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Asymptotic distributions of the MLE's and the LR test in the growth curve model with a serial covariance structure

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1. Introduction

In the growth curve model of Potthoff and Roy [10] we observe an $N \times p$ random matrix Y whose rows are independently distributed as $N_p(\cdot, \Sigma)$ with

$$(1.1) \qquad E[Y] = A\Xi B,$$

where A and B are known $N \times k$ and $q \times p$ design matrices of ranks k and $q \leq p$, respectively, and Ξ is a $k \times q$ matrix of unknown parameters. This model is also called a GMANOVA model since the model in the special case of $B = I_p$ is a MANOVA model. The model in the case when Σ is arbitrary positive definite has been studied by many authors. A comprehensive review is given by Grizzle and Allen [5].

In this paper we consider the case when Σ has a serial covariance structure, or an autoregressive structure of the first order

$$(1.2) \qquad \Sigma = \sigma^2 G(\rho) = \sigma^2(\rho^{|i-j|}), \qquad i, j = 1, 2, \dots, p,$$

where $\sigma > 0$ and $|\rho| < 1$ are unknown. In most applications of the model (1.1), p is the number of time points observed each of the N subjects, $(q - 1)$ is the degree of polynomial, and k is the number of groups. Further, p is small. For the situations, it is natural to assume (1.2) as a covariance structure. In fact, Lee [8] has pointed out that the serial covariance structure (1.2) is appropriate for three sets of real data. Fujikoshi, Kanda and Tanimura [4] studied the limiting distributions of the MLE(maximum likelihood estimate)'s of ρ and σ^2 and the LR(likelihood ratio) test for (1.2) in the situation where p and k are fixed and $N \to \infty$. The purpose of this paper is to extend the limiting results by finding the next terms in the asymptotic expansions. Some preliminary results on our asymptotic method are given in Section 2. In Section 3 we obtain an asymptotic expansion of the distribution of the MLE's of ρ and σ^2 up to the order $N^{-1/2}$. In Section 4 we discuss with refinements of chi-square approximation to the null distribution of LR statistic for (1.2).

2. Preliminaries

Throughout this paper it is assumed that Σ has the serial covariance structure (1.2). Then

$$\Sigma^{-1} = \{\sigma^2(1-\rho^2)\}^{-1}(\rho^2 D_1 - 2\rho D_2 + D_3)\,, \tag{2.1}$$

where $D_1 = \mathrm{diag}(0, 1, \dots, 1, 0)$, $D_3 = I_p =$ the identity matrix of order p, and

$$D_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & & & 0 \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & 0 \end{pmatrix}. \tag{2.2}$$

It is known (Fujikoshi, Kanda and Tanimura [4]) that the estimates $\hat{\Xi}$, $\hat{\rho}$ and $\hat{\sigma}^2$ of Ξ, ρ and σ^2 based on the maximum likelihood of Y are given as the solutions of the following equations:

$$\hat{\Xi} = (A'A)^{-1}A'Y\hat{\Sigma}^{-1}B'(B\hat{\Sigma}^{-1}B')^{-1}\,, \tag{2.3}$$

$$\hat{\sigma}^2 = \{p(1-\hat{\rho}^2)\}^{-1}(a\hat{\rho}^2 - 2b\hat{\rho} + c)\,, \tag{2.4}$$

$$(p-1)a\hat{\rho}^3 - (p-2)b\hat{\rho}^2 - (pa+c)\hat{\rho} + pb = 0\,, \tag{2.5}$$

where $\hat{\Sigma} = \hat{\sigma}^2 G(\hat{\rho})$, $a = \mathrm{tr}\, D_1 R$, $b = \mathrm{tr}\, D_2 R$, $c = \mathrm{tr}\, D_3 R$, $R = n^{-1}(Y - A\hat{\Xi}B)' \times (Y - A\hat{\Xi}B)$ and $n = N - k$. Note that the MLE's of Ξ, ρ and σ^2 are given by $\hat{\Xi}$, $\hat{\rho}$ and $(n/N)\hat{\sigma}^2$, respectively. Our asymptotic distribution theory is based on perturbation method. We shall derive stochastic expansions of the MLE's in terms of

