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Dynamics of interfaces in reaction diffusion systems

Xu-Yan CHEN

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1. Introduction

It is observed frequently in nature that a physical system develops in such an inhomogeneous way that in different spatial regions, the system is in distinctive states or it behaves in distinctive manners. Under certain circumstances, these spatial regions may be well-distinguished, and be clearly separated by certain boundaries, which are so-called *interfaces*. Such interfaces form a variety of geometrical patterns, and exhibit significant changes in size, shape and location as time passes.

The interfacial phenomena attract a lot of attention and stimulate continuing activity in natural science. From various points of view, people attempt to understand the underlying mechanism of generation of the interfaces, their internal structure and their dynamical behavior. For example, the classical Stefan problem treats the liquid freezing and the solid melting. The front of shock waves in Riemann problem for fluid flow is another type of interfaces ([16, 26]). Friedrichs [14] presented a fantastic description of many interesting interfacial phenomena arising in physics. More recently, chemists observed rotating spiral waves and expanding target patterns in the well-known Belousov-Zhabotinski reagent ([41, 40]), which leads to extensive mathematical studies of reaction diffusion systems ([10, 36] and references therein). The pigmentation patterns of the shells and the animal coats are also viewed as a kind of interfaces in a theory of biological pattern formation ([27, 30]).

In this paper we are concerned with an interfacial phenomenon in a class of reaction diffusion systems. Mathematically, we study a nonlinear partial differential equation of parabolic type:

$$(1.1a)^{\varepsilon} \qquad \frac{\partial u}{\partial t} = \varepsilon \Delta u + \frac{1}{\varepsilon} f(u, v) \qquad x \in \mathbf{R}^n,\ t > 0,$$

$$(1.1b)^{\varepsilon} \qquad \frac{\partial v}{\partial t} = D \Delta v + g(u, v) \qquad x \in \mathbf{R}^n,\ t > 0,$$

with initial condition

$$(1.2a) \qquad u(x, 0) = \phi(x) \qquad x \in \mathbf{R}^n,$$

(1.2b) $$v(x, 0) = \psi(x) \qquad x \in \mathbf{R}^n.$$

The nonlinear terms f and g in equation (1) are as follows:

(1.3a) $$f(u, v) = F(u) - v = u(1 - u)(u - a) - v,$$

(1.3b) $$g(u, v) = u - \gamma v.$$

In the above, Δ stands for the Laplacian $\sum_{i=1}^{n} \partial^2/\partial x_i^2$, $D > 0$, $a \in (0, 1)$ and $\gamma > 0$ are constants, and $\varepsilon > 0$ serves as a small paramater.

Problem $(1.1)^{\varepsilon}$–(1.2) is a diffusing and reacting system with two components, which is a prototype of modelling the propagation of chemical waves in excitable media. The unknowns in $(1.1)^{\varepsilon}$–(1.2) are real-valued functions $u(x, t)$ and $v(x, t)$, that represent—as termed by Paul Fife—the propagator and the controller in the system respectively, while $\phi(x)$ and $\psi(x)$ are given initial data.

Of essential importance in the problem are the assumption that $0 < \varepsilon \ll 1$, and the bistable property—as explained below—of the term f describing the kinetics of chemicai reaction of the propagator component u. The former implies that u diffuses quite slowly while its reaction takes place much faster. Under this situation, the development of interfaces in this system consists of two consecutive stages. The first one is a short time-period of the birth of interfaces, and next an evolutionary process of interfaces follows.

Before discussing these two stages in detail, let us look at a simple typical example of interfacial dynamical patterns, namely, the traveling wave solution of constant speed. Notice that the nullcline $\{(u, v); f(u, v) = 0\}$ of the function f consists of three branches

(1.4a) $$u = h_+(v) \qquad v < F(a_+),$$

(1.4b) $$u = h_-(v) \qquad v > F(a_-),$$

(1.4c) $$u = h_0(v) \qquad u \in [F(a_-),\ F(a_+)],$$

where the relation $h_-(v) < h_0(v) < h_+(v)$ holds for $v \in (F(a_-), F(a_+))$, and a_+ and a_- are two solutions of the algebraic equation $F'(u) = 0$ with $a_+ > a_-$. Fix arbitrarily $b \in (a_-, a_+)$. It is well-known that the nonlinear eigenvalue problem

(1.5a) $$U''(z) + cU'(z) + f(U(z), b) = 0 \qquad z \in \mathbf{R},$$

(1.5b) $$\lim_{z \to -\infty} U(z) = h_+(b),$$

(1.5c) $$\lim_{z \to \infty} U(z) = h_-(b),$$

(1.5d) $$U(0) = h_0(b)$$

has a unique solution (U, c). Here $U = U(z; b)$ is a smooth function of z given by

$$U = U(z; b) = h_-(b) + (h_+(b) - h_-(b)) \times \left(1 + \exp\left\{\frac{z + z_0}{\sqrt{2}}(h_+(b) - h_-(b))\right\}\right)^{-1}, \tag{1.6}$$

where $z_0 \in \mathbf{R}$ is a constant ensuring the condition (1.5d), and $c = W(b)$ is a real number given by

$$c = W(b) = [h_+(b) + h_-(b) - 2h_0(b)]/\sqrt{2}. \tag{1.7}$$

For a discussion of this problem in its more general setting, see [2, 3]. Moreover, for the Cauchy problem of the equation $\partial u/\partial t = \partial^2 u/\partial x^2 + f(u, b)$ ($x \in \mathbf{R}$, $t > 0$), Fife and McLeod [13] proved that the traveling wave solution U is stable subject to a fairly large class of perturbations of initial data. To relate these waves to equation $(1.1)^\varepsilon$, consider the following scalar equation:

$$\frac{\partial u}{\partial t} = \varepsilon \Delta u + \frac{1}{\varepsilon} f(u, b) \qquad x \in \mathbf{R}^n,\ t > 0, \tag*{$(1.8)^\varepsilon$}$$

where $\varepsilon > 0$ is a small parameter. Given any point $x_0 \in \mathbf{R}^n$ and any vector $\xi \in S^{n-1}$, define

$$U^\varepsilon(x, t) = U\left(\frac{(x - x_0, \xi) - ct}{\varepsilon}\right) \qquad x \in \mathbf{R}^n,\ t \in \mathbf{R}, \tag{1.9}$$

where $(\cdot\,, \cdot)$ denotes the usual scalar product in $\mathbf{R}^n$. Then U^ε satisfies $(1.8)^\varepsilon$ and

$$\lim_{\varepsilon \downarrow 0} U^\varepsilon(x, t) = \begin{cases} h_+(b) & (x - x_0, \xi) < ct, \\ h_-(b) & (x - x_0, \xi) > ct. \end{cases} \tag{1.10}$$

The transition layer of $U^\varepsilon(\cdot, t)$ is flat and has thickness of order $O(\varepsilon)$. In fact, it locates at a narrow strip along the hyperplane $\{x \in \mathbf{R}^n; (x - x_0, \xi) = ct\}$, which moves at a constant speed c in the direction ξ. The geometry and dynamics of layers of this special solution U^ε are so notably simple. The importance of traveling waves lies in that they describe the local internal structure of generic layers (see Appendix 2, especially equation (2A. 12)). The macroscopic behavior of layers in general cases, however, is far more complicated. More often than not, transition layers may be curved rather than flat, and the velocity of their propagation may vary with time and place. In order to surmount the difficulties in analyzing the exact solutions of problem $(1.1)^\varepsilon$–(1.2) for $\varepsilon > 0$, it is useful to take formal limits of equation $(1.1)^\varepsilon$ as $\varepsilon \downarrow 0$, respectively, in the first and second stages of interfacial dynamics, and then to study such reduced limiting equations.

To discuss the first stage of the interfacial dynamics for problem $(1.1)^{\varepsilon}$–(1.2), we observe that for smooth initial data ϕ and ψ without sharp jumps, the diffusion term $\varepsilon\Delta u$ in $(1.1\mathrm{a})^{\varepsilon}$ may be negligible for a time, in other words, the ordinary differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{\varepsilon} f(u, v) \qquad x \in \mathbf{R}^n,\ t > 0 \tag{1.11}$$

approximates equation $(1.1\mathrm{a})^{\varepsilon}$ and governs the behavior of u, for $t \ll 1$. Since $0 < \varepsilon \ll 1$, the change of v is much smaller as compared with that of u in the early stage, that is, $v(x, t) \approx \psi(x)$ for $t \ll 1$. Replacing v by ψ in (1.11), we obtain

$$\frac{\partial u}{\partial t} = \frac{1}{\varepsilon} f(u, \psi) \qquad x \in \mathbf{R}^n,\ t > 0. \tag{1.12}$$

For simplicity, assume that $F(a_-) + \sigma \le \psi(x) \le F(a_+) - \sigma$ $(x \in \mathbf{R}^n)$ for some $\sigma > 0$, and that zero is a regular value of the function $\phi(x) - h_0(\psi(x))$. Then by virtue of the quadrature of equation (1.12) under the initial condition (1.2a), one observes that $u(x, t)$ approaches two stable branches $h_+(\psi(x))$ or $h_-(\psi(x))$ during a short time-period, depending on whether $\phi(x) > h_0(\psi(x))$ or $\phi(x) < h_0(\psi(x))$ respectively, apart from the region where $\phi(x) \approx h_0(\psi(x))$. Thus the whole space $\mathbf{R}^n$ is decomposed into two different domains, namely, "the excited region" $\Omega_+(t)$ in which $u \approx h_+(v)$, and "the rest region" $\Omega_-(t)$ in which $u \approx h_-(v)$. What splits them is a quite thin "interfacial layer region" $\Omega_0(t)$, across which the system undergoes a sharp transition from the rest state $u = h_-(v)$ to the excited state $u = h_+(v)$, or *vise versa.* The thickness of $\Omega_0(t)$ becomes to be of order $O(\varepsilon)$ after a time-period of order $O(\varepsilon|\log \varepsilon|)$, as seen by a formal calculation using equation (1.12).

The above formal discussion concerning the first stage of emergence of interfaces will be further justified by mathematical proofs in Section 2 of this paper. We shall present there explicit estimates of the thickness of layer region and the length of time interval for formation of such sharp transition layers. The analysis relies on a scaling transformation and a probabilistic argument and is based on the study of a much simpler case, namely, that of scalar equations $\partial u/\partial t = \varepsilon\Delta u + \varepsilon^{-1} F(x, u)$. Our results are related to those of Fife and Hsiao [12], who investigated one-dimensional scalar equations. The methods we used are however quite different from theirs and are effective enough to permit us to deal with the higher dimensional case and the systems case.

In the second evolutionary stage, the interfacial region $\Omega_0(t)$ may move and deform as t evolves. It is worth stressing that the diffusion term $\varepsilon \Delta u$ can no longer be neglected near a sharp transition layer, and thus equation (1.11) is not suited to be an approximation of $(1.1a)^\varepsilon$ at the second stage. To derive a limiting equation, we rely on matched asymptotic expansion methods.

We may accept several hypotheses below on the limiting behavior of solutions as $\varepsilon \downarrow 0$:

(a) the transition layer region $\Omega_0(t)$ tends to a hypersurface $\Gamma(t)$ in $\mathbf{R}^n$, which is called *interface*;

(b) the component $u(\cdot, t)$ has jump discontinuity across $\Gamma(t)$; the relations $u = h_+(v)$ and $u = h_-(v)$ hold, respectively, in two disjoint regions $\Omega_+(t)$ and $\Omega_-(t)$ with

$$\mathbf{R}^n \backslash \Gamma(t) = \Omega_+(t) \cup \Omega_-(t); \tag{1.13}$$

(c) $\Gamma(t)$ changes smoothly as time t varies (at least in a certain, possibly short, time interval).

For simplicity we assume further that

(d) $\Gamma(t)$ is a compact hypersurface for each $t > 0$;

(e) there exists $\sigma > 0$ such that

$$a_- + \sigma \le v(x, t) \le a_+ - \sigma \qquad x \in \mathbf{R}^n,\ t > 0. \tag{1.14}$$

Under these hypotheses and by matched asymptotic methods, the following free-boundary problem can be deduced as the singular limit of equation $(1.1)^\varepsilon$ when $\varepsilon \downarrow 0$:

$$\frac{\partial v}{\partial t} = D\Delta v + g_\pm(v),\ t > 0, \tag{1.15a}$$

$$\frac{\partial \Gamma}{\partial t} = \{W(v) - \varepsilon(n-1)\kappa\} N \qquad \eta \in M,\ t > 0, \tag{1.15b}$$

$$v(\cdot\,, t) \in C^1(\mathbf{R}^n) \qquad t > 0, \tag{1.15c}$$

with the initial condition

$$v(x, 0) = \psi(x) \qquad x \in \mathbf{R}^n, \tag{1.16a}$$

$$\Gamma(\eta, t) = S(\eta) \qquad \eta \in M. \tag{1.16b}$$

Here $\Gamma(\cdot, t)\colon M \to \mathbf{R}^n$ is an imbedding of an $(n-1)$-dimensional compact manifold M into $\mathbf{R}^n$. Denote by $\Gamma(t)$ the imbedded hypersurface $\{\Gamma(\eta, t); \eta \in M\}$. In (1.15a), the functions g_+ and g_- are defined by $g_\pm(v) = g(h_\pm(v), v)$ with $h_\pm$ as in (1.4), and $\Omega_+(t)$ and $\Omega_-(t)$ are two disjoint regions satisfying (1.13). In the *interface equation* (1.15b), the function W is as in (1.7), $\kappa(\eta, t)$ is the mean curvature of the hypersurface at the point $x = \Gamma(\eta, t)$, and $N(\eta, t)$ is the unit normal vector of $\Gamma(t)$ pointing from $\Omega_+(t)$ to $\Omega_-(t)$. This equation reveals that the motion of interface $\Gamma(t)$ is affected by an interplay between the mean curvature of the interface and the value of v on it. Physically speaking, the interfacial dynamics is under the influence of the interfacial tension, together with the driving force due to the difference between the depth of two potential wells $h_+(v)$ and $h_-(v)$. Derivation of (1.15a) and (1.15c) involves the *outer* part of the layers and is concerned with a regularity analysis of solutions, while derivation of (1.15b) requires a careful analysis of the *inner* part of the layers, for which it is convenient to choose a new coordinate system. The reader is referred to Appendices 1 and 2 for more details.

