1}^{\infty} H_2(x(s-\sigma_1))\,ds \tag{24}$$

which, in view of (7) and the boundedness of $x(t)$, implies $\liminf_{t\to\infty} x(t) = 0$. We claim that $\lim_{t\to\infty} x(t) = 0$. To this end, let $\mu = \limsup_{t\to\infty} x(t)$ and let $\{t_n\}$ and $\{t_n'\}$ be two sequences of points in the interval $[t_1, \infty)$ such that

$$\lim_{n\to\infty} t_n = \infty, \qquad \lim_{n\to\infty} t_n' = \infty, \qquad \lim_{n\to\infty} x(t_n) = 0 \qquad \text{and} \qquad \lim_{n\to\infty} x(t_n') = \mu.$$

From (22) we see that $z(t_n) \leqq x(t_n)$ which implies

$$l \leqq 0. \tag{25}$$

On the other hand, by taking $\varepsilon' > 0$ and sufficiently small, we find

$$\begin{aligned} z(t_n') &\geqq x(t_n') - P(t_n')x(t_n'-\tau) - M(q_2+\eta)\int_{t_n'-\sigma_1}^{t_n'-\sigma_2} x(s)\,ds \\ &\geqq x(t_n') - P(t_n')(\mu+\varepsilon') - M(q_2+\eta)(\sigma_1-\sigma_2)(\mu+\varepsilon'). \end{aligned}$$

By taking limits as $n \to \infty$ we obtain

$$l \geqq \mu - P_0(\mu + \varepsilon') - M(q_2 + \eta)(\sigma_1 - \sigma_2)(\mu + \varepsilon').$$

As ε' is arbitrary, we conclude that

$$l \geqq \mu - P_0\mu - M(q_2 + \eta)(\sigma_1 - \sigma_2)\mu = [1 - P_0 - M(q_2 + \eta)(\sigma_1 - \sigma_2)]\mu$$

which, in view of (20) and (25), implies $l = \mu = 0$. Hence

$$\lim_{t\to\infty} z(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} x(t) = 0. \tag{26}$$

Then from (24) we see that

$$-z(t) \leqq -(q_1 - q_2 - 2\eta)\int_t^\infty H_2(x(s - \sigma_1))\, ds.$$

Now by using (22) into this inequality we obtain,

$$\begin{aligned} x(t) &\geqq P(t)G(x(t-\tau)) + (q_2 + \eta)\int_{t-\sigma_1}^{t-\sigma_2} H_2(x(s))\, ds \\ &\quad + (q_1 - q_2 - 2\eta)\int_{t-\sigma_1}^{\infty} H_2(x(s))\, ds \\ &= P(t)\frac{G(x(t-\tau))}{x(t-\tau)}x(t-\tau) + (q_2 + \eta)\int_{t-\sigma_1}^{t-\sigma_2} \frac{H_2(x(s))}{x(s)}x(s)\, ds \\ &\quad + (q_1 - q_2 - 2\eta)\int_{t-\sigma_1}^{\infty} \frac{H_2(x(s))}{x(s)}x(s)\, ds. \end{aligned}$$

Let

$$0 < \varepsilon < \frac{1}{2}\min\left\{p_0, \frac{\eta}{q_2 + \eta}\right\}. \tag{27}$$

Then by (4), (6), (7) and (26) we see that for $t \geqq t_2$ where t_2 is sufficiently large,

$$\begin{aligned} x(t) &\geqq (p_0 - \varepsilon)x(t-\tau) + (q_2 + \eta)(1 - \varepsilon)\int_{t-\sigma_1}^{t-\sigma_2} x(s)\, ds \\ &\quad + (q_1 - q_2 - 2\eta)(1 - \varepsilon)\int_{t-\sigma_1}^{\infty} x(s)\, ds. \end{aligned}$$

It follows by Lemma 3 that the equation

$$v(t) = (p_0 - \varepsilon)v(t-\tau) + (q_2 + \eta)(1 - \varepsilon)\int_{t-\sigma_1}^{t-\sigma_2} v(s)\, ds$$

$$+ (q_1 - q_2 - 2\eta)(1 - \varepsilon) \int_{t-\sigma_1}^{\infty} v(s)\, ds$$

has a continuous positive solution $v: [T - m, \infty) \to (0, \infty)$ where $T \geqq t_2$ is sufficiently large. Clearly $v(t)$ is also a positive solution of the neutral equation

$$\frac{d}{dt}[v(t) - (p_0 - \varepsilon)v(t - \tau)] + (q_1 - \varepsilon_1)v(t - \sigma_1) - (q_2 + \varepsilon_2)v(t - \sigma_2) = 0$$

where $\varepsilon_1 = \eta + \varepsilon q_1 - \eta\varepsilon$ and $\varepsilon_2 = \eta - q_2\varepsilon - \eta\varepsilon$ are positive numbers. As η and ε are arbitrarily small, it follows that ε_1 and ε_2 are also arbitrarily small. Hence, by Lemma 2, Eq.(9) has a positive solution. This contradicts the hypothesis and completes the proof of the theorem. ■

4. Existence of a positive solution

Consider the neutral delay differential equation

$$\frac{d}{dt}[y(t) - p_0 y(t - \tau)] + q_1 y(t - \sigma_1) - q_2 y(t - \sigma_2) = 0 \tag{28}$$

where

$$p_0,\ q_1,\ q_2 \in (0, \infty),\ \tau \in (0, \infty),\ \sigma_1,\ \sigma_2 \in [0, \infty)\,. \tag{29}$$

The next lemma is extracted from [3].