$$\begin{aligned} U &= (A'A)^{-1/2}A'(Y - A\Xi B)\Sigma^{-1/2}\,, \\ V &= \sqrt{n}(\Sigma^{-1/2}S\Sigma^{-1/2} - I_p)\,, \end{aligned} \tag{2.6}$$

where $S = n^{-1}Y'(I_N - A(A'A)^{-1}A')Y$. Here U and V are independent, the elements of U are independently distributed as $N(0, 1)$, and V has the limiting density given by

$$f_0(V) = \pi^{-p(p+1)/4}2^{-p(p+3)/4}\,\mathrm{etr}\,(-\tfrac{1}{4}V^2)\,. \tag{2.7}$$

We can write R in terms of U and V as

$$\begin{aligned} R &= S + n^{-1}\{I_p - \hat{\Sigma}^{-1}B'(B\hat{\Sigma}^{-1}B')^{-1}B\}'\Sigma^{1/2}U'U\Sigma^{1/2} \\ &\quad \times \{I_p - \hat{\Sigma}^{-1}B'(B\hat{\Sigma}^{-1}B')^{-1}B\} \\ &= \Sigma^{1/2}\{I_p + n^{-1/2}V + n^{-1}W\}\Sigma^{1/2} + O_p(n^{-3/2})\,, \end{aligned} \tag{2.8}$$

where $W = MU'UM$, $M = I_p - P_B$, and $P_B = \Sigma^{-1/2}B'(B\Sigma^{-1}B')^{-1}B\Sigma^{-1/2}$. This implies stochastic expansions of a, b and c as follows:

$$
\begin{aligned}
a &= a_0 + n^{-1/2}a_1 + n^{-1}a_2 + O_p(n^{-3/2}), \\
b &= b_0 + n^{-1/2}b_1 + n^{-1}b_2 + O_p(n^{-3/2}), \qquad (2.9)\\
c &= c_0 + n^{-1/2}c_1 + n^{-1}c_2 + O_p(n^{-3/2}),
\end{aligned}
$$

where (a_0, a_1, a_2), (b_0, b_1, b_2) and (c_0, c_1, c_2) are defined by $(\mathrm{tr}\,\tilde{D}_i, \mathrm{tr}\,\tilde{D}_i V, \mathrm{tr}\,\tilde{D}_i W)$, $i = 1, 2, 3$, respectively, and $\tilde{D}_i = \Sigma^{1/2}D_i\Sigma^{1/2}$, $i = 1, 2, 3$. These expansions as well as (2.4) and (2.5) are fundamental in deriving stochastic expansions of $\hat{\rho}$ and $\hat{\sigma}^2$.

We now list some formulas used in Sections 3 and 4, which are given by the following Lemmas 2.1 and 2.2. These are derived by straightforward calculations.

LEMMA 2.1. *Let Σ be the matrix defined by* (1.2), *and let $\eta = \sigma^2(1-\rho^2)^{-1}$. Then*

(1) $\mathrm{tr}\,\Sigma^2 = \eta^2\{p - 2\rho^2 - p\rho^4 + 2\rho^{2(p+1)}\}$,

(2) $\mathrm{tr}\,\Sigma^3 = \eta^3\{p + 3(p-2)\rho^2 - 3(p+2)\rho^4 - p\rho^6 + 6(p+1)\rho^{2(p+1)} - 6(p-1)\rho^{2(p+2)}\}$,

(3) $\mathrm{tr}\,\Sigma^2 D_1 = \eta^2\{p - 2 - p\rho^4 + 2\rho^{2p}\}$,

(4) $\mathrm{tr}\,\Sigma^2 D_1 \Sigma D_1 = \eta^3\{p - 2 + (3p-10)\rho^2 - 3p\rho^4 - p\rho^6 + 2p\rho^{2(p-1)} + 2(p+1)\rho^{2p} - 2(2p-5)\rho^{2(p+1)}\}$,

(5) $\mathrm{tr}\,\Sigma^3 D_1 = \eta^3\{p - 2 + 3(p-2)\rho^2 - (3p+4)\rho^4 - p\rho^6 + 2(2p+1)\rho^{2p} - 2(p-3)\rho^{2(p+1)} - 2(p-2)\rho^{2(p+2)}\}$.