REMARK 1.1. The unknowns in our limiting problem (1.15)–(1.16) are the variable v and the interface Γ. The component u is not involved directly in (1.15)–(1.16), but is determined by the equations $u = h_\pm(v)$ in $\Omega_\pm(t)$, after the problem (1.15)–(1.16) is solved.

REMARK 1.2. One may wonder why the term $\varepsilon(n-1)\kappa$ in equation (1.15b) is retained in the limit as $\varepsilon \downarrow 0$. The reason is that in certain circumstances the terms $\varepsilon(n-1)\kappa$ and $W(v)$ may be of same order. For instance, in the case of the spiral wave solutions in Belousov-Zhabotinski reactions, the curvature κ is so large near the center of the spiral (see [24, 37]) that the above mentioned two terms are both of order $O(\sqrt{\varepsilon})$. Another situation in which $\varepsilon(n-1)\kappa$ cannot be omitted, is found in a study of equilibria of reaction diffusion systems. Ohta, Mimura and Kobayashi [31] have shown that under suitable conditions, equation (1.15)–(1.16) has redially symmetric equilibrium solutions, with the radii of interfaces being of order $O(1)$. Since the relation

$$W(v) = \varepsilon(n-1)\kappa \qquad \text{on the interface } \Gamma \tag{1.17}$$

holds for equilibria, $W(v)|_\Gamma$ in this case has the same order $O(\varepsilon)$, as $\varepsilon(n-1)\kappa$ has. As a matter of fact, the interfacial tension term $\varepsilon(n-1)\kappa$ plays a primary role in the stability and instability analysis of interfaces (see [31]).

There arise naturally two problems about the limiting equation (1.15):

(P1) Prove rigorously that (1.15) holds in the limit as $\varepsilon \downarrow 0$ for solutions of the original problem $(1.1)^\varepsilon$–(1.2).

(P2) Study the free-boundary problem (1.15)–(1.16) and then compare the results with the properties of solutions of $(1.1)^\varepsilon$–(1.2).

Problem (P1) is not yet understood satisfactorily, as far as this author is aware. Although there are many results strongly supporting the validity of (1.15) as the singular limiting equation $(1.1)^\varepsilon$ when $\varepsilon \downarrow 0$, most of those are limited to scalar equations and treat only equilibrium solutions or one-dimensional case, rather than the nonstationary solutions of higher dimensional systems, in which we are chiefly interested here. For instance, Modica [29] studied a variational problem with a small parameter in the Van der Waals-Cahn-Hilliard theory of phase transitions in a fluid confined to a bounded container, and proved that the interfaces arising from the singular limits of the global minimizers satisfy an equation similar to (1.15b) (see also [19]). Mimura [28] discussed problem (P1) in detail for special solutions of one-dimensional systems (that is, $n = 1$ in problem $(1.1)^{\varepsilon}$–(1.2)), such as traveling waves and standing pulses. In one dimensional scalar equations, there is an interesting phenomenon — the "very slow dynamics" of interfaces — which is beyond the scope of our interface equation (1.15b) (see [5]) mainly due to the fact that the curvature term or the interfacial tension makes no sense when $n = 1$. More recently, it seems that de Mottoni and Schatzman [7] made some progress in (P1) for non-equilibrium solutions of scalar equations in higher dimensions.

Hilhorst, Nishiura and Mimura [21] considered (P2) and investigated the global interfacial dynamics for problem (1.15)–(1.16) in one-dimensional case. As for two-dimensional systems, recently Ikeda [22] has numerically studied the free-boundary problem (1.15)–(1.16). In the present paper we shall focus our attention on problem (P2) for higher dimensional systems. Existence (locally in time) and regularity of classical solutions to problem (1.15)–(1.16) will established in section 3. We may regard such a solvability not only as a proof of mathematical consistence of (1.15) but also as partial evidence for the validity of the limiting equation (1.15). In general one cannot expect the existence of global classical solutions to problem (1.15)–(1.16), since the interface may develop singularities or self-intersections at some moment.

A striking difference between the one-dimensional case and the higher dimensional case, is that in the former case the interface equation is an ordinary differential equation, while in the latter which we study in this paper, the interface evolves according to a partial differential equation (1.15b), therefore the curvature effects must be taken into account. It should be mentioned here that our interface equation (1.15b) is closely related to the following equation:

$$\frac{\partial \Gamma}{\partial t} = -\varepsilon(n-1)\kappa N, \tag{1.18}$$

which can be viewed as the singular limiting equation of $\partial u/\partial t = \varepsilon \Delta u + \varepsilon^{-1}F(u)$ where $F(u) = u(1-u)(u-1/2)$. This equation has been extensively studied (see [1, 4, 17, 18, 20, 33]). Equation (1.15b) differs from (1.18) in that the former involves also the component v in addition to the curvature κ. This feature causes certain marked contrasts between the behavior of solutions of (1.15) and that of solutions of (1.18). For example, when starting from a convex hypersurface, the solution of (1.18) remains convex, while the interface of the solution of (1.15) may become nonconvex in some cases (see [31, 28]). In spite of such differences, the nonexistence of global classical solutions is a common difficulty in analyzing (1.15) and (1.18). For equation (1.18), quite recently Sethian [32, 34] came up with a new idea of applying the Crandall-Lions theory of viscosity solutions to overcome this difficulty and has shown its usefulness in the numerical studies. Evans and Spruck [8] furthered his idea and proved the global existence and uniqueness along with many interesting properties of viscosity solution to the initial-value problem for equation (1.18) (see also [6] for a generalization). We hope that their approach may be possibly extended to the study of equation (1.15), at least may provide us a method to resolve (1.15)–(1.16) globally in time. Tracing the qualitative behavior of solutions to (1.15) is a far more hard task. These remain to be attractive problems for further attacks.

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2. The first stage: the emergene of transition layers

In this section we study the emergence of transition layers in problem $(1.1)^\varepsilon$–(1.2).

Let f, g and F be as in (1.3), and denote by a_+ and a_-, respectively, the local maximum and the local minimum points of the function F (see (1.4)). Define $F^*: \mathbf{R} \to \mathbf{R}$ by

$$F^*(u) = \begin{cases} F(u) & a_- \le u \le a_+, \\ F(a_+) & u > a_+, \\ F(a_-) & u < a_-. \end{cases} \tag{2.1}$$

For given initial data ϕ and ψ, define

$$\Omega_+ = \{x \in \mathbf{R}^n;\ F^*(\phi(x)) > \psi(x)\}, \tag{2.2}$$

$$\Omega_- = \{x \in \mathbf{R}^n;\ F^*(\phi(x)) < \psi(x)\}, \tag{2.3}$$

and

$$\Gamma = \{x \in \mathbf{R}^n;\ F^*(\phi(x)) = \psi(x)\}, \tag{2.4}$$

and define a function $U^*: \mathbf{R}^n \setminus \Gamma \to \mathbf{R}$ by

$$U^*(x) = \begin{cases} h_+(\psi(x)) & \text{for } x \in \Omega_+, \\ h_-(\psi(x)) & \text{for } x \in \Omega_-, \end{cases} \tag{2.5}$$

where $h_\pm$ are functions as in (1.4). Note that if ϕ and ψ are continuous, the function U^* defined by (2.5) has jumping discountinuity at the set Γ.

THEOREM 2.1. *Assume that ϕ and ψ: $\mathbf{R}^n \to \mathbf{R}$ are bounded and continuous, and let $(u^\varepsilon(x, t), v^\varepsilon(x, t))$ be the solution to problem* (1.1)$^\varepsilon$–(1.2). *Then*

$$\lim_{s\to\infty} \lim_{\varepsilon\downarrow 0} u^\varepsilon(x, \varepsilon s) = U^*(x) \qquad x \in \mathbf{R}^n \setminus \Gamma. \tag{2.6}$$

Roughly speaking, the above theorem means that there appears a jumping discontinuity of the component u^ε near the set Γ in a short period of time. The following theorem provides a more precise estimate on the width of transition layer and the length of time for the layer generation.

THEOREM 2.2. *Let ϕ and ψ: $\mathbf{R}^n \to \mathbf{R}$ be functions with bounded C^2 norms, and assume that there exist positive constants β_2, σ_2 and ρ_2 such that*

$$F(a_-) + \beta_2 \le \psi(x) \le F(a_+) - \beta_2 \tag{2.7a}$$

for all $x \in \mathbf{R}^n$ and such that

$$|\phi(x) - h_0(\psi(x))| \ge \sigma_2 \min\{\rho_2, dist(x, \Gamma)\} \tag{2.7b}$$

for any $x \in \mathbf{R}^n$. Then for any $\delta > 0$, there exist positive constants ε_2, C_2 and K_2, independent of ε, such that

$$|u^\varepsilon(x, t) - U^*(x)| < \delta \tag{2.8}$$

for $0 < \varepsilon \le \varepsilon_2$, $K_2\varepsilon|\log\varepsilon| \le t \le 3K_2\varepsilon|\log\varepsilon|$, and $x \in \mathbf{R}^n \setminus \Gamma$ with $dist(x, \Gamma) \ge C_2\varepsilon|\log\varepsilon|$.

Theorem 2.1 is proved by a simple scaling transformation. In proving Theorem 2.2, it is important to study the behavior of the set

$$\{x \in \mathbf{R}^n;\ F^*(u^\varepsilon(x, t)) = v^\varepsilon(x, t)\} \tag{2.9}$$

in a small time interval. This will be achieved by employing a large deviation result in the probability theory for the Brownian motion (see Lemma 2.8).

LEMMA 2.3. *Assume the conditions of Theorem 2.1. Then there exists a constant $C_3 = C_3(\phi, \psi) > 0$ such that*

$$|u^\varepsilon(x, t)| + |v^\varepsilon(x, t)| \le C_3 \tag{2.10}$$

for any $\varepsilon > 0$, $x \in \mathbf{R}^n$ *and* $s \ge 0$.

PROOF. Apply the standard invariant regions argument (see [35]). □

PROOF OF THEOREM 2.1. Fix arbitrarily $x_0 \in \mathbf{R}^n$. Define

$$\begin{aligned} U^\varepsilon(y, s) &= u^\varepsilon(x_0 + \varepsilon y, \varepsilon s) \qquad y \in \mathbf{R}^n,\ s \ge 0, \\ V^\varepsilon(y, s) &= v^\varepsilon(x_0 + \varepsilon y, \varepsilon s) \qquad y \in \mathbf{R}^n,\ s \ge 0. \end{aligned} \tag{2.11}$$

Then U^ε satisfies the following parabolic equation:

$$\begin{cases} \dfrac{\partial U^\varepsilon}{\partial s} = \Delta U^\varepsilon + F(U^\varepsilon) - V^\varepsilon & y \in \mathbf{R}^n,\ s > 0, \\ U^\varepsilon(y, 0) = \phi(x_0 + \varepsilon y) & y \in \mathbf{R}^n. \end{cases} \tag{2.12}$$

Recall that v^ε satisfies equation $(1.1\mathrm{b})^\varepsilon$ and the term $g(u^\varepsilon, v^\varepsilon)$ in $(1.1\mathrm{b})^\varepsilon$ is uniformly bounded (see Lemma 2.3). Using standard parabolic estimates, we see that for any $\delta > 0$ there exists $t_1 = t_1(\delta) > 0$ such that

$$|v^\varepsilon(x, t) - \psi(x)| \le \delta \tag{2.13}$$

for all $\varepsilon > 0$, $x \in \mathbf{R}^n$ and $0 \le t \le t_1$. It follows that

$$\begin{aligned} |V^\varepsilon(y, s) - \psi(x_0)| &\le |v^\varepsilon(x_0 + \varepsilon y, \varepsilon s) - \psi(x_0 + \varepsilon y)| + |\psi(x_0 + \varepsilon y) - \psi(x_0)| \\ &\le \delta + |\psi(x_0 + \varepsilon y) - \psi(x_0)| \end{aligned} \tag{2.14}$$

for all $\varepsilon > 0$, $y \in \mathbf{R}^n$ and $0 \le s \le t_1/\varepsilon$. Thus $V^\varepsilon(y, s)$ converges as $\varepsilon \downarrow 0$ to $\psi(x_0)$ uniformly on any compact subset of $\mathbf{R}^n \times [0, \infty)$. Since the solution to the parabolic equation (2.12) depends continuously on the inhomogeneous term and the initial data, such a convergence implies that $U^\varepsilon(y, s) \to U(y, s)$ as $\varepsilon \downarrow 0$, uniformly on any compact subset of $\mathbf{R}^n \times [0, \infty)$, where U is the solution of

$$\begin{cases} \dfrac{\partial U}{\partial s} = \Delta U + F(U) - \psi(x_0) & y \in \mathbf{R}^n,\ s > 0, \\ U(y, 0) = \phi(x_0) & y \in \mathbf{R}^n. \end{cases} \tag{2.15}$$

Since $U(y, 0)$ is independent of y, $U(y, s) \equiv U(s)$ satisfies an ordinary differential equation:

$$\begin{cases} U'(s) = F(U) - \psi(x_0) & s > 0, \\ U(0) = \phi(x_0). \end{cases} \tag{2.16}$$

Therefore,

(2.17) $\lim_{s\to\infty} \lim_{\varepsilon\downarrow 0} u^\varepsilon(x_0, \varepsilon s) = \lim_{s\to\infty} \lim_{\varepsilon\downarrow 0} U^\varepsilon(0, s) = \lim_{s\to\infty} U(s)$.

