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An oscillation criterion for Sturm-Liouville equations with Besicovitch almost-periodic coefficients

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Let $\boldsymbol{R}$ denote the real line. The class $\Omega \subset L^1_{loc}(\boldsymbol{R})$ of Besicovitch almost-periodic functions is the closure of the set of all finite trigonometric polynomials with the Besicovitch seminorm $\|\cdot\|_B$:

$$\|p\|_B := \limsup_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} |p(s)|ds,$$

where $p \in \Omega$. The mean value, $M\{p\}$, of $p \in \Omega$ always exists, is finite, and is uniform with respect to α for $\alpha \in \boldsymbol{R}$, where

$$M\{p\} := \lim_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} p(s+\alpha)ds,$$

for some $t_0 \geq 0$ (see [1] and [2] for details).

Consider the second order nonlinear differential equation

$$\text{(E)} \qquad x''(t) - \lambda p(t) f(x(t)) = 0,$$

where $p \in \Omega$, $f \in C(\boldsymbol{R}; \boldsymbol{R})$ and $\lambda \in \boldsymbol{R} - \{0\}$.

Equation (E) is oscillatory at $+\infty$ and $-\infty$ if every continuable solution of (E) has an infinity of zeros clustering only at $+\infty$ and $-\infty$, respectively.

Recently, A. Dzurnak and A. B. Mingarelli [3] proved the following very interesting result by using Levin's comparsion theorem [5].

THEOREM A. *Let $p \in \Omega$ and $M\{|p|\} > 0$. If f is the identity mapping, then* (E) *is oscillatory at $+\infty$ and $-\infty$ for every $\lambda \in \boldsymbol{R} - \{0\}$ if and only if $M\{p\} = 0$.*

The purpose of this note is to extend Theorem A to the nonlinear case by using the following nonlinear version of Levin's comparsion theorem which is due to Yeh [8].

THEOREM B. *Let*

(C_1) *$f \in C^1(\boldsymbol{R} - \{0\})$ such that $xf(x) > 0$ and $f'(x) > 0$ for all $x \neq 0$,*

(C_2) *f' is decreasing on $(0, \infty)$ and increasing on $(-\infty, 0)$,*

(C_3) $\int_0^x \frac{dt}{f(t)} = \infty$ *for all* $x \neq 0$,

(C_4) φ_1 *and* φ_2 *are locally Lebesgue integrable on* $[a, \infty)$.

Suppose that x_1 *and* x_2 *are non-trivial solution of*

$$x''(t) + f(x(t))\, \varphi_1(t) = 0 \tag{E_1}$$

and

$$x''(t) + f(x(t))\, \varphi_2(t) = 0, \tag{E_2}$$

respectively, on the interval $[\alpha, \beta] \subseteq [a, \infty)$. *If* $x_1(t) \neq 0$ *for all* $t \in [\alpha, \beta]$, $x_1(\alpha) = x_2(\alpha)$ *and the inequality*

$$\frac{-x_1'(\alpha)}{f(x_1(\alpha))} + \int_\alpha^t \varphi_1(s)ds > \left| \frac{-x_2'(\alpha)}{f(x_2(\alpha))} + \int_\alpha^t \varphi_2(s)ds \right| \tag{C_5}$$

hold for all $t \in [\alpha, \beta]$, *then we have the following results*:

(R_1) $x_2(t) \neq 0$ *for all* $t \in [\alpha, \beta]$,

(R_2) $\frac{-x_1'(t)}{f(x_1(t))} > \left| \frac{-x_2'(t)}{f(x_2(t))} \right|$ *for all* $t \in [\alpha, \beta]$.

For other related results, we refer to Mingarelli and Halvorsen [4, 7], and Markus and Moore [6].

In order to treat with our main result, we need the following.

LEMMA 1. *Let* (C_1), (C_2) *and* (C_3) *hold. Assume that*

(C_6) $p: [t_0, \infty) \to \boldsymbol{R}$ *is locally Lebesgue integrable and has a mean value* $M\{p\}$, *where* $t_0 \geq 0$,

(C_7) $M\{p\} = 0$,

(C_8) $f'(x) \geq k$ *for some* $k > 0$ *and for all* $x \neq 0$.