LEMMA 2.2. *Let V be a $p \times p$ symmetric random matrix with the probability density function defined by* (2.7). *Then, it holds that for any $p \times p$ symmetric matrices A, B and C,*

(1) $E[\mathrm{tr}\,AV \cdot \mathrm{tr}\,BV] = 2\,\mathrm{tr}\,AB$,

(2) $E[\mathrm{tr}\,AVBV] = \mathrm{tr}\,AB + \mathrm{tr}\,A \cdot \mathrm{tr}\,B$,

(3) $E[\mathrm{tr}\,V^4] = p(2p^2 + 5p + 5)$,

(4) $E[(\mathrm{tr}\,AV)^2\,\mathrm{tr}\,(BV)^2] = 2\{(\mathrm{tr}\,A)^2 + \mathrm{tr}\,A^2\}\,\mathrm{tr}\,B^2 + 8\,\mathrm{tr}\,(AB)^2$,

(5) $E[(\mathrm{tr}\,AV)^2\,\mathrm{tr}\,BV \cdot \mathrm{tr}\,CV] = 4\,\mathrm{tr}\,A^2 \cdot \mathrm{tr}\,BC + 8\,\mathrm{tr}\,AB \cdot \mathrm{tr}\,AC$,

(6) $E[(\mathrm{tr}\,AV)^2\,\mathrm{tr}\,V^4] = 2(2p^3 + 5p^2 + 21p + 24)\,\mathrm{tr}\,A^2 + 16(\mathrm{tr}\,A)^2$,

(7) $E[(\mathrm{tr}\, AV)^2\, \mathrm{tr}\, BV \cdot \mathrm{tr}\, V^3]$

$$= 12\{(p+1)\, \mathrm{tr}\, A^2 \cdot \mathrm{tr}\, B + 2(p+1)\, \mathrm{tr}\, AB \cdot \mathrm{tr}\, A + 4\, \mathrm{tr}\, A^2 B\},$$

(8) $E[(\mathrm{tr}\, AV)^2 (\mathrm{tr}\, V^3)^2]$

$$= 12(4p^3 + 9p^2 + 43p + 48)\, \mathrm{tr}\, A^2 + 72(p^2 + 2p + 3)(\mathrm{tr}\, A)^2.$$

Some formulas in Lemma 2.2 have been used in Nagao [9], Fujikoshi [3], etc. For the formulas in the case of $A = B = I_p$, see Hayakawa and Kikuchi [6]. The following lemma was proved by Fujikoshi [2].

LEMMA 2.3. *Let V be a symmetric random matrix defined by* (2.6), *where nS is distributed as a Wishart distribution W_p (Σ, n). Then, the probability density function of V can be expanded as*

$$f(V) = f_0(V)\{1 + n^{-1/2} q_1(V) + n^{-1} q_2(V)\} + O(n^{-3/2}), \tag{2.10}$$

where $f_0(V)$ is given by (2.7), *and*

$$q_1(V) = -\tfrac{1}{2}(p+1)\, \mathrm{tr}\, V + \tfrac{1}{6}\, \mathrm{tr}\, V^3,$$

$$q_2(V) = \tfrac{1}{2}\{q_1(V)\}^2 - \tfrac{1}{24} p(2p^2 + 3p - 1) + \tfrac{1}{4}(p+1)\, \mathrm{tr}\, V^2 - \tfrac{1}{8}\, \mathrm{tr}\, V^4 .$$

3. The distribution of $\hat{\rho}$ and $\hat{\sigma}^2$

It is known (Fujikoshi, Kanda and Tanimura [4]) that the limiting distributions of $\sqrt{n}(\hat{\rho} - \rho)$ and $\sqrt{n}(\hat{\sigma}^2 - \sigma^2)$ are $N(0, \tau_\rho^2)$ and $N(0, \tau_\sigma^2)$, respectively, where

$$\tau_\rho^2 = p\{(p-1)r\}^{-1}(1 - \rho^2)^2, \qquad \tau_\sigma^2 = 2r^{-1}(1 + \rho^2)\sigma^4, \tag{3.1}$$

and $r = p - (p-2)\rho^2$. We generalize this result by finding the terms of $n^{-1/2}$ in the asymptotic expansions of the distributions.

