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On oscillation of Volterra integral equations and first order functional differential equations

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1. Introduction

In the functional differential equations with deviating arguments, oscillatory behaviour of solutions plays an important role and has been studied by many authors (cf. [1]). To our knowledge, however, few papers are known in the oscillation theory of integral equations. For a result on the latter problem we refer to Parhi and Misra [4].

Consider the Volterra integral equation

$$x(t) = f(t) - \int_0^t a(t, s)g(s, x(s))\, ds\,, \qquad t \geqq 0\,. \tag{1}$$

In (1), $f\colon [0, \infty) \to R$ and $g\colon [0, \infty) \times R \to R$ are continuous, and $a\colon [0, \infty) \times [0, \infty) \to R$ is such that $a(t, s) = 0$ for $s > t$, $a(t, s) \geqq 0$ for $0 \leqq t < \infty$ and $0 \leqq s \leqq t$. In addition $a(t, s)$ is supposed to be continuous for $0 \leqq t < \infty$ and $0 \leqq s \leqq t$. We consider only the solutions of (1) which exist, are continuous on $[0, \infty)$, and are nontrivial in any neighbourhood of infinity. A solution $x(t)$ of (1) is said to be *oscillatory* if $x(t)$ has zeros for arbitrarily large t; otherwise, a solution $x(t)$ is said to be *nonoscillatory*. This definition is the same as in the case of differential equations.

In this paper we propose some criteria for all solutions of (1) to be oscillatory, which is not considered in [4], and also apply the idea to study the oscillation of functional differential equations of the type

$$x'(t) + \sum_{i=1}^{n} p_i(t)|x(g_i(t))|^\alpha \operatorname{sgn} x(g_i(t)) = q(t)x(t) + r(t)\,, \qquad \alpha > 0\,. \tag{2}$$

We will see that every solution of Volterra integral equation (1) or first order functional differential equation (2) is oscillatory when the strong oscillating forced terms are attached.

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2. Oscillation of integral equations

THEOREM 1. *Suppose that* $xg(t, x) > 0$ *for* $x \neq 0$ *and* $t \geqq 0$ *and that* $\int_0^\sigma a(t, s)\, ds$ *is bounded for each fixed* $\sigma > 0$. *If* $\limsup_{t\to\infty} f(t) = \infty$ *and* $\liminf_{t\to\infty} f(t) = -\infty$, *then every solution of* (1) *is oscillatory.*

PROOF. If we suppose that $x(t)$ is not an oscillatory solution of (1) on $[0, \infty)$, then there exists a $T > 0$ such that $x(t) > 0$ or < 0 for $t \geqq T$. Suppose that $x(t) > 0$ for $t \geqq T$. From (1), we have

$$\begin{aligned} 0 < x(t) &= f(t) - \int_0^t a(t, s)g(s, x(s))\, ds \\ &= f(t) - \int_0^T a(t, s)g(s, x(s))\, ds - \int_T^t a(t, s)g(s, x(s))\, ds \\ &\leqq f(t) - \int_0^T a(t, s)g(s, x(s))\, ds\,, \qquad t \geqq T, \end{aligned}$$

since $\int_T^t a(t, s)g(s, x(s))\, ds \geqq 0$ for $t \geqq T$. This inequality combined with the fact that $\int_0^T a(t, s)g(s, x(s))\, ds < L \int_0^T a(t, s)\, ds < \infty$ where $L = \sup_{t\in[0,T]} g(t, x(t))$ and $\liminf_{t\to\infty} f(t) = -\infty$ yields a contradiction to $x(t) > 0$. Next, suppose that $x(t) < 0$ for $t \geqq T$. From (1), we obtain

$$\begin{aligned} 0 > x(t) &= f(t) - \int_0^T a(t, s)g(s, x(s))\, ds - \int_T^t a(t, s)g(s, x(s))\, ds \\ &\geqq f(t) - \int_0^T a(t, s)g(s, x(s))\, ds\,, \qquad t \geqq T, \end{aligned}$$

which, in view of the assumption $\limsup_{t\to\infty} f(t) = \infty$, leads to a contradiction to $x(t) < 0$. Q.E.D.