By looking at (2.16), one can easily find that

(2.18) $$\lim_{s\to\infty} U(s) = h_+(\psi(x_0)) \quad \text{or} \quad h_-(\psi(x_0)),$$

which depends on whether $x_0 \in \Omega_+$ or $x_0 \in \Omega_-$. □

LEMMA 2.4. *Assume that conditions of Theorem 2.2. Then*
(i) *there exists a constant $C_4 > 0$ such that*

(2.19) $$|v^\varepsilon(x, t) - \psi(x)| \le C_4 t$$

for all $\varepsilon > 0$, $x \in \mathbf{R}^n$ and $t \ge 0$;
(ii) *there exists $t_4 > 0$ such that*

(2.20) $$F(a_-) + \beta_4 \le v^\varepsilon(x, t) \le F(a_+) - \beta_4$$

for all $\varepsilon > 0$, $x \in \mathbf{R}^n$ and $0 \le t \le t_4$, where $\beta_4 = \beta_2/2$ with β_2 being as in (2.7a).

PROOF. (i) By Lemma 2.3, there exists a constant $K_4 > 0$ such that

(2.21) $$|g(u^\varepsilon(x, t), v^\varepsilon(x, t))| \le K_4$$

for any ε, x and t. If C_4 was chosen sufficiently large, then the function $\tilde{v}(x, t) = \psi(x) + C_4 t$ satisfies

(2.22) $$\frac{\partial \tilde{v}}{\partial t} - D\Delta\tilde{v} - g(u^\varepsilon, v^\varepsilon) \ge C_4 - D\Delta\psi - K_4 \ge 0$$

for any ε, x and t (recall that we assumed $\|\psi\|_{C^2(\mathbf{R}^n)} < \infty$). From the maximum principle it follows that $\tilde{v}(x, t) \ge v^\varepsilon(x, t)$ for any ε, x and t. The lower bound can be derived similarly.

Statement (ii) follows from (i). □

LEMMA 2.5. *Fix arbitrarily positive constants K_5, σ_5 and ρ_5. Suppose that two functions $J: \mathbf{R}^n \times [0, \infty) \to \mathbf{R}$ and $\Phi: \mathbf{R}^n \to \mathbf{R}$ and a subset G of $\mathbf{R}^n$ satisfy*

(2.23) $$\begin{cases} |J(x, t)| + |\Phi(x)| \le K_5 & x \in \mathbf{R}^n,\ t > 0, \\ \Phi(x) \ge \sigma_5 \min\{\rho_5, dist(x, G)\} & x \notin G. \end{cases}$$

Let $w^\varepsilon(x, t)$ be the solution to the Cauchy problem

(2.24) $$\begin{cases} \dfrac{\partial w^\varepsilon}{\partial t} = \varepsilon\Delta w^\varepsilon + \varepsilon^{-1} J(x, t) w^\varepsilon & x \in \mathbf{R}^n,\ t > 0, \\ w^\varepsilon(x, 0) = \Phi(x) & x \in \mathbf{R}^n. \end{cases}$$

Then there exist positive constants ε_5, t_5 and C_5, depending only on K_5, σ_5 and

ρ_5, such that

$$w^{\varepsilon}(x, t) > 0 \tag{2.25}$$

for any $0 < \varepsilon \leq \varepsilon_5$, $0 \leq t \leq t_5$ and $x \in \mathbf{R}^n$ with $dist(x, G) \geq C_5(t + \varepsilon|\log \varepsilon|)$.

Proof. We shall use a probabilistic argument similiar to that found in [15].

For a point $x \in \mathbf{R}^n$, denote by $\mathfrak{X}_x$ the space of all continuous curves $X: [0, \infty) \to \mathbf{R}^n$ with $X(0) = x$. For each $\delta > 0$, let $\mathfrak{P}_x^{\delta}$ be the probability measure on $\mathfrak{X}_x$ induced by the diffusion process corresponding to the equation $\partial p/\partial t = \delta \Delta p$.

By equation (2.24) and the Feynman-Kac formula, we obtain

$$w^{\varepsilon}(x, t) = \mathfrak{E}_x^{\varepsilon}\left[\Phi(X(t)) \cdot \exp\left\{\int_0^t \varepsilon^{-1} J(X(s), t - s)\, ds\right\}\right], \tag{2.26}$$

where $\mathfrak{E}_x^{\varepsilon}$ is the mathematical expectation with respect to $\mathfrak{P}_x^{\varepsilon}$.

For each (x, t), we decompose the space of sample paths $\mathfrak{X}_x$ into two parts $\mathfrak{X}'_x$ and $\mathfrak{X}''_x$ as follows:

$$\begin{aligned} \mathfrak{X}'_x &= \{X \in \mathfrak{X}_x;\ \max_{0 \leq s \leq t} |X(s) - x| \leq L(t + \varepsilon|\log \varepsilon|)\}, \\ \mathfrak{X}''_x &= \mathfrak{X}_x \backslash \mathfrak{X}'_x, \end{aligned} \tag{2.27}$$

where $L > 0$ is a large constant to be chosen later (see (2.34)). This induces a decomposition of the right hand side of equation (2.26):

$$w^{\varepsilon}(x, t) = \mathfrak{E}_x^{\varepsilon}[\,]|_{\mathfrak{X}'_x} + \mathfrak{E}_x^{\varepsilon}[\,]|_{\mathfrak{X}''_x}. \tag{2.28}$$

The first term is easily controlled. In fact, we have

$$\Phi(X(s)) \geq \sigma_5 \min\{\rho_5, dist(x, G) - Lt - L\varepsilon|\log \varepsilon|\} \tag{2.29}$$

for any $x \in \mathbf{R}^n \backslash G$, $0 \leq s \leq t$ and $X \in \mathfrak{X}'_x$. From this it follows that

$$\mathfrak{E}_x^{\varepsilon}[\,]|_{\mathfrak{X}'_x} \geq \exp\left\{-\frac{tK_5}{\varepsilon}\right\} \sigma_5 L(t + \varepsilon|\log \varepsilon|), \tag{2.30}$$

provided that $x \in \mathbf{R}^n$ and $t > 0$ satisfy $t + \varepsilon|\log \varepsilon| < \rho_5/L$ and $dist(x, G) \geq 2L(t + \varepsilon|\log \varepsilon|)$. To estimate the second term on the right hand side of (2.28), we recall the following result (see [39] or [38]):

For any $L > 0$, there exists a constant $\delta_5 = \delta_5(L) > 0$ such that

$$\mathfrak{P}_x^{\delta}[X \in \mathfrak{X}_x;\ \sup_{0 \leq s \leq 1} |X(s) - x| > L] \leq \exp\{-L^2/8\delta\} \tag{2.31}$$

for any $\delta \in (0, \delta_5]$ and $x \in \mathbf{R}^n$.

According to this and in view of

$$\varepsilon t(t + \varepsilon|\log\varepsilon|)^{-2} \le (4|\log\varepsilon|)^{-1} \qquad \text{for all } t > 0, \tag{2.32}$$

we find that

$$\begin{aligned} \mathfrak{P}_x^\varepsilon[\mathfrak{X}_x''] &= \mathfrak{P}_x^\varepsilon[X \in \mathfrak{X}_x;\ \sup\nolimits_{0\le s\le t}|X(s) - x| > L(t + \varepsilon|\log\varepsilon|)] \\ &= \mathfrak{P}_x^{\varepsilon t(t+\varepsilon|\log\varepsilon|)^{-2}}[X \in \mathfrak{X}_x;\ \sup\nolimits_{0\le s\le 1}|X(s) - x| > L] \\ &\qquad\qquad\qquad \text{(by a change of variables)} \\ &\le \exp\left\{-\frac{L^2(t + \varepsilon|\log\varepsilon|)^2}{8\varepsilon t}\right\}, \end{aligned} \tag{2.33}$$

for any x and t, provided $(4|\log\varepsilon)^{-1} \le \delta_5$. This implies that

$$\begin{aligned} \mathfrak{E}_x^\varepsilon[\,]|_{\mathfrak{X}_x''} &\ge -K_5\cdot\exp\left\{\frac{tK_5}{\varepsilon}\right\}\cdot\exp\left\{-\frac{L^2(t + \varepsilon|\log\varepsilon|)^2}{8\varepsilon t}\right\} \\ &\ge -K_5\varepsilon^4\exp\left\{-\frac{tK_5}{\varepsilon}\right\}, \end{aligned} \tag{2.34}$$

for any x and t, provided $L^2 = 16(K_5 + 1)$ and $0 < \varepsilon \le \exp(-\delta_5/4)$. Combining inequalities (2.30) and (2.34) completes the proof of Lemma 2.5. □

LEMMA 2.6. *Assume the conditions of Theorem 2.2. Then for any $M > 0$, there exist positive constants ε_6, t_6 and C_6 such that*

$$u^\varepsilon(x, t) \ge h_0(\psi(x)) + M(\tau + \varepsilon|\log\varepsilon|) \tag{2.35}$$

for any $0 < \varepsilon \le \varepsilon_6$, $0 \le t \le \tau \le t_6$ and $x \in \Omega_+$ with dist$(x, \Gamma) \ge C_6(\tau + \varepsilon|\log\varepsilon|)$.

PROOF. Take two small numbers $\beta_6 > 0$ and $\gamma_6 > 0$ such that

$$\begin{cases} F'(u) \ge \beta_6 & \text{for all } u \in [a_- + \gamma_6, a_+ - \gamma_6]; \\ a_- + 2\gamma_6 \le h_0(\psi(x)) \le a_+ - 2\gamma_6 & \text{for all } x \in \mathbf{R}^n. \end{cases} \tag{2.36}$$

Fix an arbitrarily large $M > 0$. Without loss of generality, we suppose that

$$\beta_6 M \ge C_4 + \|\Delta(h_0 \circ \psi)\|_{L^\infty(\mathbf{R}^n)}, \tag{2.37}$$

where C_4 is as in Lemma 2.4.

For each $\tau > 0$, consider a function

$$\tilde{w}(x, t) = \tilde{w}(x, t; \tau, \varepsilon) = u^\varepsilon(x, t) - h_0(\psi(x)) - M(\tau + \varepsilon|\log\varepsilon|). \tag{2.38}$$

It is easy to check that

$$\frac{\partial\tilde{w}}{\partial t} = \varepsilon\Delta\tilde{w} + \varepsilon^{-1}J(x, t)\tilde{w} + b(x, t), \tag{2.39}$$

where J and $b\colon \mathbf{R}^n \times [0, \infty) \to \mathbf{R}$ are defined, respectively, as follows:

$$\begin{aligned} J(x, t) &= J(x, t; \tau, \varepsilon) \\ &= \tilde{w}^{-1}(x, t)\{F(u^\varepsilon(x, t)) - F(u^\varepsilon(x, t) - \tilde{w}(x, t))\}, \end{aligned} \tag{2.40}$$

$$\begin{aligned} b(x, t) &= b(x, t; \tau, \varepsilon) \\ &= \varepsilon^{-1}\{F(h_0 \circ \psi(x) + M\tau + M\varepsilon|\log\varepsilon|) - v^\varepsilon(x, t)\} \\ &\quad + \varepsilon\Delta(h_0 \circ \psi)(x). \end{aligned} \tag{2.41}$$

Let $w(x, t; \tau, \varepsilon)$ be the solution to the problem

$$\begin{cases} \dfrac{\partial w}{\partial t} = \varepsilon\Delta w + \varepsilon^{-1}J(x, t)w & x \in \mathbf{R}^n,\ t > 0, \\ w(x, 0; \tau, \varepsilon) = \tilde{w}(x, 0; \tau, \varepsilon) & x \in \mathbf{R}^n. \end{cases} \tag{2.42}$$

First we show that if ε_6 and t_6 are sufficiently small constants, then

$$\tilde{w}(x, t; \tau, \varepsilon) \geq w(x, t; \tau, \varepsilon) \tag{2.43}$$

for any $0 < \varepsilon \leq \varepsilon_6$, $0 \leq t \leq \tau \leq t_6$ and $x \in \mathbf{R}^n$. To this end, let $\varepsilon_6 \in (0, \frac{1}{2})$ and $t_6 > 0$ be so small that

$$M(t_6 + \varepsilon_6|\log\varepsilon_6|) \leq \gamma_6 \tag{2.44}$$

where γ_6 is as in (2.36). Then for any $0 < \varepsilon \leq \varepsilon_6$, $0 \leq t \leq \tau \leq t_6$ and $x \in \mathbf{R}^n$, we have

$$a_- + \gamma_6 \leq h_0(\psi(x)) + M\tau + M\varepsilon|\log\varepsilon| \leq a_+ - \gamma_6, \tag{2.45}$$

and hence.

$$\begin{aligned} b(x, t; \tau, \varepsilon) &\geq \varepsilon\Delta(h_0 \circ \psi)(x) + \varepsilon^{-1}\{F(h_0 \circ \psi(x)) \\ &\qquad \beta_6 M(\tau + \varepsilon|\log\varepsilon|) - \psi(x) - C_4 t\} \qquad \text{(by (2.36))} \\ &\geq \{\beta_6 M|\log\varepsilon| - \varepsilon\|\Delta(h_0 \circ \psi)\|_{L^\infty(\mathbf{R}^n)}\} \\ &\quad + \varepsilon^{-1}\{\beta_6 M\tau - C_4 t\} \\ &\geq 0. \qquad \text{(by (2.37))} \end{aligned} \tag{2.46}$$

Combining this with (2.39) and (2.42) and using the comparison theorem, we obtain (2.43).