LEMMA 3.1. *Let $\hat{\rho}$ and $\hat{\sigma}^2$ be the estimates defined by* (2.3) $\sim$ (2.5). *Then*

(1) $\hat{\rho} = \rho + n^{-1/2}\rho_1 + n^{-1}\rho_2 + O_p(n^{-3/2})$,

(2) $\hat{\sigma}^2 = \sigma^2 + n^{-1/2}\sigma_1 + n^{-1}\sigma_2 + O_p(n^{-3/2})$,

where

$$\rho_1 = -\{(p-1)r\sigma^2\}^{-1}\{(r - \rho^2)\rho a_1 - r b_1 + \rho c_1\},$$

$$\begin{aligned} \rho_2 = {} & 2r^{-1}(p-2)\rho\rho_1^2 - \{(p-1)r\sigma^2\}^{-1}\rho_1[\{p - 3(p-1)\rho^2\}a_1 \\ & - 2(p-2)\rho b_1 + c_1] \\ & - \{(p-1)r\sigma^2\}^{-1}[(r - \rho^2)\rho a_2 - r b_2 + \rho c_2], \end{aligned}$$

$$\sigma_1 = \sigma^2\{p(1-\rho^2)\}^{-1}\{2(p-1)\rho\rho_1 + (1-\rho^2)\,\mathrm{tr}\,V\}\,,$$
$$\sigma_2 = 2\sigma^2\{p(1-\rho^2)\}^{-1}\{(p-1)(1+\rho^2)(1-\rho^2)^{-1}\rho_1^2$$
$$+ (p-1)\rho\rho_2 + \rho\rho_1\,\mathrm{tr}\,V + \rho_1\sigma^{-2}(a_1\rho - b_1)$$
$$+ (2\sigma^2)^{-1}(a_2\rho^2 - 2b_2\rho + c_2)\}\,.$$

PROOF. The first result (1) is obtained by substituting (2.9) into (2.5) and finding the solution of $\hat{\rho}$ in an expanded form. This result and the equation (2.4) yield the second result (2).

We now use the following method frequently used in finding asymptotic expansions. Let ϕ_ρ be the characteristic function of $\sqrt{n}(\hat{\rho} - \rho)$. Then we may write ϕ_ρ as

$$\phi_\rho(t) = E\left[\exp(it\rho_1)\left\{1 + \frac{1}{\sqrt{n}}it\rho_2\right\}\right] + O(n^{-1}) \tag{3.2}$$

which will be evaluated as

$$\exp\left(-\frac{1}{2}\tau_\rho^2 t^2\right)\left[1 + \frac{1}{\sqrt{n}}\{(it)g_1 + (it)^3 g_3\}\right] + O(n^{-1})\,. \tag{3.3}$$

An asymptotic expansion can be obtained by formally inverting (3.3). The validity of this method has been established under certain regularity conditions (see, e.g., Bhattacharya and Ghosh [1]). We can write ρ_1 and ρ_2 as

$$\rho_1 = \alpha_1\,\mathrm{tr}\,C_1V, \qquad \rho_2 = \alpha_2\,\mathrm{tr}\,C_1V\cdot\mathrm{tr}\,C_2V + \alpha_1\,\mathrm{tr}\,C_1W, \tag{3.4}$$

where

$$\alpha_1 = -(1-\rho^2)\{2(p-1)\rho\}^{-1}\,, \qquad \alpha_2 = p(1-\rho^2)^2\{2(p-1)^2 r\rho\sigma^2\}^{-1}\,,$$
$$C_1 = I_p - p(r\sigma^2)^{-1}Q\,, \qquad Q = \Sigma - \rho^2\tilde{D}_1\,, \tag{3.5}$$
$$C_2 = \tilde{D}_1 + \{p + (p-2)\rho^2\}\{p(1-\rho^2)r\}^{-1}Q\,.$$

Using some formulas with respect to V (see, e.g., Siotani, Hayakawa and Fujikoshi [11]) and noting that $E[W] = kM$, it is seen that $\phi_\rho(t)$ can be expressed as (3.3) with

$$\begin{aligned} g_1 &= 2\alpha_2\,\mathrm{tr}\,C_1C_2 + \alpha_1 k\,\mathrm{tr}\,C_1M\,, \\ g_3 &= \tfrac{4}{3}\alpha_1^3\,\mathrm{tr}\,C_1^3 + 4\alpha_1^2\alpha_2\,\mathrm{tr}\,C_1^2\cdot\mathrm{tr}\,C_1C_2\,. \end{aligned} \tag{3.6}$$

The coefficients g_1 and g_3 can be simplified by using Lemma 2.1. The final result is given in Theorem 3.1.

