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Holomorphic functions on the nilpotent subvariety of symmetric spaces

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Introduction

Let $\mathfrak{g}$ be a complex reductive Lie algebra and let $\mathfrak{g}_{\boldsymbol{R}}$ be a non compact real form of $\mathfrak{g}$. Let $\mathfrak{g}_{\boldsymbol{R}} = \mathfrak{k}_{\boldsymbol{R}} \oplus \mathfrak{p}_{\boldsymbol{R}}$ be a Cartan decomposition of $\mathfrak{g}_{\boldsymbol{R}}$ and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the direct sum obtained by complexifying $\mathfrak{k}_{\boldsymbol{R}}$ and $\mathfrak{p}_{\boldsymbol{R}}$. G denotes the adjoint group of $\mathfrak{g}$ and we put $K_\theta = \{a \in G; \theta a = a\theta\}$, where $\theta: \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automrophism of order 2 defined by $\theta = 1$ on $\mathfrak{k}$, $\theta = -1$ on $\mathfrak{p}$. K denotes the identity component of K_θ. S denotes the symmetric algebra on $\mathfrak{p}$ and we put $J = \{u \in S; au = u$ for any $a \in K_\theta\}$ and $J_+ = \{u \in J; \partial(u)1 = 0\}$. J' denotes the ring of K-invariant polynomials and we put $J'_+ = \{f \in J'; f(0) = 0\}$. $\mathcal{O}(\mathfrak{p})$ deotes the space of holomorphic functions on $\mathfrak{p}$. We put $\mathcal{O}_0(\mathfrak{p}) = \{F \in \mathcal{O}(\mathfrak{p});$ $\partial(u)F = 0$ for any $u \in J_+\}$ and $\mathfrak{N} = \{x \in \mathfrak{p}; h(x) = 0$ for any $h \in J'_+\}$. The space $\mathcal{O}(\mathfrak{N})$ of holomorphic functions on the analytic set $\mathfrak{N}$ (cf. [2]) is equal to $\mathcal{O}(\mathfrak{p})|_{\mathfrak{N}}$ by the Oka-Cartan Theorem.

Consider the restriction mapping $\mathscr{R}: F \to F|_{\mathfrak{N}}$ of $\mathcal{O}_0(\mathfrak{p})$ to $\mathcal{O}(\mathfrak{N})$. In our previous paper [4] we showed that $\mathscr{R}$ is a linear isomrophism of $\mathcal{O}_0(\mathfrak{p})$ onto $\mathcal{O}(\mathfrak{N})$ when $\mathfrak{g} = \mathfrak{so}(d, 1)$ $(d \geqslant 3)$. In this paper we will show that we obtain the same result for any complex reductive Lie algebra.

1. Preliminaries.

Let S' be the ring of all polynomial functions on $\mathfrak{p}$ and S'_n be the homogeneous subspace of S' of degree n for $n \in \boldsymbol{Z}_+ = \{0, 1, \cdots\}$. For $f \in S'$ and $a \in K_\theta$, $af \in S'$ is given by $(af)(x) = f(a^{-1}x)$. It is known that any element of J' is invariant under K_θ (see [1] Proposition 10). It is also known that J' has homogeneous generators $P_1, \cdots, P_r$ such that $P_j|_{\mathfrak{p}_{\boldsymbol{R}}}$ is real valued $(j = 1, \cdots, r)$, where $r = \dim \mathfrak{a}_{\boldsymbol{R}}$ and $\mathfrak{a}_{\boldsymbol{R}}$ is a maximal abelian subalgebra of $\mathfrak{p}_{\boldsymbol{R}}$. $\mathscr{H} = \{f \in S'; \partial(u)f = 0$ for any $u \in J_+\}$ denotes the space of harmonic polynomials on $\mathfrak{p}$. The following lemma is known.

LEMMA 1.1 ([1] *Theorem* 14 *and Lemma* 18). (i) *If* $f \in S'$ *and* $f = 0$ *on* $\mathfrak{N}$, *then* $f \in J'_+S'$, *where* $J'_+S' = \sum_{j=1}^{r} S'P_j$.

(ii) *For any* $k \in \boldsymbol{Z}_+$ *we have* $S'_k = (J'_+S')_k \oplus \mathscr{H}_k$, *where* $(J'_+S')_k = J'_+S' \cap S'_k$.