Next we claim that there exist positive constants $\tilde{\varepsilon}_6$, $\tilde{t}_6$ and $\tilde{C}_6$ such that

$$w(x, t; \tau, \varepsilon) \geq 0 \tag{2.47}$$

for any $0 < \varepsilon \leq \tilde{\varepsilon}_6$, $0 \leq t \leq \tau \leq \tilde{t}_6$ and $x \in \Omega_+$ with $dist(x, \Gamma) \geq \tilde{C}_6(\tau + \varepsilon|\log \varepsilon|)$. For this, we put

$$\Phi(x) = \tilde{w}(x, 0, \tau, \varepsilon) = \phi(x) - h_0(\psi(x)) - M(\tau + \varepsilon|\log \varepsilon|),$$
$$G = \Omega_- \cup \{x \in \mathbf{R}^n;\ dist(x, \Gamma) \leq 2\sigma_2^{-1} M(\tau + \varepsilon|\log \varepsilon|)\}, \tag{2.48}$$

where σ_2 is an in (2.7b). If we can choose positive constants K_5, ρ_5 and σ_5 such that J, Φ and G satisfy the conditions of Lemma 2.5, then (2.47) follows immediately from Lemma 2.5 (note that in Lemma 2.5 the constants ε_5, t_5 and C_5 depend on J, G and Φ only through the constants K_5, ρ_5 and σ_5). Since J and Φ here are uniformly bounded, K_5 is easily found. Now we seek ρ_5 and σ_5. Observe that any $x \notin G$ is contained in Ω_+ and satisfies that

$$dist(x, \Gamma) > 2\sigma_2^{-1} M(\tau + \varepsilon|\log \varepsilon|), \tag{2.49}$$

which along with the assumption (2.7b) implies that

$$\begin{aligned}\Phi(x) &\geq \sigma_2 \min\{\rho_2,\ dist(x, \Gamma)\} - M(\tau + \varepsilon|\log \varepsilon|)\\ &\geq \sigma_2 \min\{\rho_2,\ \tfrac{1}{2} dist(x, G) + \sigma_2^{-1} M(\tau + \varepsilon|\log \varepsilon|)\}\\ &\quad - M(\tau + \varepsilon|\log \varepsilon|)\\ &\geq \sigma_2 \min\{\rho_2 - \sigma_2^{-1} M(\tau + \varepsilon|\log \varepsilon|),\ \tfrac{1}{2} dist(x, G)\}\\ &\geq \tfrac{1}{2}\sigma^2 \min\{\rho_2,\ dist(x, G)\},\end{aligned} \tag{2.50}$$

for any $\tau + \varepsilon|\log \varepsilon| \leq (2M)^{-1}\rho_2\sigma_2$ and $x \notin G$. Therefore it suffices to take $\sigma_5 = \sigma_2/2$ and $\rho_5 = \rho_2$ for applying Lemma 2.5. The inequality (2.47) is proved.

Lemma 2.6 follows from (2.43) and (2.47). □

LEMMA 2.7. *Let A and B be positive numbers. Consider an ordinary differential equation for $R(t)$*

$$\begin{cases} R'(t) = A[B - R(t)]R(t) & t > 0,\\ R(0) = R_0 \in (0, B). \end{cases} \tag{2.51}$$

Then,

$$B > R(t) > B - R_0^{-1} B^2 \exp(-ABt) \qquad t \geq 0. \tag{2.52}$$

PROOF. A simple quadrature gives

$$B - R(t) = \frac{B(B - R_0)}{R_0 \exp(ABt) + (B - R_0)}. \qquad \square \tag{2.53}$$

LEMMA 2.8. *For constants $\varepsilon > 0$, $\rho > 0$ and $\eta \in (F(a_-), F(a_+))$, consider an initial-boundary value problem of parabolic equation:*

$$\text{(2.54)}\qquad \begin{cases} \dfrac{\partial \tilde{u}}{\partial t} = \varepsilon \Delta \tilde{u} + \varepsilon^{-1}[F(\tilde{u}) - \eta] & |x| < \rho,\ t > 0, \\ \tilde{u} = h_0(\eta) & |x| = \rho,\ t > 0, \\ \tilde{u} = h_0(\eta) + \varepsilon|\log \varepsilon| & |x| \le \rho,\ t = 0. \end{cases}$$

Denote the solution by $\tilde{u}(x, t; \varepsilon, \eta, \rho)$. Then for any $\delta > 0$ there exist positive ε_8, σ_8, K_8, L_8 and M_8 such that

$$\text{(2.55)}\qquad \begin{aligned} &\tilde{u}(0, t; \varepsilon, \eta, \rho) \ge h_+(\eta) - \delta - \frac{M_8}{\varepsilon|\log \varepsilon|}\exp(-\sigma_8 t/\varepsilon) && t > 0, \\ &\tilde{u}(0, t; \varepsilon, \eta, \rho) \ge h_+(\eta) - 2\delta && t \ge K_8 \varepsilon |\log \varepsilon|, \end{aligned}$$

for all $0 < \varepsilon \le \varepsilon_8$, $\rho \ge L_8 \varepsilon$ and $F(a_-) + \beta_4 \le \eta \le F(a_+) - \beta_4$.

PROOF. For any $\delta > 0$ there are $\sigma_8 > 0$ and $M_8 > 0$ such that

$$\text{(2.56)}\qquad \begin{gathered} F(\xi) - F(h_0(\eta)) \ge 2\sigma_8[\xi - h_0(\eta)] \\ h_+(\eta) - h_0(\eta) \le \sqrt{M_8} \end{gathered}$$

for all ξ and η with $h_0(\eta) \le \xi \le h_+(\eta) - \delta$ and $F(a_-) + \beta_4 \le \eta \le F(a_+) - \beta_4$, where β_4 is as in Lemma 2.4.

Let $R(t)$ be the solution to problem (2.51) with $A = \varepsilon^{-1} M_8^{-1/2} \sigma_8$, $B = h_+(\eta) - h_0(\eta) - \delta$ and $R_0 = \varepsilon|\log \varepsilon|$. Let $\varphi_1(x)$ be the positive eigenfunction of the principal eigenvalue λ_1 of the Laplacian in the unit disk, that is,

$$\text{(2.57)}\qquad \begin{cases} \Delta \varphi_1 + \lambda_1 \varphi_1 = 0 & |x| < 1, \\ \varphi_1(x) = 0 & |x| = 1. \end{cases}$$

We further assume that

$$\text{(2.58)}\qquad \varphi_1(0) = 1.$$

Note that $0 \le \varphi_1(x) \le 1$ for all $|x| \le 1$.

Define

$$\text{(2.59)}\qquad u_8(x, t) = h_0(\eta) + R(t)\varphi_1\left(\frac{x}{\rho}\right) \qquad |x| \le 1,\ t \ge 0,$$

and denote $\tilde{u}(x, t) = \tilde{u}(x, t; \varepsilon, \eta, \rho)$ for brevity. Then it is easy to see

$$\text{(2.60)}\qquad \begin{aligned} &u_8(x, 0) \le \tilde{u}(x, 0) && |x| \le 1, \\ &u_8(x, t) = \tilde{u}(x, t) = h_0(\eta) && |x| = 1. \end{aligned}$$

Moreover,

$$
\begin{aligned}
&\varepsilon\frac{\partial u_8}{\partial t} - \varepsilon^2 \Delta u_8 - [F(u_8) - \eta] \\
&\qquad = \varepsilon R'\varphi_1 - \varepsilon^2\rho^2 R\Delta\varphi_1 - F(h_0(\eta) + R\varphi_1) + F(h_0(\eta)) \\
&\qquad \le R\varphi_1\left[\frac{\sigma_8}{\sqrt{M_8}}\{h_+(\eta) - h_0(\eta) - \delta - R\} + \varepsilon^2\rho^{-2}\lambda_1 - 2\sigma_8\right] \qquad (2.61)\\
&\qquad \le R\varphi_1(\varepsilon^2\rho^{-2}\lambda_1 - \sigma_8) \\
&\qquad \le 0,
\end{aligned}
$$

provided $\rho \ge \varepsilon\sqrt{\lambda_1/\sigma_8}$. Thus u_8 is a subsolution to problem (2.54) if ρ is as above, and therefore, by the comparison theorem and Lemma 2.8, we obtain that

$$
\begin{aligned}
\tilde{u}(0, t) &\ge u_8(0, t) \\
&= h_0(\eta) + R(t) \qquad (2.62)\\
&\ge h_+(\eta) - \delta - M_8\{\varepsilon|\log\varepsilon|\}^{-1}\exp(-\sigma_8 t/\varepsilon)
\end{aligned}
$$

for any $t \ge 0$. □

LEMMA 2.8′. *For any $F(a_-) < \eta < F(a_+)$, $\Lambda \in \mathbf{R}^n$ and $\rho > 0$, consider an initial-boundary value problem*

$$
(2.63)\qquad \begin{cases}
\dfrac{\partial \bar{u}}{\partial t} = \varepsilon\Delta\bar{u} + \varepsilon^{-1}[F(\bar{u}) - \eta] & |x| < \rho,\ t > 0, \\
\bar{u} = \Lambda & |x| = \rho,\ t > 0, \\
\bar{u} = \Lambda & |x| \le \rho,\ t = 0.
\end{cases}
$$

Denote the solution by $\bar{u}(x, t; \varepsilon, \eta, \rho, \Lambda)$. Then for any $\delta > 0$ and $\Lambda \in \mathbf{R}$ there exist positive constants ε_8', K_8' such that

$$
(2.64)\qquad \bar{u}(0, t; \varepsilon, \eta, \rho, \Lambda) \le h_+(\eta) + \delta,
$$

for all $0 < \varepsilon \le \varepsilon_8'$, $t \ge K_8'\varepsilon|\log\varepsilon|$, $\rho \ge L_8'\varepsilon$ and $F(a_-) + \beta_4 \le \eta \le F(a_+) - \beta_4$.

PROOF. Take constants $\Lambda_8' \in \mathbf{R}$ and $\sigma_8' > 0$ such that

$$
(2.65)\qquad \begin{aligned}
&\Lambda_8' \ge \max\{\Lambda, h_+(\eta)\}, \\
&F(\xi) - F(h_+(\eta)) \le -\sigma_8'[\xi - h_+(\eta)],
\end{aligned}
$$

for all ξ and η with $h_+(\eta) \le \xi \le \Lambda_8'$ and $F(a_-) + \beta_4 \le \eta \le F(a_+) - \beta_4$.

Let $\bar{R}(t)$ be the solution to problem (2.51) with $R_0 = 1$, A and B defined by

$$(2.66) \qquad \begin{aligned} A &= \varepsilon^{-1}(\Lambda'_8 + 1)^{-1}\alpha'_8, \\ B &= \Lambda'_8 + 1 - h_+(\eta) - \delta, \end{aligned}$$

and let φ_1 satisfy (2.57)–(2.58). Define

$$(2.67) \qquad \bar{u}_8(x, t) = \Lambda'_8 + 1 - \bar{R}(t)\varphi_1(x/\rho) \qquad \mathrm{D}x| \le \rho,\ t \ge 0.$$

The rest of the proof goes quite analogously to that of Lemma 2.8. In fact, if α'_8 is sufficiently small and L'_8 is sufficiently large, then $\bar{u}_8(x, t)$ defined as above is a supersolution to problem (2.63) for $\rho \ge \varepsilon L'_8$. Hence, $\bar{u} \le \bar{u}_8$. However, by Lemma 2.7,

$$(2.68) \qquad \begin{aligned} \bar{u}_8(0, t) &= \Lambda'_8 + 1 - \bar{R}(t) \\ &\le h_+(\eta) + \delta + (\Lambda'_8 + 1)^2 \exp(-\alpha'_8 t/\varepsilon), \end{aligned}$$

which is smaller than $h_+(\eta) + 2\delta$, if $t \ge \varepsilon K'_8$ with K'_8 being sufficiently large. $\square$

LEMMA 2.9. *Assume the condiions of Theorem 2.2. Then for any $\delta > 0$ there exist positive constants ε_9, t_9, C_9 and K_9 such that*

$$(2.69) \qquad |u^\varepsilon(x, t) - h_+(\psi(x))| \le \delta$$

for $0 < \varepsilon \le \varepsilon_9$, $K_9\varepsilon|\log\varepsilon| \le t \le t_9$ and $x \in \Omega_+$ with $dist(x, \Gamma) \ge C_9(t + \varepsilon|\log\varepsilon|)$.

PROOF. For any $\varepsilon > 0$, $\rho > 0$, $y \in \mathbf{R}^n$ and $\tau > 0$, define

$$(2.70) \qquad u_9(x, t) = \tilde{u}(x - y, t; \varepsilon, \eta, \rho) \qquad |x - y| \le \rho,\ t \ge 0,$$

where $\tilde{u}$ is as in Lemma 2.8 and η is as follows:

$$(2.71) \qquad \eta = \eta(y, \rho, \tau) = \psi(y) + \tau C_4 + \rho\|\nabla\psi\|_{L^\infty(\mathbf{R}^n)}.$$

By Lemma 2.4, we find that

$$(2.72) \qquad \begin{aligned} \varepsilon\frac{\partial u^\varepsilon}{\partial t}(x, t) - \varepsilon^2\Delta u^\varepsilon(x, t) - F(u^\varepsilon(x, t)) &= -v^\varepsilon(x, t) \\ \ge -\eta &= \frac{\partial u_9}{\partial t} - \varepsilon^2\Delta u_9(x, t) - F(u_9(x, t)), \end{aligned}$$

for all $x \in \mathbf{R}^n$ with $|x - y| < \rho$ and $t \in (0, \tau]$. By the assumption (2.7b), Lemma 2.6 and the Lipschitz continuity of the function $h_0(\eta)$ on the interval $\eta \in [F(a_-) + \beta_4, F(a_+) - \beta_4]$, we find that if ε_9, t_9 and C_9^{-1} are sufficiently small positive numbers, then for $0 < \varepsilon \le \varepsilon_9$, $0 < \rho \le \varepsilon|\log\varepsilon|$, $0 < \tau \le t_9$ and $y \in \Omega_+$ with $dist(y, \Gamma) \ge C_9(\tau + \varepsilon|\log\varepsilon|)$, it holds that

$$\begin{aligned} u^\varepsilon(x, t) &\geq u_9(x, t) \qquad |x - y| = \rho,\ 0 \leq t \leq \tau, \\ \phi(x) &\geq u_9(x, 0) \qquad |x - y| \leq \rho. \end{aligned} \tag{2.73}$$