EXAMPLE 1. Consider the integral equation

$$x(t) = f(t) - \int_0^t a(t, s)g(s, x(s))\, ds\,, \tag{3}$$

where $f(t) = (t \sin t + \cos t - 1)(t + 1)^{-1/2} + (\cos t)^3$, $a(t, s) = 0$ if $s > t$, $a(t, s) = (t + 1)^{-1/2}s^{2/3}$ for $0 \leqq t < \infty$, $0 \leqq s \leqq t$ and $g(s, x(s)) = (sx(s))^{1/3}$. All conditions of Theorem 1 are satisfied, so that every solution of (3) is oscillatory. In fact, $x(t) = (\cos t)^3$ is one such solution of (3).

REMARK. Parhi and Misra [4] did not give any example of a real oscillating solution of (1).

THEOREM 2. *Let* $g(t, x)$ *be monotone increasing in* x *for fixed* t *and* $xg(t, x) > 0$ *if* $x \neq 0$. *Let* $g(t, x)$ *be bounded for* $0 \leqq t < \infty$, $|x| \leqq \alpha$ *for every fixed* $\alpha > 0$. *Suppose that* $-\infty < \lim_{t\to\infty} \int_\delta^t a(t, s)g(s, -K)\, ds < 0$ *for every fixed* $\delta > 0$ *and* $K > 0$ *and that* $h(t) = \int_0^\sigma a(t, s)\, ds$, $0 \leqq t < \infty$, *is bounded for every fixed* $\sigma > 0$. *If* $\lim_{t\to\infty} f(t) = -\infty$, *then all bounded solutions of* (1) *are oscillatory.*

PROOF. Let $x(t)$, $t \in [0, \infty)$, be a bounded solution of (1) which is not oscillatory. So there exist constants $K > 0$ and $T \geqq 0$ such that $|x(t)| \leqq K$ for $t \in [0, \infty)$ and $x(t) > 0$ or <0 for $t \geqq T$. Suppose first that $0 < x(t) \leqq K$ for $t \geqq T$. Then

$$\begin{aligned} 0 < x(t) &= f(t) - \int_0^T a(t, s)g(s, x(s))\, ds - \int_T^t a(t, s)g(s, x(s))\, ds \\ &\leqq f(t) + \left| \int_0^T a(t, s)g(s, x(s))\, ds \right| \\ &\leqq f(t) + L \int_0^T a(t, s)\, ds \\ &\leqq f(t) + L \cdot B\,, \end{aligned} \tag{4}$$

where $L = \sup_{0 \leqq t \leqq T} |g(t, x(t))|$ and $B = \sup_{t \geqq T} \int_0^T a(t, s)\, ds < \infty$. Because of (4) and $\lim_{t\to\infty} f(t) = -\infty$, this gives a contradiction. Next suppose that $x(t) < 0$ for $t \geqq T$. Then

$$\begin{aligned} -K \leqq x(t) &= f(t) - \int_0^T a(t, s)g(s, x(s))\, ds - \int_T^t a(t, s)g(s, x(s))\, ds \\ &\leqq f(t) + L \int_0^T a(t, s)\, ds - \int_T^t a(t, s)g(s, -K)\, ds \\ &\leqq f(t) + L \cdot B + D\,, \end{aligned} \tag{5}$$

where $K > 0$, $L = \sup_{0 \leqq t \leqq T} |g(t, x(t))|$, $B = \sup_{t \geqq T} \int_0^T a(t, s)\, ds$ and $D = -\lim_{t\to\infty} \int_T^t a(t, s)g(s, -K)\, ds$. Taking the limit as $t \to \infty$ in (5) we get a contradiction. Q.E.D.

REMARK. Theorem 2 is a variant of Theorem 2.2 of [4].