THEOREM 3.1. *The distribution function of $\sqrt{n}(\hat{\rho}-\rho)/\tau_\rho$ can be expanded as*

$$P\left(\frac{\sqrt{n}(\hat{\rho}-\rho)}{\tau_\rho}\leq x\right)=\Phi(x)-\frac{1}{\sqrt{n}}\left\{\frac{g_1}{\tau_\rho}\Phi^{(1)}(x)+\frac{g_3}{\tau_\rho^3}\Phi^{(3)}(x)\right\}+O(n^{-1}),$$

where $\Phi^{(j)}(x)$ denotes the j-th derivative of the standard normal distribution function $\Phi(x)$, τ_ρ is given by (3.1), *and*

$$\begin{aligned}g_1&=-2r^{-1}(1-\rho^2)\rho+\alpha_1 k\operatorname{tr}C_1M,\\ g_3&=\tfrac{8}{3}r^{-3}\alpha_1^3 p(p-1)\rho^4\{3p^2-(p-2)(3p-2)\rho^2\}.\end{aligned}\tag{3.7}$$

Similarly we can derive an asymptotic expansion of the distribution of $\sqrt{n}(\hat{\sigma}^2-\sigma^2)$ by expanding its characteristic function

$$\phi_\sigma(t)=E\left[\exp(it\sigma_1)\left\{1+\frac{1}{\sqrt{n}}it\sigma_2\right\}\right]+O(n^{-1}).\tag{3.8}$$

The evaluation of (3.8) can be done by the same way as in the case of $\phi_\rho(t)$. We note only that

$$\begin{aligned}\sigma_1&=r^{-1}\operatorname{tr}QV,\\ \sigma_2&=\alpha_3\operatorname{tr}C_1V\cdot\operatorname{tr}C_3V+r^{-1}\operatorname{tr}QW,\end{aligned}\tag{3.9}$$

where

$$\alpha_3=\{(p-1)r\}^{-1}(1-\rho^2),\qquad C_3=\tilde{D}_1-r^{-1}(p-2)Q.\tag{3.10}$$

THEOREM 3.2. *The distribution function of $\sqrt{n}(\hat{\sigma}^2-\sigma^2)/\tau_\sigma$ can be expanded as*

$$P\left(\frac{\sqrt{n}(\hat{\sigma}^2-\sigma^2)}{\tau_\sigma}\leq x\right)=\Phi(x)-\frac{1}{\sqrt{n}}\left\{\frac{h_1}{\tau_\sigma}\Phi^{(1)}(x)+\frac{h_3}{\tau_\sigma^3}\Phi^{(3)}(x)\right\}+O(n^{-1}),$$

where τ_σ is given by (3.1), *and*

$$\begin{aligned}h_1&=kr^{-1}\operatorname{tr}QM,\\ h_3&=\tfrac{4}{3}r^{-3}\{p+3r\rho^2-(p-2)\rho^6\}\sigma^6.\end{aligned}\tag{3.11}$$

The coefficients g_1 and h_1 in the MANOVA case of $B=I_p$ can be simplified as

$$g_1=-2r^{-1}(1-\rho^2)\rho\qquad\text{and}\qquad h_1=0,\tag{3.12}$$

respectively, since $M=O$.