From this, (2.72) and the comparison theorem, it follows that

$$u^\varepsilon(x, t) \geq u_9(x, t) \qquad |x - y| \leq \rho,\ 0 \leq t \leq \tau, \tag{2.74}$$

for all ε, ρ, τ and y as above; in particular, for $\tau \in [K_9\varepsilon|\log\varepsilon|, t_9]$ we have

$$\begin{aligned} u^\varepsilon(y, \tau) &\geq u_9(y, \tau) \\ &= \tilde{u}(0, \tau; \varepsilon, \eta, \rho) \\ &\geq h_+(\eta) - 2\delta \\ &\geq h_+(\psi(y)) - 2\delta - M_9(\tau C_4 + \rho\|\nabla\psi\|_{L^\infty(\mathbf{R}^n)}), \end{aligned} \tag{2.75}$$

where M_9 stands for

$$M_9 = \max\{|h'_+(\eta)|;\ F(a_-) + \beta_4 \leq \eta \leq F(a_+) - \beta_4\}. \tag{2.76}$$

Therefore

$$u^\varepsilon(y, \tau) \geq h_+(\psi(y)) - 3\delta, \tag{2.77}$$

provided ε_9 and t_9 were chosen sufficiently small. Similarly one can prove

$$u^\varepsilon(y, \tau) \leq h_+(\psi(y)) + 3\delta, \tag{2.78}$$

by using Lemma 2.8′. Since $\delta > 0$ is arbitrary, the lemma is established. □

Proof of Theorem 2.2. By Lemma 2.9, when $t \in [K_9\varepsilon|\log\varepsilon|, 3K_9\varepsilon|\log\varepsilon|]$, inequality (2.8) holds for $0 < \varepsilon \leq \varepsilon_9$ and $x \in \Omega_+$ with

$$dist(x, \Gamma) \geq C_9(3K_9 + 1)\varepsilon|\log\varepsilon|. \tag{2.79}$$

The part of (2.8) involving $x \in \Omega_-$ follows from an analogue of Lemma 2.9. □

Remark 2.10. (i) Assuming the conditions of Theorem 2.2, one can give estimates of other types. The following is an example:

for any $\delta > 0$ *and* $\alpha \in (0, 1)$, *there exist positive constants* ε_{10} *and* C_{10} *such that*

$$|u^\varepsilon(x, t) - U^*(x)| < \delta$$

for $0 < \varepsilon < \varepsilon_{10}$, $\varepsilon^\alpha \leq t \leq 3\varepsilon^\alpha$, *and* $x \in \mathbf{R}^n$ *with* $dist(x, \Gamma) \geq C_{10}\varepsilon^\alpha$.

Note also that the number 3 is unimportant here and in Theorem 2.2; in fact it can be replaced by an arbitrary number.

(ii) The arguments in this section are applicable to the Cauchy problem for scalar equations and to the Neumann boundary value problems for scalar equations and systems in smooth bounded domains of $\mathbf{R}^n$. In the Dirichlet boundary value problems our method can still be directly employed in the analysis of internal layers and the treatment of boundary layers needs little modifications. One can also consider nonlinearities more general than those given by (1.3). We shall not get involved further in these details here.

3. The second stage: the interfacial motion

In this section we are concerned with a free-boundary problem which describes the interfacial dynamics for the reaction diffusion system $(1.1)^\varepsilon$. As discussed intuitively in section 1, the singular limiting problem of $(1.1)^\varepsilon$ reduces to equation (1.15). This equation involves the component v and the interface Γ which interact each other in a highly nonlinear way. Here the interface, or the free-boundary, is an imbedded hypersurface $\Gamma(t)$ separating the whole space $\mathbf{R}^n$ into two regions $\Omega_+(t)$ and $\Omega_-(t)$ and propagating as time evolves. We shall show in this section that the initial-value problem for (1.15)–(1.16) is solvable at least in a short interval of time.

To be more precise, fix a smooth compact hypersurface M of $\mathbf{R}^n$, which is the boundary of a bounded domain $\Omega_+(0)$. Let $S: M \to \mathbf{R}^n$ denote the inclusion map and let $\Omega_-(0) = \mathbf{R}^n \backslash (\Omega_+(0) \cup M)$. [1] Given a C^1 function $\psi: \mathbf{R}^n \to \mathbf{R}$ and the hypersurface M, our problem is to find a function $v: \mathbf{R}^n \times [0, T] \to \mathbf{R}$ and a mapping $\Gamma: M \times [0, T] \to \mathbf{R}^n$ satisfying the following equations:

(3.1a) $$\frac{\partial v}{\partial t} = D\Delta v + g_+(v) \qquad x \in \Omega_+(t),\ 0 < t \le T,$$

(3.1b) $$\frac{\partial v}{\partial t} = D\Delta v + g_-(v) \qquad x \in \Omega_-(t),\ 0 < t \le T,$$

(3.2) $$\left(\frac{\partial \Gamma}{\partial t}, N\right) = W(v) - \varepsilon(n-1)\kappa \qquad \eta \in M,\ 0 < t \le \mathrm{T},$$

(3.3) $$v(\cdot, t) \in C^1(\mathbf{R}^n) \qquad 0 < t \le T,$$

(3.4) $$v(x, 0) = \psi(x) \qquad x \in \mathbf{R}^n,$$

(3.5) $$\Gamma(\eta, 0) = S(\eta) \qquad \eta \in M,$$

where g_+ and $g_-: \mathbf{R} \to \mathbf{R}$ are smooth functions, The equation of motion for

[1] A topological remark: any connected compact hypersurface separates $\mathbf{R}^n$ into two connected components.

the interface is (3.2), in which $W: \mathbf{R} \to \mathbf{R}$ is a smooth function, $\kappa = \kappa(\eta, t)$ is the mean curvature of the hypersurface $\Gamma(t) = \{\Gamma(\eta, t); \eta \in M\}$ at the point $x = \Gamma(\eta, t)$, $N = N(\eta, t)$ is the unit normal vector field of $\Gamma(t)$ pointing from $\Omega_+(t)$ to $\Omega_-(t)$, and $(\cdot, \cdot)$ is the Euclidean inner product of $\mathbf{R}^n$. The free-boundary condition (3.3) requires that $v(\cdot, t)$ and its derivatives $\partial u/\partial x_i (1 \le i \le n)$ are continuous across the interface $\Gamma(t)$, for each $t > 0$.

Definition 3.1. We call a pair (v, Γ) a classical solution to problem (3.1)–(3.5) in the time interval $[0, T]$ if

(i) $\Gamma: M \times [0, T] \to \mathbf{R}^n$ is of class C^2 in $\eta \in M$ and of class C^1 in $t \in [0, T]$;

(ii) there exists a C^1 function $\Xi: \mathbf{R}^n \times [0, T] \to \mathbf{R}$ satifying

$$\begin{aligned}\Gamma(t) &:= \{(\Gamma(\eta, t), t); \eta \in M, t \in [0, T]\} \\ &= \{(x, t) \in \mathbf{R}^n \times [0, T]; \Xi(x, t) = 0\},\end{aligned} \tag{3.6}$$

$$\nabla_x \Xi(x, t) \neq 0 \text{ for any } (x, t) \in \Gamma(t), \tag{3.7}$$

$$\Omega_+(t) = \{x \in \mathbf{R}^n; \Xi(x, t) > 0\} \qquad t \in [0, T], \tag{3.8}$$

$$\Omega_-(t) = \{x \in \mathbf{R}^n; \Xi(x, t) < 0\} \qquad t \in [0, T], \tag{3.9}$$

(iii) $v: \mathbf{R}^n \times [0, T] \to \mathbf{R}$ is a bounded continuous function; moreover, v is of class C^1 in t and C^2 in x in the set

$$\Omega := \bigcup_{0 \le t \le T} [(\Omega_+(t) \cup \Omega_-(t)) \times \{t\}]; \tag{3.10}$$

(iv) each of equations (3.1)–(3.5) are satisfied in the usual sense.

The main result in this section is the local (in time) solvability of problem (3.1)–(3.5):

Theorem 3.2. *Assume that g_+, g_- and $W: \mathbf{R} \to \mathbf{R}$ are of class C^1. Let M be a compact hypersurface in $\mathbf{R}^n$ of class $C^{2+\alpha}$ with $\alpha \in (0, 1)$ and let $S: M \to \mathbf{R}^n$ be the inclusion map. Suppose that $\psi: \mathbf{R}^n \to \mathbf{R}$ is of class $C^{1+\alpha}$ and that $\|\psi\|_{C^{1+\alpha}(\mathbf{R}^n)} < \infty$. Then there exists a classical solution (v, Γ) to problem* (3.1)–(3.5) *in a time interval* $[0, T]$ *with* $T = T(\psi, M) > 0$.

Now we describe the outline of the proof of the above theorem.

Firstly it is appropriate to translate equation (3.2) into an evolution equation for scalar functions on M. To this end we take a C^∞ imbedding $\iota_0: M \to \mathbf{R}^n$ and denote by $N_0(\eta)$ the unit outward normal vector of the hypersurface $\iota_0(M)$ at the point $\iota_0(\eta)$. If $L > 0$ is small, then the geodesic map

$$\iota: M \times (-L, L) \longrightarrow \mathbf{R}^n \tag{3.11}$$

which is defined by

(3.12) $$\imath(\eta, \zeta) = \imath_0(\eta) + \zeta N_0(\eta) \qquad \eta \in M,$$

maps $M \times (-L, L)$ diffeomorphically onto an open tubular neighborhood $\imath(M \times (-L, L))$ of $\imath_0(M)$. For $\imath_0$ sufficiently close to the inclusion map S in C^2 topology, S can be represented by a function $z_0: M \to (-L/4, L/4)$ with

(3.13) $$\imath(\eta, z_0(\eta)) = S(\eta) \qquad \eta \in M.$$

Since S is of class $C^{2+\alpha}$, so is z_0. Moreover, if $\Gamma(\cdot, t): M \to \mathbf{R}^n$ is a $C^{k+\beta}(k \geq 1, \beta \in [0, 1))$ imbedding sufficiently close to S in C^1 topology, then it also corresponds to a $C^{k+\beta}$ function $z(\cdot, t): M \to (-L/2, L/2)$ such that

(3.14) $$\imath(\eta, z(\eta, t)) = \Gamma(\eta, t) \qquad \eta \in M.$$

Thus, for finding the (local in time) solution $\Gamma(\cdot, t)$ of equation (3.2) starting from S, it suffices to find a family of functions $z(\cdot, t): M \to \mathbf{R}$ which satisfies a certain evolution equation.

For $\zeta \in (-L, L)$, let g_ζ be the Riemannian metric on M induced by the imbedding

(3.15) $$M \longrightarrow \mathbf{R}^n, \qquad \eta \longrightarrow \imath(\eta, \zeta).$$

Then the outward unit normal vector field is given by

(3.16) $$N(\eta, t) = \frac{1}{\sqrt{1 + \|dz\|_{g_z}^2}} \left(\frac{\partial}{\partial \zeta} - \mathrm{grad}_z z \right),$$

where grad_ζ is the gradient operator with respect to the metric g_ζ. The normal velocity of $\Gamma(t)$ is

(3.17) $$\left(\frac{\partial \Gamma}{\partial t}, N \right) = \frac{\partial z / \partial t}{\sqrt{1 + \|dz\|_{g_z}^2}},$$

and the mean curature κ can be written as follows:

(3.18) $$-(n-1)\kappa(\eta, t) = \mathrm{div}_z \left\{ \frac{\mathrm{grad}_z z}{\sqrt{1 + \|dz\|_{g_z}^2}} \right\} - \frac{tr_z J_z}{2\sqrt{1 + \|dz\|_{g_z}^2}},$$

where div_z and tr_z are, respectively, the divergence operator and the trace operator with respect to the metric g_z, and J_z is a symmetric bilinear form on the tangent boundle TM defined by:

(3.19) $$J_z(X, Y) = \left. \frac{\partial}{\partial \zeta} \right|_{\zeta = z} (X, Y)_{g_\zeta} \qquad X, Y \in T_\eta M,$$

with $(\cdot, \cdot)_{g_\zeta}$ being the inner product associated with the metric g_ζ. Therefore, at least for a short period of time, in terms of z, equation (3.2) is:

$$\frac{\partial z}{\partial t} = \sqrt{1 + \|dz\|_{g_z}^2}\,\mathrm{div}_z \left\{ \frac{\mathrm{grad}_z z}{\sqrt{1 + \|dz\|_{g_z}^2}} \right\} - \tfrac{1}{2}\mathrm{tr}_z J_z - W(v(\imath(\eta, z), t))\sqrt{1 + \|dz\|_{g_z}^2}, \tag{3.20}$$

or, equivalently,

$$\frac{\partial z}{\partial t} = \Delta_z z - \frac{1}{1 + \|dz\|_{g_z}^2}((\nabla_z)_{\mathrm{grad}_z z}\mathrm{grad}_z z, \mathrm{grad}_z z)_{g_z} - \tfrac{1}{2}\mathrm{tr}_z J_z - W(v(\tau(\eta, z), t))\sqrt{1 + \|dz\|_{g_z}^2}. \tag{3.21}$$

It is more convenient to consider a C^∞ Riemannian metric $\tilde{g}$ on $M \times \mathbf{R}$ and a C^∞ imbedding $\tilde{\imath}$: $M \times \mathbf{R} \to \mathbf{R}^n$ such that

$$\tilde{g} = g_{\xi(\zeta)} + d\zeta^2 \tag{3.22}$$

$$\tilde{\imath}(\eta, \zeta) = \imath(\eta, \xi(\zeta)) \qquad \eta \in M,\ \zeta \in \mathbf{R}, \tag{3.23}$$

where ξ: $\mathbf{R} \to \mathbf{R}$ is a C^∞ increasing function satisfying

$$\begin{aligned} \xi(\zeta) &= \zeta & |\xi| &\le L/2, \\ \xi(\zeta) &= 3L/4 & \xi &\ge 4, \\ \xi(\zeta) &= -3L/4 & \xi &\le -3L/4. \end{aligned} \tag{3.24}$$