If $x(t) \neq 0$ *is a solution of the differential equation*

$$x''(t) - p(t) f(x(t)) = 0 \tag{E_3}$$

on $[t_0, \infty)$, *then* $\lim_{t \to \infty} \frac{1}{t} \int_{t_0}^t f'(x(s)) \left\{ \frac{x'(s)}{f(x(s))} \right\}^2 ds = 0.$

PROOF. Define

$$z(t) := \frac{-x'(t)}{f(x(t))} \quad \text{for all } t \in [t_0, \infty).$$

It follows from (E_3) that $z(t)$ is a solution of

(E_4) $$z'(t) - f'(x(t))z^2(t) + p(t) = 0$$

on $[t_0, \infty)$. Since $f'(x(t))z^2(t) \geq 0$ on $[t_0, \infty)$, it suffices to show that

$$\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} f'(x(s))z^2(s)ds = 0.$$

Assume, on the contrary, that

$$\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} f'(x(s))z^2(s)ds > 0. \tag{1}$$

Integrating (E_4) from t_0 to t and dividing it by t, we have

$$\frac{z(t)}{t} = \frac{z(t_0)}{t} - \frac{1}{t}\int_{t_0}^{t} p(s)ds + \frac{1}{t}\int_{t_0}^{t} f'(x(s))z^2(s)ds \tag{2}$$

for all $t > t_0$. It follows from (1), (2) and (C_7) that there exist a positive constant m and an increasing sequence $\{t_n\}_{n=1}^{\infty}$ of (t_0, ∞) with $\lim_{n\to\infty} t_n = \infty$ such that

$$\frac{z(t_n)}{t_n} > m^2 \quad \text{for all } n \text{ large enough.} \tag{3}$$

It follows from (C_7) that there exists t^* large enough such that

$$\left|\int_{t_0}^{t} p(s)ds\right| < \frac{m^2 t}{4} \quad \text{for all } t \geq t^*. \tag{4}$$

Using (4), we have

$$\int_{t_n}^{t} p(s)ds = \int_{t_0}^{t} p(s)ds - \int_{t_0}^{t_n} p(s)ds < \frac{m^2 t}{4} + \frac{m^2 t_n}{4} \tag{5}$$

for all $t \geqslant t_n \geq t^*$. It follows from (3) and (5) that

$$\begin{aligned} z(t_n) - \int_{t_n}^{t} p(s)ds &> z(t_n) - \frac{m^2 t_n}{4} - \frac{m^2 t}{4} \\ &\geq z(t_n) - \frac{m^2 t_n}{4} - \frac{m^2(3t_n)}{4} > m^2 t_n - m^2 t_n = 0 \end{aligned} \tag{6}$$

for all $t \in [t_n, 3t_n] \subset [t^*, \infty)$. Since $f \in C^1([t_n, 3t_n])$ for all n such that $[t_n, 3t_n] \subset [t^*, \infty)$, for such n, the equation

(E_5) $$x_n''(t) - \frac{f(x_n(t))m^2}{4} = 0$$

has a unique solution $x_n(t)$ on $[t_n, 3t_n]$ satisfies $x_n(t_n)$ and

$$\frac{-x_n'(t_n)}{f(x_n(t_n)} = z(t_n) - \frac{m^2 t_n}{2}.$$

It follows from (5) and (6) that

$$\begin{aligned}
&\frac{-x'(t_n)}{f(x(t_n))} - \int_{t_n}^{t} p(s)ds = z(t_n) - \int_{t_n}^{t} p(s)ds \\
&> z(t_n) - \frac{m^2 t_n}{4} - \frac{m^2 t}{4} = \left\{z(t_n) - \frac{m^2 t_n}{2}\right\} - \left\{\frac{m^2 t}{4} - \frac{m^2 t_n}{4}\right\} \\
&= \frac{-x_n'(t_n)}{f(x_n(t_n))} - \int_{t_n}^{t} \frac{m^2}{4} ds \geq 0 \quad \text{on } [t_n, 3t_n] \subset [t^*, \infty).
\end{aligned}$$

Using Theorem B, we have

$$\frac{-x'(t)}{f(x(t))} > \left|\frac{-x_n'(t)}{f(x_n(t))}\right| \quad \text{on } [t_n, 3t_n] \subset [t^*, \infty). \tag{7}$$