LEMMA 4([3]). *Assume that* (29) *holds,*

$$0 < p_0 < 1\,, \qquad q_1 > q_2 \qquad \text{and} \qquad 1 - p_0 - q_2(\sigma_1 - \sigma_2) > 0\,. \tag{30}$$

Then every nonoscillatory solution of Eq.(28) *tends to* 0 *as* $t \to \infty$.

Now, consider the nonlinear neutral delay differential equation

$$\frac{d}{dt}[x(t) - p(t)x(t - \tau)] + q_1 H(x(t - \sigma_1)) - q_2 H(x(t - \sigma_2)) = 0 \tag{31}$$

where

(32)
$p \in C[[t_0, \infty), R^+]$, $H \in C[R, R]$, $q_1, q_2 \in (0, \infty)$, $\tau \in (0, \infty)$ and $\sigma_1, \sigma_2 \in [0, \infty)$.

The following theorem is a partial converse of Theorem 1 and shows that, under appropriate hypotheses, Eq.(31) has a positive solution provided that an associated linear equation has a positive solution.

THEOREM 2. *Assume that* (32), (30) *holds,*

$$0 < p(t) \leqq p_0\,, \qquad \sigma_1 \geqq \sigma_2 \tag{33}$$

and that there exists a positive constant δ *such that*

$$\text{either} \quad 0 \leqq H(u) \leqq u \quad \text{for} \quad 0 \leqq u \leqq \delta \tag{34}$$

$$\text{or} \quad 0 \geqq H(u) \geqq u \quad \text{for} \quad -\delta \leqq u \leqq 0\,. \tag{35}$$

Suppose also $H(u)$ *is nondecreasing in a neighborhood of the origin and that the characteristic equation of Eq.*(28)

$$\lambda - p_0 \lambda e^{-\lambda\tau} + q_1 e^{-\lambda\sigma_1} - q_2 e^{-\lambda\sigma_2} = 0 \tag{36}$$

has a real root. Then Eq.(31) *has a nonoscillatory solution.*

PROOF. Assume that $0 \leqq H(u) \leqq u$ for $0 \leqq u \leqq \delta$. The case where $0 \geqq H(u) \geqq u$ for $-\delta \leqq u \leqq 0$ is similar and will be omitted.

Let λ_0 be a real root of Eq.(36) and set $y(t) = \exp(\lambda_0 t)$. Then $y(t)$ is a nonoscillatory solution of Eq.(28). By Lemma 4, $y(t)$ tends to zero as $t \to \infty$ which implies that $\lambda_0 < 0$. Hence, there exists a sufficiently large T such that $0 < y(t) \leqq \delta$ for $t \geqq T - \bar{\delta}$ and H is nondecreasing in $[0, \delta]$.

By integrating both sides of (28) from $t \geqq T$ to ∞ we obtain

$$p_0 y(t-\tau) + q_1 \int_t^\infty y(s-\sigma_1)\,ds - q_2 \int_t^\infty y(s-\sigma_2)\,ds = y(t)$$

or

$$p_0 y(t-\tau) + q_2 \int_{t-\sigma_1}^{t-\sigma_2} y(s)\,ds + (q_1 - q_2) \int_{t-\sigma_1}^\infty y(s)\,ds = y(t)\,, \qquad t \geqq T.$$

In view of (30), (32) and (33) this equation yields

$$p(t) y(t-\tau) + q_2 \int_{t-\sigma_1}^{t-\sigma_2} H(y(s))\,ds + (q_1 - q_2) \int_{t-\sigma_1}^\infty H(y(s))\,ds \leqq y(t)\,, \qquad t \geqq T.$$

By Lemma 3 it follows that the corresponding equation

$$p(t) x(t-\tau) + q_2 \int_{t-\sigma_1}^{t-\sigma_2} H(x(s))\,ds + (q_1 - q_2) \int_{t-\sigma_1}^\infty H(x(s))\,ds = x(t)$$

also has a continuous positive solution. Hence, Eq.(31) has a positive solution. The proof of the theorem is complete. ■

Finally, by combining Theorems 1 and 2 we obtain the following necessary and sufficient condition for the oscillation of every solution of Eq.(31).

COROLLARY. *Assume that* (30), (32) *hold,*

$$\sigma_1 \geqq \sigma_2\,, \qquad 0 < p(t) \leqq p_0 = \lim_{t\to\infty} p(t)\,, \qquad 0 < \frac{H(u)}{u} \leqq 1$$

and $H(u)$ is nondecreasing in a neighborhood of the origin. Then every solution of Eq.(31) *oscillates if and only if every solution of the linear equation* (28) *oscillates, that is, if and only if Eq.* (36) *has no negative real roots.*

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