Assume that

$$\{(z_0(\eta), dz_0(\eta));\ \eta \in M\} \subset \{(\zeta, \omega) \in \mathbf{R} \times T^*M;\ |\zeta| + \|\omega\|_{g_0} \le C_0\}, \tag{3.25}$$

where g_0 is $g_\zeta|_{\zeta=0}$ and $C_0 > 0$ is a constant. Let J: $M \times \mathbf{R} \to \mathbf{R}$ be a C^∞ function such that

$$\begin{aligned} J(\eta, z) &= \mathrm{tr}_z J_z(\eta) & \eta &\in M,\ |z| \le L/2, \\ J(\eta, z) &= 0 & \eta &\in M,\ |z| \ge 3L/4. \end{aligned} \tag{3.26}$$

Consider further a function Ξ: $\mathbf{R} \times T^*M \to \mathbf{R}$ such that

$$\begin{aligned} 1 \ge \Xi(\zeta, \omega) &\ge 0 && \text{for all } (\zeta, \omega) \in \mathbf{R} \times T^*M; \\ \Xi(\zeta, \omega) &\equiv 1 && \text{for } (\zeta, \omega) \in \mathbf{R} \times T^*M \text{ with } |\zeta| + \|\omega\|_{g_0} \le 2C_1; \\ \Xi(\zeta, \omega) &\equiv 0 && \text{for } (\zeta, \omega) \in \mathbf{R} \times T^*M \text{ with } |\zeta| + \|\omega\|_{g_0} \ge 4C_1; \end{aligned} \tag{3.27}$$

and define A: $\mathbf{R} \times T^*M \to \mathbf{R}$ and B: $\mathbf{R} \times T^*M \to TM$ by

$$A(\zeta, \omega) = \Xi\left(\frac{\zeta}{2}, \frac{\omega}{2}\right)\sqrt{1 + \|\omega\|_{\tilde{g}_\zeta}} \tag{3.28a}$$

$$(B(\zeta, \omega), X)_{\tilde{g}_\zeta} = \Xi(\zeta, \omega)\,\omega(X) \tag{3.28b}$$

for all $\zeta \in \mathbf{R}$, $\omega \in T^*_\eta M$ and $X \in T_\eta M$ with $\eta \in M$. Given a function $v(x, t)$, $\zeta \in \mathbf{R}$, $\eta \in M$ and $\omega \in T^*_\eta M$, we define a differential operator $\mathscr{A}$ by

$$\begin{aligned}\mathscr{A}(\eta, v, \zeta, \omega, t)[z] = \tilde{\Delta}_\zeta z &- \frac{1}{A(\zeta, \omega)^2}((\tilde{\nabla}_\zeta)_{B(\zeta,\omega)}\, \widetilde{\operatorname{grad}}_\zeta z,\ B(\zeta, \omega))_{\tilde{g}_\zeta} \\ &- J(\eta, \zeta) - W(v(\tilde{\imath}(\eta, z), t))\,A(\zeta, \omega).\end{aligned} \tag{3.29}$$

Since we are concerned only with the local solution, equation (3.17) can be replaced by the following equation:

$$\frac{\partial z}{\partial t} = \mathscr{A}(\eta, v, z, dz, t)[z]. \tag{3.30}$$

In fact, because of the construction, equation (3.30) coincides with (3.20) if $|z| \le L/2$ and $|z| + \|dz\|_{g_0} \le 2C_1$, which is satisfied by any solution to (3.30) in a sufficiently small period of time.

In terms of z, we should understand that the domains $\Omega_+(t)$ and $\Omega_-(t)$ in equation (3.1) are determined by

$$\begin{aligned}\Omega_+(t) &= \Omega^z_+(t) \\ &= [\Omega_+(0)\backslash\tilde{\imath}(M \times \mathbf{R})] \cup \{\tilde{\imath}(\eta, \zeta);\ \zeta < z(\eta, t)\},\end{aligned} \tag{3.31a}$$

$$\begin{aligned}\Omega_-(t) &= \Omega^z_-(t) \\ &= [\Omega_-(0)\backslash\tilde{\imath}(M \times \mathbf{R})] \cup \{\tilde{\imath}(\eta, \zeta);\ \zeta < z(\eta, t)\},\end{aligned} \tag{3.31b}$$

and that the interface $\Gamma(t)$ is given by

$$\Gamma(t) = \Gamma^z(t) = \{\tilde{\imath}(\eta, z(\eta, t));\ \eta \in M\}. \tag{3.31c}$$

In conclusion, problem (3.1)–(3.5) is reduced to the following:

$$\frac{\partial v}{\partial t} = D\Delta v + g_+(v) \qquad x \in \Omega_+(t),\ t > 0, \tag{3.32a}$$

$$\frac{\partial v}{\partial t} = D\Delta v + g_-(v) \qquad x \in \Omega_-(t),\ t > 0, \tag{3.32b}$$

$$\frac{\partial z}{\partial t} = \mathscr{A}(\eta, v, z, dz, t)[z] \qquad \eta \in M,\ t > 0, \tag{3.33}$$

$$(3.34) \qquad v(\cdot, t) \in C^1(\mathbf{R}^n) \qquad t > 0,$$

$$(3.35) \qquad v(x, 0) = \psi(x) \qquad x \in \mathbf{R}^n,$$

$$(3.36) \qquad z(\eta, 0) = z_0(\eta) \qquad \eta \in M.$$

In what follows we shall assume that

$$(3.37) \qquad g_+,\ g_- \text{ and } W\colon \mathbf{R} \to \mathbf{R} \text{ are of compact support.}$$

This can hardly be a restriction, as far as only the local solutions are concerned.

By the above observation, we find that Theorem 3.2 follows from the following:

THEOREM 3.2′. *Let $z_0\colon M \to \mathbf{R}$ be a function of class $C^{2+\alpha}$ with $\alpha \in (0, 1)$ and let $\psi\colon \mathbf{R}^n \to \mathbf{R}$ be of class $C^{1+\alpha}$ such that $\|\psi\|_{C^{1+\alpha}(\mathbf{R}^n)} < \infty$. Then there exists a classical solution pair (v, z) of problem* (3.32)–(3.36).

For $\beta \in (0, 1)$, let

$$(3.38a) \qquad \mathscr{X}_\beta(R) = C^{\beta,\beta/2}(\overline{B(R)} \times [0, T]) \times C^{1+\beta,(1+\beta)/2}(M \times [0, T]).$$

and let

$$(3.38b) \qquad \|(v, z)\|_{\mathscr{X}_\beta(R)} = \|v\|_{C^{\beta,\beta/2}(\overline{B(R)} \times [0,T])} + \|z\|_{C^{1+\beta,(1+\beta)/2}(M \times [0,T])}$$

For $\lambda \in [0, 1]$, $R > 0$ and $(\mathscr{V}, \mathscr{Z}) \in \mathscr{X}_\beta(R)$ consider the following problm (3.39)–(3.44):

$$(3.39a) \qquad \frac{\partial v}{\partial t} = D\Delta v + \lambda g_+(\mathscr{V}) \qquad x \in \Omega_+^{\mathscr{Z}}(t) \cap B(R),\ t \in (0, T],$$

$$(3.39b) \qquad \frac{\partial v}{\partial t} = D\Delta v + \lambda g_-(\mathscr{V}) \qquad x \in \Omega_-^{\mathscr{Z}}(t) \cap B(R),\ t \in (0, T],$$

$$(3.40) \qquad \frac{\partial z}{\partial t} = \lambda \mathscr{A}(\eta, \mathscr{V}, \mathscr{Z}, d\mathscr{Z}, t)z + (1 - \lambda)\Delta_0 z \qquad \eta \in M,\ t \in (0, T],$$

$$(3.41) \qquad v(\cdot, t) \in C^1(B(R)) \qquad t > 0,$$

$$(3.42) \qquad v(x, t) = \psi(x) \qquad |x| = R,\ t \in (0, T],$$

$$(3.43) \qquad v(x, 0) = \psi(x) \qquad |x| \le R.$$

$$(3.44) \qquad z(\eta, 0) = z_0(\eta) \qquad \eta \in M.$$

In (3.39) the sets $\Omega_\pm^{\mathscr{Z}}(t)$ are as in (3.31). We begin with the solving of the problem (3.39)–(3.44). For each pair $(\mathscr{V}, \mathscr{Z}) \in \mathscr{X}_\beta(R)$, denote by $(v, z) = \Phi_{\lambda,R}(\mathscr{V}, \mathscr{Z})$ the solution of equations (3.39)–(3.44). This determines a map $\Phi_{\lambda,R}: \mathscr{X}_\beta(R) \to \mathscr{X}_\beta(R)$. We shall next seek a fixed point (v_R, z_R) of the map $\Phi_{1,R}$ for each fixed $R > 0$, by applying the continuation method to the one-parameter family of maps $\{\Phi_{\lambda,R}\}_{\lambda \in [0,1]}$. Then we shall show that (a subsequence of) (v_R, z_R) converges as $R \to \infty$ and the limit is a solution of problem (3.32)–(3.36).

Problem (3.39)–(3.44) is resolved in the lemma below for given $(\mathscr{V}, \mathscr{Z}) \in \mathscr{X}_\beta(R)$ and $R > 0$.

LEMMA 3.3. *Let $z_0: M \to \mathbf{R}$ be a function of class $C^{2+\beta}$ with $\beta \in (0, 1)$ and let $R > 0$ be so large that $\tilde{\imath}(M \times \mathbf{R}) \subset B(R)$. Fix constants $\lambda \in [0, 1]$ and $T > 0$. Then for any $\psi \in C^1(\overline{B(R)})$ and any pair $(\mathscr{V}, \mathscr{Z}) \in \mathscr{X}_\beta(R)$, there exists a unique solution pair (v, z) of problem (3.39)–(3.44). Moreover,*

$$(3.45) \qquad \|v\|_{C^{1+\gamma,(1+\gamma)/2}(\overline{B(R)} \times [0,T])} + \|z\|_{C^{2+\gamma,(2+\gamma)/2}(M \times [0,T])} \le C_2,$$

where $\gamma \in (0, 1)$ and $C_2 > 0$ depend only on

$$\beta,\ T,\ \|z_0\|_{C^{2+\beta}(M)},\ \|\psi\|_{C^{1+\beta}(\overline{B(R)})}, \quad \textit{and} \quad \|(\mathscr{V}, \mathscr{Z})\|_{\mathscr{X}_\beta(R)},$$

and are independent of λ and R.

PROOF. A unique solution z to equations (3.40), (3.44) is easily found by the classical theory for parabolic equations.

To find the solution v to equations (3.39), (3.41)–(3.43), we define a function $G: \mathbf{R}^n \times [0, T] \to \mathbf{R}$ by

$$(3.47) \qquad G(x, t, v) = \begin{cases} g_+(\mathscr{V}(x, t)) & x \in \Omega_+^{\mathscr{Z}}(t) \\ g_-(\mathscr{V}(x, t)) & x \in \Omega_-^{\mathscr{Z}}(t), \end{cases}$$

and let $G^\delta (\delta > 0)$ be approximations of G satisfying

$$(3.48) \qquad |G^\delta(x, t)| \le C_3 \text{ for all } x \in \mathbf{R}^n,\ t \in [0, T], \text{ and } \delta > 0;$$

$$(3.49) \qquad G^\delta: \mathbf{R}^n \times [0, T] \to \mathbf{R} \text{ is a class } C^1 \text{ for each } \delta > 0;$$

$$(3.50) \qquad G^\delta \equiv G \text{ for } x \in \mathbf{R}^n,\ t \in [0, T] \text{ and } v \in \mathbf{R} \text{ with } dist(x, \Gamma^{\mathscr{Z}}(t)) \ge \delta.$$

Consider the following equation appoximating (3.39):

$$(3.39)^\delta \qquad \frac{\partial v^\delta}{\partial t} = D\Delta v^\delta + \lambda G^\delta(x, t) \qquad x \in \mathbf{R}^n,\ t \in (0, T].$$

By the existence theorem for parabolic equations possessing smooth coefficients and smooth nonlinear terms, we obtain the classical solution v^δ of equations $(3.39)^\delta$, (3.42), (3.43).