Now, define

$$z_n(t) := \frac{-x_n'(t)}{f(x_n(t))} \quad \text{on } [t_n, 3t_n] \subset [t^*, \infty).$$

It is clear that $z_n(t)$ is a solution of the differential equation

$$(\mathrm{E}_6) \qquad z_6'(t) - f'(x_n(t)) z_n^2(t) + \frac{m^2}{4} = 0$$

on $[t_n, 3t_n] \subseteq [t^*, \infty)$ with $z_n(t_n) = z(t_n) - \dfrac{m^2 t_n}{2}$. Let

$$r_n := \frac{1}{z_n(t_n) - \dfrac{m}{2\sqrt{k}}}$$

and

$$w_n(t) := \frac{m}{2\sqrt{k}} + \frac{1}{k(t_n - t) + r_n}$$

on $\left[t_n, t_n + \dfrac{r_n}{k}\right) \subseteq [t^*, \infty)$, where n is large enough such that $z_n(t_n) > \dfrac{m}{2\sqrt{k}}$. It is clear that $w_n(t_n) = z_n(t_n)$ and

$$w_n'(t) - \mathrm{k}w_n^2(t) + \frac{m^2}{4} < 0 \le z_n'(t) - kz_n^2(t) + \frac{m^2}{4}$$

for all $t \in [t_n, 3t_n] \cap \left[t_n, t_n + \frac{r_n}{k}\right) \subseteq [t^*, \infty)$. A simple comparsion argument shows that

$$w_n(t) \le z_n(t) \quad \text{on } [t_n, 3t_n] \cap \left[t_n, t_n + \frac{r_n}{k}\right) \subseteq [t^*, \infty).$$

It follows from $z_n(t_n) = z(t_n) - \frac{m^2 t_n}{2} > \frac{m^2 t_n}{2}$ that $t_n + \frac{r_n}{k} \in [t_n, 3t_n]$ for n large enough. By the definitin of $w_n(t)$, we see that

$$\lim_{t \to \left(t_n + \frac{r_n}{k}\right)^-} w_n(t) = \infty \quad \text{for } n \text{ large enough.}$$

Hence,

$$\lim_{t \to \left(t_n + \frac{r_n}{k}\right)^-} z_n(t) = \infty \quad \text{for } n \text{ large enough.} \tag{8}$$

Now, take n_0 large enough such that

$$t_{n_0} + \frac{r_{n_0}}{k} \in [t_{n_0}, 3t_{n_0}].$$

Clearly, there exists a positive constant M such that

$$\frac{-x'(t)}{f(x(t))} \le M < \infty \text{ on } [t_{n_0}, 3t_{n_0}] \subseteq [t^*, \infty).$$

It follows from (7) and (8) that

$$\infty = \lim_{t \to \left(t_{n_0} + \frac{r_{n_0}}{k}\right)^-} z_n(t) \le \lim_{t \to \left(t_{n_0} + \frac{r_{n_0}}{k}\right)^-} \left\{\frac{-x'(t)}{f(x(t))}\right\} \le M < \infty,$$

which is a contradiction. Thus the proof is complete.

THEOREM 2. *Let* (C_1), (C_2), (C_3), *and* (C_8) *hold. If* $p \in \Omega$ *such that* (C_7) *and* $M\{|p|\} > 0$ *hold, then* (E) *is oscillatory at* $+\infty$ *and* $-\infty$ *for every* $\lambda \in \boldsymbol{R} - \{0\}$.

PROOF. Without loss of generality, we only show that (E_3) is oscillatory at $+\infty$. Assume, on the contrary, that (E_3) has a solution $x(t)$ which is nonoscillatory at $+\infty$. Thus, we can assume that there exists $t_0 > 0$ such that $x(t) > 0$ on $[t_0, \infty)$. Define

$$z(t) := \frac{-x'(t)}{f(x(t))} \quad \text{for all } t \in [t_0, \infty).$$

It is clear $z(t)$ is a solution of (E_4) on $[t_0, \infty)$. Hence, for any fixed $\delta > 0$, we have

$$\frac{1}{\delta}\int_t^{t+\delta} p(s)ds = \frac{1}{\delta}\int_t^{t+\delta} f'(x(s))z^2(s)ds - \frac{z(t+\delta)}{\delta} + \frac{z(t)}{\delta} \quad \text{on } [t_0, \infty). \tag{9}$$