4. The distributions of LR statistics

4.1. The LR test. We consider the problem of testing

$$H: \Sigma = \sigma^2 G(\rho) \tag{4.1}$$

against $\Sigma > 0$. The MLE's of Ξ and Σ under $\Sigma > 0$ was obtained by Khatri [7]. The LR test is equivalent to reject the hypothesis H if

$$\begin{aligned} T &= -n \log \{|\hat{\Sigma}_\Omega|/|\hat{\sigma}^2 G(\hat{\rho})|\} \\ &= -n \log [|\hat{\Sigma}_\Omega|/\{(\hat{\sigma}^2)^p (1 - \hat{\rho}^2)^{p-1}\}] , \end{aligned} \tag{4.2}$$

is large, where $\hat{\Xi}_\Omega = (A'A)^{-1}A'YS^{-1}B'(BS^{-1}B')^{-1}$ and

$$\hat{\Sigma}_\Omega = n^{-1}(Y - A\hat{\Xi}_\Omega B)'(Y - A\hat{\Xi}_\Omega B) . \tag{4.3}$$

It is known (Fujikoshi, Kanda and Tanimura [4]) that the limiting null distribution of T is a chi-square distribution with the degrees of freedom $f = \frac{1}{2}p(p+1) - 2$. We shall investigate refinements of the chi-square approximation.

4.2. The MANOVA case. The testing problem in the MANOVA case of $B = I_p$ is equivalent to test (4.1) against $\Sigma > 0$, based on S, where S is distributed as a Wishart distribution $W_p(\Sigma, n)$. The LR statistic T can be written as

$$T = -n \log [|S|/\{(\hat{\sigma}^2)^p (1 - \hat{\rho}^2)^{p-1}\}] . \tag{4.4}$$

Here $\hat{\sigma}^2$ is given by (2.4), but $\hat{\rho}$ is defined as the solution of

$$\begin{aligned} &(p-1)(\mathrm{tr}\, D_1 S)\hat{\rho}^3 - (p-2)(\mathrm{tr}\, D_2 S)\hat{\rho}^2 \\ &\quad - (p\, \mathrm{tr}\, D_1 S + \mathrm{tr}\, S)\hat{\rho} + p(\mathrm{tr}\, D_2 S) = 0 . \end{aligned} \tag{4.5}$$

Recently Wakaki, Eguchi and Fujikoshi [12] have obtained an asymptotic expansion of the null distribution of a class of tests for a general covariance structure, based on a Wishart matrix. From their result it follows that

$$P(T \le x) = P(\chi_f^2 \le x) + \frac{\ell}{n}\{P(\chi_{f+2}^2 \le x) - P(\chi_f^2 \le x)\} + O(n^{-3/2}) , \tag{4.6}$$

where ℓ is a constant not depending on n. This suggests to use $\tilde{T} = \{1 - 2\ell(fn)^{-1}\}T$ as a better chi-square approximation, since

$$P(\tilde{T} \le x) = P(\chi_f^2 \le x) + O(n^{-3/2}) . \tag{4.7}$$

We are concerned with a simple expression for ℓ. A general expression for ℓ has been given by Wakaki, Eguchi and Fujikoshi [12]. However, it is difficult to simplify it in this case. We shall determine ℓ by evaluating the expectation of T. Based on stochastic expansions of $\log|S|$, $\hat{\rho}$ and $\hat{\sigma}^2$, we can expand T as

$$T = T_0 + n^{-1/2}T_1 + n^{-1}T_2 + O_p(n^{-3/2}), \tag{4.8}$$

where

$$\begin{aligned}
T_0 &= \tfrac{1}{2}\{\mathrm{tr}\, V^2 - p^{-1}(\mathrm{tr}\, V)^2\} - \tfrac{1}{4}r\{p(p-1)\rho^2\}^{-1}(\mathrm{tr}\, C_1 V)^2, \\
T_1 &= -\tfrac{1}{3}\{\mathrm{tr}\, V^3 - p^{-2}(\mathrm{tr}\, V)^3\} + \{(1-\rho^2)\sigma^2\}^{-1}a_1\rho_1^2 \\
&\quad + \{p^2(1-\rho^2)^3\}^{-1}[(1-\rho^2)\{pr + 2(p-2)\rho^2\}\rho_1^2\,\mathrm{tr}\, V \\
&\quad + \tfrac{8}{3}(p-1)(p-2)\rho^3\rho_1^3], \\
T_2 &= \tfrac{1}{4}\{\mathrm{tr}\, V^4 - p^{-3}(\mathrm{tr}\, V)^4\} \\
&\quad + \{p^3(1-\rho^2)^3\}^{-1}[(1-\rho^2)\{-(p^2-(p-2)^2\rho^2)\rho_1^2(\mathrm{tr}\, V)^2 \\
&\quad + p^2(p-1)r\bar{\rho}_2^2 + 2p(p+(p-2)\rho^2)\rho_1\bar{\rho}_2\,\mathrm{tr}\, V\} \\
&\quad + 2\rho_1^2\{2p(p-1)\rho(p+(p-2)\rho^2)\bar{\rho}_2 \\
&\quad + (p^2-(p-2)^2\rho^2)\rho\rho_1\,\mathrm{tr}\, V\}],
\end{aligned}$$