Suppose $F \in \mathcal{O}(\mathfrak{p})$. Let $F = \sum_{k=0}^{\infty} F_k$ be the development of F by the series of homogeneous polynomial F_k of degree k. Then $\sum F_k$ converges to F uniformly on each compact set in $\mathfrak{p}$ and F_k is given by the following formula:

$$F_k(x) = \frac{1}{2\pi i} \oint_{|t|=\rho} \frac{F(tx)}{t^{k+1}} dt \qquad \text{for } x \in \mathfrak{p}, \tag{1.1}$$

where $\rho > 0$ and the right hand side of (1.1) does not depend on ρ.

Let d be a positive integer and $d \geqslant 2$. $\mathcal{O}(\boldsymbol{C}^d)$ denotes the space of entire functions on $\boldsymbol{C}^d$. $P(\boldsymbol{C}^d)$ denotes the space of the polynomials on $\boldsymbol{C}^d$ and $H_k(\boldsymbol{C}^d)$ denotes the space of the homogeneous polynomials of degree k on $\boldsymbol{C}^d$. For $\alpha = (\alpha_1, \dots, \alpha_d) \in \boldsymbol{Z}^d_+$, we put $z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}$ and $\alpha! = \alpha_1! \cdots \alpha_d!$, where $z = (z_1, \dots, z_d) \in \boldsymbol{C}^d$. For $z \in \boldsymbol{C}^d$, we define

$$\langle z^\alpha, z^\beta \rangle = \begin{cases} 0 & (\alpha \neq \beta) \\ \alpha! & (\alpha = \beta). \end{cases}$$

Then we can extend $\langle\ ,\ \rangle$ to the inner product on $P(\boldsymbol{C}^d)$. For $f \in P(\boldsymbol{C}^d)$, we define $\|f\| = \langle f, f \rangle^{1/2}$.

Let $P_1, P_2, \dots, P_s$ be arbitrary homogeneous polynomials on $\boldsymbol{C}^d$ with real coefficients. We put $\mathscr{H}_k(\boldsymbol{C}^d) = \{F \in H_k(\boldsymbol{C}^d);\ P_j(D)F = 0 \text{ for } j = 1, 2, \dots, s\}$ and $J'_k(\boldsymbol{C}^d) = \{\sum_{j=1}^{s} \phi_j P_j \in H_k(\boldsymbol{C}^d);\ \phi_1, \dots, \phi_s$ are some homogeneous polynomials on $\boldsymbol{C}^d\}$. The following lemma is known.

LEMMA 1.2 ([3] *Remark and Lemma* 1). (i) *For any* $k \in \boldsymbol{Z}_+$ *it is valid that* $H_k(\boldsymbol{C}^d) = \mathscr{H}_k(\boldsymbol{C}^d) \oplus J_k(\boldsymbol{C}^d)$ *and that* $\mathscr{H}_k(\boldsymbol{C}^d) \perp J_k(\boldsymbol{C}^d)$ *with respect to the inner product* $\langle\ ,\ \rangle$.

(ii) *Let* $F \in \mathcal{O}(\boldsymbol{C}^d)$ *and* $F = \sum_{k=0}^{\infty} F_k\ (F_k \in H_k(\boldsymbol{C}^d))$. *Then we have*

$$\limsup_{n \to \infty} (\|F_n\| / \sqrt{n!})^{1/n} = 0. \tag{1.2}$$

Conversely, if we have a sequence $\{F_k \in H_k(\boldsymbol{C}^d);\ k \in \boldsymbol{Z}_+\}$ *which satisfies* (1.2), *then* $\sum F_k$ *converges to some* $F \in \mathcal{O}(\boldsymbol{C}^d)$ *uniformly on each compact set in* $\boldsymbol{C}^d$.