Applying the Besicovitch semi-norm $\|\cdot\|_{B'}$, essentially a restriction of $\|\cdot\|_B$ to the interval $[t_0, \infty)$, defined by

$$\|g\|_{B'} := \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^t |g(s)|ds,$$

to (9), we find

$$\begin{aligned} 0 &\le \left\| \frac{1}{\delta}\int_t^{t+\delta} p(s)ds \right\|_{B'} \\ &\le \left\| \frac{1}{\delta}\int_t^{t+\delta} f'(x(s))z^2(s)ds \right\|_{B'} + \left\| \frac{z(t+\delta)}{\delta} \right\|_{B'} + \left\| \frac{z(t)}{\delta} \right\|_{B'} \quad \text{for all } \delta > 0. \end{aligned} \tag{10}$$

It follows from Lemma 1 and (C_8) that $M(z^2\} = 0$, thus, $\|z\|_{B'} = \|z(t+\delta)\|_{B'} = 0$ for all $\delta > 0$. Using Fubini's theorem, we have

$$\begin{aligned} &\frac{1}{t\delta}\int_{t_0}^t \int_s^{s+\delta} f'(x(r))z^2(r)drds \\ &= \frac{1}{t\delta}\int_{t_0}^t \int_0^{\delta} f'(x(u+s))z^2(u+s)duds \\ &= \frac{1}{t\delta}\int_0^{\delta} \int_{t_0}^t f'(x(u+s))z^2(u+s)dsdu \\ &\le \frac{1}{t\delta}\int_0^{\delta} \int_{t_0}^{t+\delta} f'(x(s))z^2(s)dsdu \\ &= \frac{1}{t}\int_{t_0}^{t+\delta} f'(x(s))z^2(s)ds \quad \text{for any fixed } \delta > 0. \end{aligned} \tag{11}$$

Using (11) and Lemma 1, we have

$$\left\| \frac{1}{\delta}\int_t^{t+\delta} f'(x(s))z^2(s)ds \right\|_{B'} = 0 \quad \text{for any fixed } \delta > 0. \tag{12}$$

Applying (12) and $\|z\|_{B'} = \|z(t+\delta)\|_{B'} = 0$ to (10), we see that

$$\left\| \frac{1}{\delta} \int_t^{t+\delta} p(s)ds \right\|_{B'} = 0 \quad \text{for all } \delta > 0. \tag{13}$$

Since p is Besicovitch almost periodic, it follows from Besicovitch [1, p.97] that

$$\lim_{\delta \to 0} \left\| p(t) - \frac{1}{\delta} \int_t^{t+\delta} p(s)ds \right\|_{B'} = 0.$$

This and (13) imply $M\{|p|\} = \|p\|_{B'} = 0$, which is a contradiction. Thus the proof is complete.

EXAMPLE. Consider the differential equation

$$x''(t) - \lambda(\sin t) f(x) = 0, \tag{14}$$

where $f(x) := \operatorname{sgn}(x) ln(|x| + 1)$ satisfies (C_1), (C_2), (C_3) and (C_8). A simple computation shows that $p(t) := \sin t$ satisfies

$$M\{p\} = \lim_{t \to \infty} \frac{1}{t} \int_0^t p(s)ds = \lim_{t \to \infty} \left(\frac{-\cos t + 1}{t} \right) = 0$$

and

$$\begin{aligned} M\{|p|\} &= \lim_{t \to \infty} \frac{1}{t} \int_0^t |p(s)|\,ds \\ &= \lim_{n \to \infty} \frac{1}{2(n+1)\pi} \int_0^{2(n+1)\pi} |\sin s|\,ds \\ &= \lim_{n \to \infty} \frac{1}{(n+1)\pi} \left\{ \int_0^{\pi} (\sin s)ds + \int_{2\pi}^{3\pi} (\sin s)ds + \cdots + \int_{2n\pi}^{(2n+1)\pi} (\sin s)ds \right\} \\ &= \lim_{n \to \infty} \frac{2(n+1)}{(n+1)\pi} = \frac{2}{\pi} > 0. \end{aligned}$$

It follows from Theorem 2 that for each $\lambda \in \boldsymbol{R} - \{0\}$, (14) is oscillatory at $+\infty$ and $-\infty$.

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