$\bar{\rho}_2$ is the one obtained from ρ_2 by putting $a_2 = b_2 = c_2 = 0$. Using Lemma 2.3 we can write

$$\begin{aligned}
E[T] &= E_V\left[T_0 + \frac{1}{\sqrt{n}}\{q_1(V)T_0 + T_1\}\right. \\
&\quad \left. + \frac{1}{n}\{q_2(V)T_0 + q_1(V)T_1 + T_2\}\right] + O(n^{-3/2}),
\end{aligned} \tag{4.9}$$

where E_V denotes the expectation with respect to V with the probability density function $f_0(V)$ in (2.7). It is easily seen that

$$E_V[T_0] = f, \qquad E_V[q_1(V)T_0 + T_1] = 0.$$

Therefore,

$$2\ell = E_V[q_2(V)T_0 + q_1(V)T_1 + T_2]. \tag{4.10}$$

Each of the expectations in (4.10) is evaluated by Lemma 2.2. After much simplification, we obtain

$$\ell = \{24p(p-1)r^3\}^{-1}(\ell_0 + \ell_1\rho^2 + \ell_2\rho^4 + \ell_3\rho^6), \tag{4.11}$$

where

$$\begin{aligned}
\ell_0 &= p^3(2p^5 + p^4 - 4p^3 + p^2 + 2p - 20), \\
\ell_1 &= -3p^2(p-2)(2p^5 + p^4 - 4p^3 + p^2 + 2p + 4),
\end{aligned}$$

$$\ell_2 = 3p^2(p-2)(2p^5 - 3p^4 - 6p^3 + 9p^2 + 8),$$

$$\ell_3 = -(p-2)^2(2p^6 - 3p^5 - 6p^4 + 9p^3 - 16).$$

In the special case of $p = 2$, we have $f = 1$ and $\ell = \frac{3}{4}$. Hence the corrected LR statistic is given by

$$\tilde{T} = \left(1 - \frac{3}{2n}\right)T. \tag{4.12}$$

This result agrees with the one (see, e.g., Siotani, Hayakawa and Fujikoshi [11, p. 374]) for testing an intraclass correlation structure.

4.3. The GMANOVA case. It is interesting to see how the correction term $\{1 - 2\ell(fn)^{-1}\}$ in the MANOVA case should be modified in the case rank$(B) < p$. For this purpose, we shall obtain a constant $\tilde{\ell}$ such that

$$E[T] = f + 2(\ell + \tilde{\ell})n^{-1} + O(n^{-3/2}), \tag{4.13}$$

where ℓ is given by (4.11). We note that a corrected LR statistic

$$\tilde{T} = \{1 - 2(\ell + \tilde{\ell})(fn)^{-1}\}T \tag{4.14}$$

gives a better chi-square approximation, in a sense that $E[\tilde{T}] = f + O(n^{-3/2})$.

The LR statistic T in the general model (1.1) with (1.2) can be expanded as

$$T = T_0 + n^{-1/2}(T_1 + \tilde{T}_1) + n^{-1}(T_2 + \tilde{T}_2) + O_p(n^{-3/2}), \tag{4.15}$$