(3.48) implies that $\{v^\delta\}$ are uniformly bounded:

$$(3.51) \qquad |v^\delta(x, t)| \leq \|\psi\|_{L^\infty(B(R))} + C_3 t \qquad |x| \leq R,\ 0 \leq t \leq T,\ \delta > 0.$$

Clearly, the solution v^δ can be decomposed into two terms:

$$(3.52) \qquad v^\delta = v_1 + v_2^\delta,$$

where v_1 and v_2^δ satisfy

$$(3.53) \qquad \begin{cases} \dfrac{\partial v_1}{\partial t} = D\Delta v_1 & |x| < R,\ t \in (0, T], \\ v_1(x, t) = \psi(x) & |x| = R,\ t > 0, \\ v_1(x, 0) = \psi(x) & |x| \leq R; \end{cases}$$

$$(3.54) \qquad \begin{cases} \dfrac{\partial v_2^\delta}{\partial t} = D\Delta v_2^\delta + \lambda G^\delta(x, t) & |x| < R,\ t \in (0, T], \\ v_2^\delta(x, t) = 0 & |x| = R,\ t > 0, \\ v_2^\delta(x, 0) = 0 & |x| \leq R. \end{cases}$$

Since the inhomogeneous term G^δ in (3.54) is uniformly bounded, the standard estimate provides the following (see [25, Chapter V, §4]):

$$(3.55) \qquad \|v_2^\delta\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{B(R)}\times[0,T])} \leq C_4,$$

where $\alpha \in (0, 1)$ and $C_4 > 0$ are independent of δ. For $d > 0$ let

$$(3.56) \qquad \Omega^{\mathscr{T},d} = \{(x, t) \in \overline{B(R)} \times [0, T];\ dist(x, \Gamma^{\mathscr{T}}(t) > d\}.$$

From (3.50) it follows that

$$(3.57) \qquad \|G^\delta\|_{C^{\alpha,\alpha/2}(\overline{\Omega^{\mathscr{T},d}})} \leq C_5(d) \qquad \text{for } 0 < \delta \leq d,$$

where $C_5(d) > 0$ is a constant. Combining Schauder's interior and boundary estimates we obtain

$$(3.58) \qquad \|v_2^\delta\|_{C^{2+\alpha,(2+\alpha)/2}(\overline{\Omega^{\mathscr{T},d}})} \leq C_6(d)$$

if $0 < \delta \leq d$. By virtue of the estimates (3.55), (3.58), one can find a subsequence $\delta_k \downarrow 0$ and a function v_2: $\overline{B(R)} \times [0, T] \to \mathbf{R}$ such that

$$v_2^{\delta_k} \longrightarrow v_2 \qquad \text{in } C^{1+\gamma,(1+\gamma)/2}(\overline{B(R)} \times [0, T]) \tag{3.59}$$
$$\qquad \text{in } C^{2+\gamma,(2+\gamma)/2}(\overline{\Omega^{\mathscr{X},d}})$$

for any $d > 0$ and $\gamma \in (0, \alpha)$. Therefore

$$v_2^{\delta_k} \longrightarrow v_1 + v_2 \qquad \text{in } C^{1+\gamma,(1+\gamma)/2}(\overline{B(R)} \times [0, T]) \tag{3.60}$$
$$\qquad \text{in } C^{2+\gamma,(2+\gamma)/2}(\overline{\Omega^{\mathscr{X},d}} \cap \overline{B(R)} \times [\tau, T])),$$

for any $d > 0$, $\gamma \in (0, \alpha)$ and $\tau \in (0, T)$. In view of (3.50), we conclude that v_2 satisfies the equations

$$\begin{cases} \dfrac{\partial v_2}{\partial t} = D\Delta v_2 + \lambda g_\pm(\mathscr{V}) & x \in \Omega_\pm^{\mathscr{X}}(t) \cap B(R),\ t \in (0, T], \\ v_2(x, t) = 0 & |x| = R,\ t \in (0, T], \\ v_2(x, 0) = 0 & |x| \le R. \end{cases} \tag{3.61}$$

From this it follows that the function $v := v_1 + v_2$ is the required solution to problem (3.39), (3.41)–(3.43). The existence of the solution is established.

The uniqueness of the solution v of (3.39), (3.41)–(3.43) follows readily from the maximun principle.

It remains to show the inequality (3.45). The $C^{1+\gamma,(1+\gamma)/2}$ norm of v has been already estimated in the above proof of the existence. As for z, we note first that the structure of $\mathscr{A}(\eta, \mathscr{V}, \mathscr{X}, d\mathscr{X}, t)$ ensures a uniform L^∞ estimate. Then one can use the interior estimates (see [25, Chapter V. § 5]) to derive the bound of $C^{2+\gamma,(2+\gamma)/2}$ norm of z. □

By the above lemma, one can define a map $\Phi_{\lambda,R}: \mathscr{X}_\beta(R) \to \mathscr{X}_\beta(R)$ by $\Phi_{\lambda,R}(\mathscr{V}, \mathscr{X}) := (v, z)$ with (v, z) being the solution of (3.39)–(3.44).

LEMMA 3.4. *Let $z_0: \mathbf{R}^n \to \mathbf{R}$ be a function of class $C^{2+\alpha}$ with $\alpha \in (0, 1)$ and let $R > 0$ be so large that $\tilde{\imath}(M \times \mathbf{R}) \subset B(R/2)$. Assume that $\psi \in C^{1+\alpha}(\overline{B(R)})$. Then there exists a fixed point $(v_R, z_R) \in \mathscr{X}_\beta(R)$ of the map $\Phi_{\lambda,R}$ if $\beta \in (0, 1)$ is chosen sufficiently smalll. Moreover,*

$$\| v_R \|_{C^{1+\gamma,(1+\gamma)/2}(\overline{B(R)}\times[0,T])} + \| z_R \|_{C^{2+\gamma,(2+\gamma)/2}(M\times[0,T])} \le C_7, \tag{3.62}$$

where $\gamma \in (0, 1)$ and $C_7 > 0$ are independent of R (and depend only on

α, T, $\|z_0\|_{C^{2+\alpha}(M)}$ *and* $\|\psi\|_{C^{1+\alpha}(\overline{B(R)})}$).

PROOF. We use Leray-Schauder's fixed point theorem. It suffices to prove the following *a priori* estmates (i)-(iv) when $\beta \in (0, 1)$ is sufficiently small:

(i) for each $\lambda \in [0, 1]$ the map $\Phi_{\lambda,R}$ maps any bounded set of $\mathscr{X}_\beta(R)$ into a relatively compact set of $\mathscr{X}_\beta(R)$;

(ii) for any bounded set $B \subset \mathscr{X}_\beta(R)$, the family of maps $\{\Phi_{\lambda,R}|_B\}_{\lambda \in [0,1]}$ is equicontinuous in λ;

(iii) $\sup\{\|v, z)\|_{\mathscr{X}_\beta(R)};\ \Phi_{\lambda,R}(v, z) = (v, z),\ \lambda \in [0, 1]\} < \infty$;

(iv) there exists a unique $(\tilde{v}, \tilde{z}) \in \mathscr{X}_\beta(R)$ such that $\Phi_{0,R}(\tilde{v}, \tilde{z}) = (\tilde{v}, \tilde{z})$.

The property (i) follows from (3.45) in Lemma 3.3 and Ascoli-Arzelà's theorem. The property (iv) can be easily obtained, since the equations (3.39) and (3.40) are linear ones when $\lambda = 0$.

PROOF OF (ii): Let λ, $\lambda' \in [0, 1]$ and $(\mathscr{V}, \mathscr{Z}) \in B$. Put

$$(v_*, z_*) = \Phi_{\lambda,R}(\mathscr{V}, \mathscr{Z}) - \Phi_{\lambda',R}(\mathscr{V}, \mathscr{Z}). \tag{3.63}$$

It suffices to show that v_* and z_* are small if $|\lambda - \lambda'|$ is small. The part involving z_* can be directly proved by a standard estimate for the quasilinear parabolic equations (see [25, Chapter V, §6]). To estimate the term v_*, we look at the equation satisfied by v_*:

$$\frac{\partial v_*}{\partial t} = D\Delta v_* + (\lambda - \lambda')g_\pm(\mathscr{V}(x, t)) \qquad x \in \Omega_\pm^{\mathscr{Z}}(t) \cap B(R),\ t \in (0, T], \tag{3.64}$$

$$v_*(x, t) = 0 \qquad |x| = R,\ t \in (0, T], \tag{3.65}$$

$$v_*(x, t) = 0 \qquad |x| \le R. \tag{3.66}$$

Since $|g_\pm(\mathscr{V})|$ are uniformly bounded by a constant $C_8(B)$ for $\mathscr{V} \in B$, by the L^p estimate and the Sobolev imbedding theorem we obtain:

$$\|v_*\|_{C^{1+\beta,(1+\beta)/2}(\overline{B(R)} \times [0,T])} \le |\lambda - \lambda'| C_9(B). \tag{3.67}$$

This establishes the property (ii).

PROOF OF (iii): Suppose that $\Phi_{\lambda,R}(v, z) = (v, z) \in \mathscr{X}_\beta(R)$. By the condition (3.37), $|g_\pm(v(x, t))|$ are bounded. Using *a priori* estimates we have

$$\|v\|_{C^{1+\beta,(1+\beta)/2}(\overline{B(R)} \times [0,T])} \le C_{10}, \tag{3.68}$$

where the constant C_{10} is independent of R. Then a standard argument found in [25, Chapter V, §6] yields that when β is sufficiently small, there exists $\gamma\in(0, 1)$ and $C_{11}>0$, independent of R, such that

$$\|z\|_{C^{2+\gamma,(2+\gamma)/2}(M\times[0,T])}\le C_{11}. \tag{3.69}$$

The above inequalities (3.68) and (3.69) imply the property (iii) along with (3.62). □

PROOF OF THEOREM 3.2′. For each $R>0$ sufficiently large and $\beta\in(0, 1)$ sufficiently small, we obtain in the preceeding lemma a fixed point (v_R, z_R) of the map $\Phi_{\lambda,R}\colon \mathscr{X}_\beta(R)\to\mathscr{X}_\beta(R)$ satisfying (3.62). Let

$$\Omega^{z_R,d}(R)=\{(x, t);\,|x|\le R,\ 0\le t\le T,\ dist(x, \Gamma^{z_R}(t))>d\}, \tag{3.70}$$

Then applying Schauder's interior estimate to equation (3.39) gives

$$\|v_R\|_{C^{2+\gamma,(2+\gamma)/2}(\overline{\Omega^{z_R,d}(R)\cap B(R/2)}\times[\tau,T])}\le C_{12}(d, \tau), \tag{3.71}$$

where $d\in(0, R/4)$ and $\tau\in(0, T)$. Combining this with (3.56) and (3.62), one can find a subsequence $R_k\to\infty$ and functions $v\colon \mathbf{R}^n\times[0, T]\to\mathbf{R}$ and $z\colon M\times[0, T]\to\mathbf{R}$ such that

$$v_{R_k}\longrightarrow v \quad\text{in } C^{1+\delta,(1+\delta)/2}_{loc}(\mathbf{R}^n\times[0, T]) \quad\text{and}\quad \text{in } C^{2+\delta,(2+\delta)/2}_{loc}(\Omega), \tag{3.72}$$

and

$$z_{R_k}\longrightarrow z \qquad\text{in } C^{2+\delta,(2+\delta)/2}(M\times[0, T]), \tag{3.73}$$

where $\delta\in(0, \gamma)$ is a constant and $\Omega:=\{(x, t)\in\mathbf{R}^n\times[0, T];\, x\notin\Gamma(t)\}$. Clearly, (v, z) is a classical solution to problem (3.32)–(3.36). □

Appendix 1. Derivations of (1.15a) and (1.15c)

In this appendix we shall derive equations (1.15a) and (1.15c) as the limit of the original equation $(1.1)^\varepsilon$ when $\varepsilon\downarrow 0$. In what follows we denote by $B(\mathbf{R}^n)$ the space of all bounded continuous functions $w\colon\mathbf{R}^n\to\mathbf{R}$ with the norm

$$\|w\|_B=\sup\nolimits_{x\in\mathbf{R}^n}|w(x)| \tag{1A.1}$$

and by $B^1(\mathbf{R}^n)$ the space of all C^1-functions $w\colon\mathbf{R}^n\to\mathbf{R}$ such that w and $\partial w/\partial x_i (1\le i\le n)$ are both in $B(\mathbf{R}^n)$, with the norm

$$\|w\|_{B^1}=\|w\|_B+\sum_{i=1}^n\left\|\frac{\partial w}{\partial x_i}\right\|_B. \tag{1A.2}$$

all continuous functions $w\colon [0, T] \to B(\mathbf{R}^n)$(resp. $B^1(\mathbf{R}^n)$) with the norm

(1A.3) $\quad \|w\|_{[0,T];B} = \max_{0\le t\le T}\|w(t)\|_B$ (resp. $\|w\|_{[0,T];B^1} = \max_{0\le t\le T}\|w(t)\|_{B^1}$).

The following proposition gives (1.15a) and (1.15c). The estimates used in the proof are similar to that used in the proof of Lemma 3.3.

PROPOSITION 1A.1. *Let $(u^\varepsilon, v^\varepsilon)(\varepsilon > 0)$ be a family of solutions of* $(1.1)^\varepsilon$ *in* $\mathbf{R}^n \times [0, T]$ *with initial data* $(\phi^\varepsilon, \psi^\varepsilon)$. *Suppose the condition* (e) *in section* 1 *along with that there exist a constant* $C_1 > 0$ *and a function* $\psi \in B^1(\mathbf{R}^n)$ *such that*

$$\|\phi^\varepsilon\|_B + \|\psi^\varepsilon\|_{B^1} \le C_1 \qquad \varepsilon > 0, \tag{1A.4}$$

$$\psi^\varepsilon \longrightarrow \psi \ \textit{in}\ B^1(\mathbf{R}^n)\ \textit{as}\ \varepsilon \downarrow 0, \tag{1A.5}$$

and that there exists a smooth family of compact hypersurfaces $\Gamma(t)\,(0 \le t \le T)$ *in* $\mathbf{R}^n$ *such that* $u^\varepsilon - h_\pm(v^\varepsilon) \to 0$ *uniformly in the sets*

$$\{(x, t);\ x \in \Omega_\pm(t),\ 0 \le t \le T,\ dist(x, \Gamma(t)) > \delta\} \tag{1A.6}$$

for each $\delta > 0$, *where* $\Omega_+(t)$ *and* $\Omega_-(t)$ *are two disjoint regions as in* (1.13). *Then,*

(i) v^ε *converges to a function* v *uniformly in* $\mathbf{R}^n \times [0, T]$;

(ii) *moreover,* $v(\cdot, t) \in B^1(\mathbf{R}^n)$ *and* $v^\varepsilon \to v$ *in* $C([0, T]; B^1(\mathbf{R}^n))$ *as* $\varepsilon \downarrow 0$;

(iii) *the limit* v *satisfies equation* (1.15a) *for* $x \in \Omega_\pm(t)$ *and* $t \in (0, T]$.

PROOF. We shall prove the statements (i), (ii) and (iii) with T replaced by a constant $T_1 \in (0, T]$, which depends only on the Lipschitz constants of the nonlinear terms $v \mapsto g_\pm(v)$. Using this repeatedly, we obtain the original statements.