3. Oscillation of differential equations

THEOREM 3. *Assume that*
(i) $p_i, r \in C[R_+, R]$, $p_i(t) \geqq 0$, $i \in I_n = \{1, 2, \ldots, n\}$;
(ii) $g_i \in C^1[R_+, R]$, $\lim_{t\to\infty} g_i(t) = \infty$, $g_i'(t) \geqq 0$, $i \in I_n$;

(iii) $q \in C[R_+, R]$;

(iv) $\limsup_{t\to\infty} \int_{C_1}^t p_i(u) \exp(\int_{C^*}^{g_i(u)} q(s)\,ds)^\alpha \exp(-\int_{C^*}^u q(s)\,ds)\,du = \infty$ *for some* $i \in I_n$ *and any fixed* $C^* > 0$ *and* $C_1 > 0$;

(v) *there exists a function* $Q \in C^1[R_+, R]$ *such that*

$$Q'(t) = r(t) \exp\left(-\int_C^t q(u)\,du\right), \qquad t > 0, \qquad \text{where } C > 0 \text{ is a constant}.$$

Then $\lim_{t\to\infty} Q(t) = 0$ *implies that every solution* $x(t)$ *of* (2) *is either oscillatory or else satisfies* $\lim_{t\to\infty} x(t) \exp(-\int_C^t q(s)\,ds) = 0$.

PROOF. Set $z(t) = x(t)\exp(-\int_C^t q(s)\,ds)$. Then, in view of the assumptions (i)–(iii), (2) becomes

$$x'(t) \exp\left(-\int_C^t q(s)\,ds\right) + x(t)(-q(t)) \exp\left(-\int_C^t q(s)\,ds\right)$$

$$+ \sum_{i=1}^n p_i(t) \exp\left(-\int_C^t q(s)\,ds\right)\left(\exp\left(\int_C^{g_i(t)} q(s)\,ds\right)\right)^\alpha |z(g_i(t))|^\alpha \operatorname{sgn} z(g_i(t))$$

$$= r(t) \exp\left(-\int_C^t q(s)\,ds\right), \qquad t \geqq C_2 \geqq C,$$

and it follows that

$$z'(t) + \sum_{i=1}^n L_i(t)|z(g_i(t))|^\alpha \operatorname{sgn} z(g_i(t)) = Q'(t), \tag{6}$$

where

$$L_i(t) = p_i(t)\left(\exp\left(\int_C^{g_i(t)} q(s)\,ds\right)\right)^\alpha \exp\left(-\int_C^t q(s)\,ds\right)$$

and

$$Q'(t) = r(t) \exp\left(-\int_C^t q(s)\,ds\right).$$

Let $x(t)$ be a nonoscillatory solution of (2) which may be supposed positive for sufficiently large t. In this case, $z(t)$ is also a nonoscillatory solution of (6) and is positive for sufficiently large t. Set $\bar{x}(t) = z(t) - Q(t)$. Then $\bar{x}(t)$ satisfies

$$\bar{x}'(t) + \sum_{i=1}^n L_i(t)(\bar{x}(g_i(t)) + Q(g_i(t)))^\alpha = 0, \tag{7}$$

so that $\bar{x}'(t) < 0$ for large t, because of (i), (ii) and the fact that $z(t) = \bar{x}(t) + Q(t) > 0$. Consequently, we have $\lim_{t\to\infty} \bar{x}(t) = k$, where k is a constant. If $k < 0$, then we get the contradiction that $z(t) < 0$ for sufficiently large t. If

$k > 0$, then we obtain

$$z(g_i(t)) = \bar{x}(g_i(t)) + Q(g_i(t)) \geqq k/2\,, \qquad \text{for all} \quad i \in I_n\,,$$

for sufficiently large t. From (7) it then follows that

$$\bar{x}'(t) + \sum_{i=1}^{n} L_i(t)(k/2)^{\alpha} \leqq 0\,. \tag{8}$$

Integrating (8) from C_1 to t, $C_1 \geqq C_2$, we have

$$\bar{x}(t) - \bar{x}(C_1) + \left(\sum_{i=1}^{n} \int_{C_1}^{t} L_i(s)\,ds\right)(k/2)^{\alpha} \leqq 0\,. \tag{9}$$

By taking the upper limit of (9) as $t \to \infty$, we get a contradiction to (iv). Hence we conclude that $k = 0$, which implies