where T_i's are given by (4.8),

$$\begin{aligned}
\tilde{T}_1 &= \mathrm{tr}\, VW - p^{-1}\,\mathrm{tr}\, V \cdot \mathrm{tr}\, W + \rho^{-1}\rho_1\,\mathrm{tr}\,(I_p - \tau Q)W - 2\rho(1-\rho^2)^{-1}\rho_1\,\mathrm{tr}\, W,\\
\tilde{T}_2 &= -\mathrm{tr}\, V^2W + \tfrac{1}{2}\,\mathrm{tr}\, W^2 - \tfrac{1}{2}p^{-1}(\mathrm{tr}\, W)^2 + p^{-2}(\mathrm{tr}\, V)^2\,\mathrm{tr}\, W\\
&\quad - \tfrac{1}{2}\{p^2(p-1)\rho^2(1-\rho^2)^2\}^{-1}[2p^2(1-\rho^2)^3\rho\tau a_1\rho_1\\
&\quad + \tfrac{1}{2}pr(1-\rho^2)^2\,\mathrm{tr}\, C_1W + 4(p-1)(2p+r)\rho^2\rho_1^2\\
&\quad - r(1-\rho^2)^2(\mathrm{tr}\, V)^2 + \{p(4-p) + 2(p-2)^2\rho^2\}(1-\rho^2)\rho\rho_1\,\mathrm{tr}\, V\\
&\quad + p\tau(1-\rho^2)^3\,\mathrm{tr}\, V\cdot\mathrm{tr}\, QV - 4p(1-\rho^2)^2\rho\tau\rho_1\,\mathrm{tr}\, QV]\cdot\mathrm{tr}\, C_1W\\
&\quad + \tau a_2\rho_1^2 - 2\{p\rho(1-\rho^2)\}^{-1}\{r\rho_1\,\mathrm{tr}\, V\cdot\mathrm{tr}\, W + \tau\rho\rho_1^2\,\mathrm{tr}\, QW\}\\
&\quad + \{p^2(1-\rho^2)^2\}^{-1}\{p(4-p) + (p-2)^2\rho^2\}\rho_1^2\,\mathrm{tr}\, W\\
&\quad + (p\rho)^{-1}\tau\rho_1(\mathrm{tr}\, V\cdot\mathrm{tr}\, QW + \mathrm{tr}\, QV\cdot\mathrm{tr}\, W) + 2\,\mathrm{tr}\, P_BVWV\\
&\quad + \mathrm{tr}\, P_B(KWK - VWV) + 2\rho^{-1}\rho_1\,\mathrm{tr}\, KP_B(I_p - \tau Q)W,
\end{aligned}$$

$K = -(p\rho)^{-1}\{r(1-\rho^2)^{-1}\rho_1 - \rho \operatorname{tr} V\}I_p + \rho^{-1}\tau\rho_1 Q$ and $\tau^{-1} = (1-\rho^2)\sigma^2$. By the same way as in (4.9), we have

$$2\tilde{\ell} = E_V E_W[q_1(V)\tilde{T}_1 + \tilde{T}_2]\,. \tag{4.16}$$

Noting that W is distributed as a Wishart distribution $W_p(\Sigma, n)$, first we take the expectation with respect to W. Next we take the expectation with respect to V with the use of Lemma 2.2. After much simplification, we obtain

$$\begin{aligned}\tilde{\ell} = \tfrac{1}{2}k[\tilde{\ell}_0 &+ \{(p-1)r\sigma^2\}^{-1}p(1-\rho^2) \operatorname{tr} M\Sigma^{1/2}D_1\Sigma^{1/2}\\ &- 2\{p(p-1)\}^{-1}(p-2) \operatorname{tr} MC_1\\ &+ \tfrac{1}{4}\{p(p-1)\rho^2\}^{-1}r\{2 \operatorname{tr}(MC_1)^2 - 4 \operatorname{tr} MC_1^2\\ &- k(\operatorname{tr} MC_1)^2\}]\,,\end{aligned}$$

where $M = I_p - \Sigma^{-1/2}B'(B\Sigma^{-1}B')^{-1}B\Sigma^{-1/2}$, and

$$\begin{aligned}\tilde{\ell}_0 = \tfrac{1}{2}p^{-1}(p-q)[(k+p)q &- \{(p-1)r\}^{-1}(p-2)\{p(p^2+2p-1)\\ &- (p^3-5p+6)\rho^2\}]\,.\end{aligned}$$

When $B = I_p$, we have $\tilde{\ell} = 0$ since $p = q$ and $M = O$.

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