First we recall some well-known properties of the heat kernel

$$H(x, t) = (4\pi Dt)^{-n/2}\exp(-|x|^2/4Dt). \tag{1A.7}$$

For any bounded function $w\colon \mathbf{R}^n \times [0, T] \to \mathbf{R}$, define a function $w^{**}(x, t)$ by

$$w^{**}(x, t) = \int_0^t \int_{\mathbf{R}^n} H(x - y,\ t - s)w(y, s)dyds \qquad x \in \mathbf{R}^n,\ t \in [0, T]. \tag{1A.8}$$

Then w^{**} is in $B^1(\mathbf{R}^n)$ for each $t \in [0, T]$, and w^{**} satisfies

$$\|w^{**}\|_{[0,t];B^1} \le C_2\sqrt{t}\,\|w\|_{[0,t];B}; \tag{1A.9}$$

moreover if $t \mapsto w(\cdot, t)$ is a continuous function $[0, T]$ into the space $L^p(\mathbf{R}^n)$, where $p > n$ is a constant, then there exist constants $C_3 > 0$ and $\theta > 0$ with $2\theta + (n/p) < 1$ such that

$$\|w^{**}\|_{[0,t];B^1} \le C_3 t^\theta\,\|w\|_{[0,t];L^p}. \tag{1A.10}$$

By the "variation of constants" formula,

$$v^\varepsilon(x, t) = \int_{\mathbf{R}^n} H(x-y, t)\psi^\varepsilon(y)dy + \{g(u^\varepsilon, v^\varepsilon)\}^{**}(x, t) \tag{1A.11}$$

for $(x, t) \in \mathbf{R}^n \times [0, T]$.

Let us prove that $\{v^\varepsilon\}_{\varepsilon>0}$ is a Cauchy net in the space $C([0, T_1]; B^1)$, where the constant T_1 will be specified later. For each $\delta > 0$, let $R^\delta_+, R^\delta_- : \mathbf{R}^n \times [0, T] \to \mathbf{R}$ be two smooth functions satisfying

$$0 \le R^\delta_\pm(x, t) \le 1 \qquad x \in \mathbf{R}^n,\ t \in [0, T], \tag{1A.12}$$

$$R^\delta_\pm(x, t) = \begin{cases} 1 & \text{if } x \in \Omega_\pm(t) \text{ and } dist(x, \Gamma(t)) > 2\delta \\ 0 & \text{if } x \notin \Omega_\pm(t) \text{ or } dist(x, \Gamma(t)) < \delta, \end{cases} \tag{1A.13}$$

and write

$$R^\delta_0(x, t) := 1 - R^\delta_+(x, t) - R^\delta_-(x, t). \tag{1A.14}$$

We decompose v^ε into

$$v^\varepsilon = \{\psi^\varepsilon\}^* + I^{\varepsilon,\delta} + J^{\varepsilon,\delta} + K^{\varepsilon,\delta}, \tag{1A.15}$$

where

$$\{\psi^\varepsilon\}^*(x, t) = \int_{\mathbf{R}^n} H(x-y, t)\psi^\varepsilon(y)dy, \tag{1A.16}$$

$$I^{\varepsilon,\delta} = \{[u^\varepsilon - h_+(v^\varepsilon)]R^\delta_+ + [u^\varepsilon - h_-(v^\varepsilon)]R^\delta_-\}^{**}, \tag{1A.17}$$

$$J^{\varepsilon,\delta} = \{u^\varepsilon R^\delta_0\}^{**}, \tag{1A.18}$$

$$K^{\varepsilon,\delta} = \{h_+(v^\varepsilon)R^\delta_+ + h_-(v^\varepsilon)R^\delta_- - \gamma v^\varepsilon\}^{**}. \tag{1A.19}$$

By the maximum principle,

$$\| \{\psi^\varepsilon\}^* - \psi^* \|_{[0,T];B^1} \le \| \psi^\varepsilon - \psi \|_{B^1}. \tag{1A.20}$$

Moreover, in view of (1A.9) and (1A.10) one finds that

$$\begin{aligned} \| I^{\varepsilon,\delta} \|_{[0,t];B^1} \le {} & C_2\sqrt{t}\, \sup_{\substack{y \in \operatorname{supp} R^\delta_+(\cdot,s) \\ 0 \le s \le t}} |u^\varepsilon(y, s) - h_+(v^\varepsilon(y, s))| \\ & + C_2\sqrt{t}\, \sup_{\substack{y \in \operatorname{supp} R^\delta_-(\cdot,s) \\ 0 \le s \le t}} |u^\varepsilon(y, s) - h_-(v^\varepsilon(y, s))|, \end{aligned} \tag{1A.21}$$

$$\| J^{\varepsilon,\delta} \|_{[0,t];B^1} \le C_3 t^\theta \| u^\varepsilon R^\delta_0 \|_{[0,t];L^p} \le C_4 t^\theta \delta. \tag{1A.22}$$

In the last inequality, we used the fact that $u^\varepsilon(x, t)$ is uniformly bounded as $\varepsilon > 0$, $x \in \mathbf{R}^n$ and $t \in [0, T]$ vary (see Lemma 2.3). The above estimates implies the following convergences in the topology of $C([0, T]; B^1)$:

$$\{\psi^\varepsilon\} \longrightarrow \psi^* \qquad \text{as } \varepsilon \downarrow 0;$$

$$(1A.23) \qquad I^{\varepsilon,\delta} \longrightarrow 0 \qquad \text{as } \varepsilon \downarrow 0, \text{ uniformly in } \delta \in (0, 1];$$

$$J^{\varepsilon,\delta} \longrightarrow 0 \qquad \text{as } \delta \downarrow 0, \text{ uniformly in } \varepsilon \in (0, 1].$$

The last term $K^{\varepsilon,\delta}$ in equation (1A.15) satisfies

$$(1A.24) \qquad \| K^{\varepsilon,\delta} - K^{\varepsilon',\delta} \|_{[0,t];B^1} \le C_5(\sigma, \gamma)\sqrt{t}\, \| v^\varepsilon - v^{\varepsilon'} \|_{[0,t];B},$$

where the constant C_5 depends on the Lipschitz constants of the functions $v \mapsto g_\pm(v)$. Choose $T_1 = \min\{T, (2C_5)^{-2}\}$. Then,

(1A.25)
$$\| v^\varepsilon - v^{\varepsilon'} \|_{[0,T_1];B^1} \le 2 \| \{\psi^\varepsilon\}^* - \{\psi^{\varepsilon'}\}^* + I^{\varepsilon,\delta} - I^{\varepsilon',\delta} + J^{\varepsilon,\delta} - J^{\varepsilon',\delta} \|_{[0,T_1];B_1}.$$

Note that the constants C_i do not depend on ε and δ. From the above observations it follows that $\{v^\varepsilon\}$ is a Cauchy net and therefore converges in the space $C([0, T_1]; B^1)$. The statements (i) and (ii) are proved.

It remains to prove statement (iii). Denote the limit function of v^ε by $v\colon \mathbf{R}^n \times [0, T_1] \to \mathbf{R}$. Taking the limit as $\varepsilon \downarrow 0$ in equation (1A.15), one observes that $v(x, t)$ satisfies the integral equation

$$(1A.26) \qquad v = \psi^* + \{h_+(v)\chi_+ + h_-(v)\chi_- - \gamma v\}^{**}$$

for $(x, t) \in \mathbf{R}^n \times [0, T_1]$, where the functions $\chi_\pm$ are defined by

$$(1A.27) \qquad \chi_\pm(x, t) = \begin{cases} 1 & x \in \Omega_\pm(t) \\ 0 & x \notin \Omega_\pm(t). \end{cases}$$

Thus v is a weak solution of the parabolic equation

$$(1A.28) \qquad v_t = D\Delta v + h_+(v)\chi_+ + h_-(v)\chi_- - \gamma v$$

in $\mathbf{R}^n \times [0, T_1]$. Since the nonlinear term of this equation

$$(1A.29) \qquad (x, t, v) \mapsto h_+(v)\chi_+(x, t) + h_-(v)\chi_-(x, t) - \gamma v$$

is smooth for $x \in \Omega_\pm(t)$, $0 < t \le T_1$ and $M_- + \sigma \le v \le M_+ - \sigma$, using the standard Schauder estimate we find that $v(x, t)$ is smooth for $x \in \Omega_\pm(t)$ and $0 < t \le T_1$ and satisfies equation (1.15a) classically for $x \in \Omega_\pm(t)$, $0 < t \le T_1$. □

Appendix 2. Derivation of (1.15b)

In this appendix we shall derive the interface equation (1.15b) from the original equation $(1.1)^{\varepsilon}$ by passing to the limit as $\varepsilon \downarrow 0$. The heuristic arguments we used below are essentially due to [11, 23, 24, 37, 42].

Fix a point $(x^*, t^*) \in \Omega_0(t^*)$ with

$$u(x^*, t^*) = h_0(v(x^*, t^*)) \tag{2A.1}$$

and consider a moving local coordinate system $(\eta_1, \dots, \eta_{n-1}, \xi, \tau)$ in a neighborhood of (x^*, t^*):

$$\eta_i = \eta_i(x, t)\ (i = 1, \dots, n-1), \qquad \xi = \xi(x, t), \qquad \tau = t, \tag{2A.2}$$

satisfying the following:

(C1) the sets $\{x; \xi(x, t) = \text{constant}\}$ are level surfaces of the function $u(\cdot, t)$;

(C2) the location of interface is $\Gamma(t) = \{x; \xi(x, t) = 0\} = \{x; u(x, t) = h_0(v(x^*, t))\}$;

(C3) $\xi(x, t) < 0$ for $x \in \Omega_+(t)$ and $\xi(x, t) > 0$ for $x \in \Omega_-(t)$;

(C4) $(\eta, \xi, \tau) = (\eta^*, 0, \tau^*)$ corresponds to the given point (x^*, t^*);

(C5) $\eta = (\eta_1, \dots, \eta_{n-1})$ is an orthogonal coordinate system on the lovel surface of $u(\cdot, t)$;

(C6) the normal vector $\nabla_x \xi$ is of unit length at point (η^*, ξ, τ), or equivalently,

$$\left| \frac{\partial x}{\partial \xi}(\eta^*, \xi, \tau) \right| = 1; \tag{2A.3}$$

(C7) the orientation of $(\eta_1, \dots, \eta_{n-1}, \xi)$ agrees with that of $(x_1, \dots, x_n)$.

Moreover we define

$$\rho = \varepsilon^{-1} \xi. \tag{2A.4}$$

As pointed out in Introduction, the component u develops very quickly a transition layer $\Omega_0(t)$ with thickness of order $O(\varepsilon)$, while the variable v varies rather smoothly. This indicates that in analyzing the internal structure of the layer of u the coordinate system (η, ρ, τ) is more suitable, while in describing the behavior of v, (η, ξ, τ) is convenient. Write for brevity that

$$u(x, t) = u(\eta, \rho, \tau), \qquad v(x, t) = v(\eta, \xi, \tau). \tag{2A.5}$$

We first compute

$$\varepsilon \frac{\partial u}{\partial t} = \varepsilon \frac{\partial u}{\partial \tau} + \frac{\partial \xi}{\partial t} \frac{\partial u}{\partial \rho} \qquad \text{for } (\eta^*, \rho, \tau) \tag{2A.6}$$

and

(2A.7) $$\varepsilon^2 \Delta u = \frac{\partial^2 u}{\partial \rho^2} + \varepsilon(n-1)\tilde{\kappa}\frac{\partial u}{\partial \rho} \qquad \text{for } (\eta^*, \rho, \tau),$$

where $\tilde{\kappa}$ is the mean curvature of the level surface of $u(\cdot, t)$ at the point (η^*, ξ, τ). Then equation $(1.1\mathrm{a})^\varepsilon$ becomes

(2A.8) $$\varepsilon\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \rho^2} + \left\{\varepsilon(n-1)\tilde{\kappa} - \frac{\partial \xi}{\partial t}\right\}\frac{\partial u}{\partial \rho} + f(u, v(\eta^*, \varepsilon\rho, \tau)).$$

Because of the assumptions (a)-(d) in Introduction, we tend to impose that $\partial u/\partial \tau$ is uniformly bounded with respct to ε. Passing to the limit of the above equation, we see that $u(\eta^*, \rho, \tau^*)$ satisfies

(2A.9) $$\frac{\partial^2 u}{\partial \rho^2} + \left\{\varepsilon(n-1)\kappa + \left(\frac{\partial \Gamma}{\partial t}, N\right)\right\}\frac{\partial u}{\partial \rho} + f(u, v(\eta^*, 0, \tau^*)) = 0,$$

where κ, Γ and N are as in equation (1.15). To match this to the condition that $u = h_\pm(v)$ in $\Omega_\pm(t)$, we further require that

(2A.10) $$\lim_{\rho\to-\infty} u(\eta^*, \rho, \tau^*) = h_+(v(\eta^*, 0, \tau^*)),$$

(2A.11) $$\lim_{\rho\to+\infty} u(\eta^*, \rho, \tau^*) = h_-(v(\eta^*, 0, \tau^*)),$$

In view of these along with (C2), comparing equations (2A.9)–(2A.11) with equation (1.5) and recalling the stability result of Fife and McLeod, we are led to the identities

(2A.12) $$u(\eta^*, \rho, \tau^*) = U(\rho; v(\eta^*, 0, \tau^*))$$

and

(2A.13) $$\varepsilon(\eta-1)\kappa(\eta^*, \tau^*) + \left(\frac{\partial \Gamma}{\partial t}(\eta^*, \tau^*), N\right) = W(v(\eta^*,0,\tau^*)),$$

where U and W are as in (1.6) and (1.7) respectively. Eqution (2A.12) gives a local description of the internal structure of transition layers, and equation (2A.13) relates the normal velocity of interface with its curvature and the value of v on it. Since the velocities in the tangential directions of interface do not contribute the deformation of the shape of interface but only to the change of its parametrization, equation (2A.13) is geometrically equivalent to (1.15b). As pointed out in Remark 1.2, one should not ignore the term $\varepsilon(n-1)\kappa$.

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Department of Mathematics,
Faculty of Science,
*Hiroshima University**)

*) Present address: Center for Dynamical Systems and Nonlinear Studies, Georgia Institute of Technology, Atlanta, GA30332, U.S.A.