2. Statement of the result and its proof.

The purpose of this paper is to prove the following

THEOREM 2.1. *The restriction mapping* $F \to F|_{\mathfrak{N}}$ *defines the following bijection*:

$$\mathscr{R}: \mathcal{O}_0(\mathfrak{p}) \xrightarrow{\sim} \mathcal{O}(\mathfrak{N}). \tag{2.1}$$

PROOF. Suppose $\dim \mathfrak{p} = d$ and $f \in \mathcal{O}(\mathfrak{N})$. Then there exists some $F \in \mathcal{O}(\mathfrak{p})$ such that $F = f$ on $\mathfrak{N}$ because $\mathcal{O}(\mathfrak{N}) = \mathcal{O}(C^d)|_{\mathfrak{N}}$. If we put $F = \sum_{n=0}^{\infty} F_n$ $(F_n \in S'_n)$, there exist $H_n \in \mathscr{H}_n$ and $G_n \in (J'_+ S')_n$ which satisfy $F_n = H_n + G_n$ for any $n \in \boldsymbol{Z}_+$ by Lemma 1.1 (ii). Let $B_{\mathfrak{p}}$ be a K_θ-invariant nondegenerate symmetric bilinear form on $\mathfrak{p}$ such that $B_{\mathfrak{p}}|_{\mathfrak{p}_R}$ is positive definite and $\{e_1, \cdots, e_d\} \subset \mathfrak{p}_{\boldsymbol{R}}$ be a basis of $\mathfrak{p}$ such that $B_{\mathfrak{p}}(e_i, e_j) = \delta_{i,j}$ $(1 \leqslant i, j \leqslant d)$ (see [1] p. 799). Now we define the mapping $\varphi: \mathfrak{p} \to \boldsymbol{C}^d$ by $\varphi(\sum_{j=1}^{d} x_j e_j) = (x_1, \cdots, x_d)$ $(x_j \in \boldsymbol{C},\ j = 1, \cdots, d)$. Let $P_1, \cdots, P_r$ be homogeneous generators of J' such that $P_j|_{\mathfrak{p}_R}$ are real valued $(1 \leqslant j \leqslant d)$. Then $\tilde{P}_j = P_j \circ \varphi^{-1}$ is a homogeneous polynomial on $\boldsymbol{C}^d$ with real coefficients. Since $H_n \circ \varphi^{-1} \in \mathscr{H}_n(\boldsymbol{C}^d)$ and $G_n \circ \varphi^{-1} \in J_n(\boldsymbol{C}^d)$ with respect to $\tilde{P}_1, \cdots, \tilde{P}_r$, we get $\|F_n \circ \varphi^{-1}\| \geqslant \|H_n \circ \varphi^{-1}\|$ from Lemma 1.2 (i) and this and Lemma 1.2 (ii) imply that $\sum H_n \circ \varphi^{-1}$ converges to some $\tilde{H} \in \mathcal{O}(\boldsymbol{C}^d)$ uniformly on each compact set in $\boldsymbol{C}^d$. If we put $\tilde{H} \circ \varphi = H$, $\sum H_n$ converges to H on each compact set in $\mathfrak{p}$ and therefore $\partial(u)H = \sum_{n=0}^{\infty} \partial(u)H_n = 0$ for any $u \in J_+$. Hence H belongs to $\mathcal{O}_0(\mathfrak{p})$. We can see that $\mathscr{R}H = f$ because $F_n = H_n$ and $f = F$ on $\mathfrak{N}$. So $\mathscr{R}$ is surjective.

Next, suppose $F \in \mathcal{O}_0(\mathfrak{p})$ and $\mathscr{R}F = 0$. If we put $F = \sum_{n=0}^{\infty} F_n$ $(F_n \in H_n(\mathfrak{p}),\ n \in Z_+)$ and $u_1, \cdots, u_r$ are homogeneous generators of J, then $\partial(u_j)F = \sum_{n=0}^{\infty} \partial(u_j)F_n = 0$ for $j = 1, 2, \cdots, r$. Therefore $\partial(u_j)F_n = 0$ because $\partial(u_j)F_n$ is a homogeneous polynomial. Hence we have $F_n \in \mathscr{H}_n$ for any $n \in \boldsymbol{Z}_+$. Furthermore, from (1.1) we can see that $F_n = 0$ on $\mathfrak{N}$ because $F = 0$ on $\mathfrak{N}$ and $t\mathfrak{N} \subset \mathfrak{N}$ for any $t \in \boldsymbol{C}$. So Lemma 1.1 implies that $F_n \in \mathscr{H}_n \cap (J'_+ S')_n = \{0\}$. Therefore we obtain $F \equiv 0$ and $\mathscr{R}$ is injective. q.e.d.

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