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} x(t) \exp\left(-\int_{C}^{t} q(s)\,ds\right) = 0\,. \qquad \text{Q.E.D.}$$

THEOREM 4. *In addition to* (i) *and* (iii) *of Theorem* 3 *assume that*

(ii-1) $g_i \in C^1[R_+, R]$, $\lim_{t\to\infty} g_i(t) = \infty$, $i \in I_n$;
(v-1) *there exists a function* $Q \in C^1[R_+, R]$ *such that*

$$Q'(t) = r(t) \exp\left(-\int_{C}^{t} q(u)\,du\right), \qquad t > 0\,,$$

where $C > 0$ *is a constant*, $\limsup_{t\to\infty} Q(t) = \infty$ *and* $\liminf_{t\to\infty} Q(t) = -\infty$. *Then every solution of* (2) *is oscillatory.*

PROOF. Suppose that a solution $x(t)$ of (2) is nonoscillatory. We assume that $x(t) > 0$ for sufficiently large t; say $t \geqq T$. Set $z(t) = x(t) \exp(-\int_T^t q(u)\,du)$, then using the assumptions (i), (iii), (ii-1) and (v-1), we see that $z(t)$ satisfies

$$z'(t) + \sum_{i=1}^{n} L_i(t)(z(g_i(t)))^{\alpha} = Q'(t)\,, \tag{10}$$

where

$$L_i(t) = p_i(t)\left(\exp\left(\int_{T}^{g_i(t)} q(s)\,ds\right)\right)^{\alpha} \exp\left(-\int_{T}^{t} q(s)\,ds\right)$$

and

$$Q'(t) = r(t) \exp\left(-\int_{T}^{t} q(s)\,ds\right), \qquad t \geqq T^* \geqq T\,.$$

From (10), we obtain $z'(t) \leqq Q'(t)$, $t \geqq T$. By integrating this we have

$0 < z(t) \leqq z(T^*) + Q(t) - Q(T^*)$ for $t \geqq T^*$, but this is a contradiction, since $\liminf_{t\to\infty} Q(t) = -\infty$. If we suppose that $x(t) < 0$, then we obtain $0 > z(t) \geqq z(T_1) + Q(t) - Q(T_1)$, for $t \geqq T_1 \geqq T$. This also leads to a contradiction, since $\limsup_{t\to\infty} Q(t) = \infty$. Q.E.D.

COROLLARY 5. *In addition to* (i), (iii) *and* (ii-1) *of Theorem* 4 *assume that*

(vi) $f \in C[R, R]$, $xf(x) > 0$ *for* $x \neq 0$; $f(x)$ *is nondecreasing*;

(v-2) *there exists a function* $Q \in C^1[R_+, R]$ *such that* $Q'(t) = r(t)$, $t > 0$, $\limsup_{t\to\infty} Q(t) = \infty$ *and* $\liminf_{t\to\infty} Q(t) = -\infty$.

Then every solution of the equation

$$x'(t) + \sum_{i=1}^{n} p_i(t)f(x(g_i(t))) = r(t)$$

is oscillatory.

REMARK. Corollary 5 is an extension of Theorem 3 of [2].

EXAMPLE 2. Consider the equation

$$x'(t) + tx(t + 2\pi)^{1/3} = t\cos t - 3(\cos t)^2 \sin t\,, \qquad \text{for } t \geqq \pi\,. \tag{11}$$

Since the function

$$Q(t) - Q(C) = \int_C^t Q'(u)\,du = \int_C^t (u\cos u - 3(\cos u)^2 \sin u)\,du$$

$$= t\sin t + \cos t + \cos^3 t - C\sin C - \cos C - \cos^3 C\,, \qquad C \geqq \pi\,,$$

satisfies the conditions of Theorem 4, every soluion of (11) is oscillatory. In fact $x(t) = \cos^3 t$ is such an oscillatory solution.

REMARK. Theorems 3 and 4 are (closely) related to Theorem 3.7.1 of [1], but equation (12) is not covered by Theorem 3.7.1.

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