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The projection method for accelerated life test model in bivariate exponential distributions

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1. Introduction

Marshal and Olkin [6] have introduced the bivariate exponential distribution ($BVED$) with survival distribution

$$P(X \geq x, Y \geq y) = \bar{F}(x, y) = \exp\{-\lambda_1 x - \lambda_2 y - \lambda_0 \max(x, y)\},$$

where λ_0, λ_1 and λ_2 are parameters. Let U_0, U_1 and U_2 be independently distributed from univariate exponential distributions with failure rates λ_0, λ_1 and λ_2, respectively. Then (X, Y) can be written as $X = \min(U_1, U_0)$ and $Y = \min(U_2, U_0)$. Thus X and Y come from the shock times U_1 and U_2, respectively and are simultaneously governed by the fatal shock causing at the time U_0. For a sample from a $BVED$ the likelihood equation system for parameters λ_0, λ_1 and λ_2 is a simple algebraic one but the solution is intractable, cf. Arnold [1]. So one has to use some iteration method, e.g., the Fisher scoring method to seek the numerical value of the maximum likelihood estimator (MLE). Arnold introduced the unbiased estimator of the simple form, which has fairly less relative efficiency to the MLE on a part of the parameter space. Proschan and Sullo [7] proposed the intuitive estimator with high relative efficiency over all the space, which is not fully efficient. We give an efficient estimator by the projection method, see Eguchi [4] for formal derivation and several applications of the method. The estimator form is less simple than other non-iterative estimators but the construction is necessary as the first stage in the following further analysis.

Ebrahimi [3] has considered a bivariate accelerated life test model, see also Basu and Ebrahimi [2] for nonparametric approaches and Mann, Shafer and Singpuwala [5] for general notion as power rule model.

The main purpose of this paper is to establish the projection method of testing and estimation for the accelerated life test model. Let (X_j, Y_j) be independently distributed from a $BVED$ for $j = 1, \ldots, J$ with parameters λ_{0j}, λ_{1j}, λ_{2j} which satisfy

$$\lambda_{ij} = C_i V_j^P \qquad (i = 0, 1, 2) \tag{1.1}$$

Here V_j is a controllable variable which denotes the stress level at the j-th stage, C_0, C_1 and C_2 are unknown parameters associated with the three types of the shocks and P describes a relation of failure rates with the stress levels. Thus the parameter structure yields a 4-dimensional surface with coordinates (C_0, C_1, C_2, P) in the space of all the $3J$ failure rates λ_{ij}'s. We note that the corresponding surface becomes flat after the log-transformation of λ_{ij}'s. Based on the estimates $\hat{\lambda}_{ij}$ of λ_{ij} via the projection method in the first stage we project the statistics $\hat{\lambda}_{ij}$'s into the flat surface through log-transformation.

In Section 2 we introduce the projection method to the accelerated life test model as stated above. The goodness-of-fit test and the estimator are shown to be asymptotically equivalent to the maximum likelihood method. Section 3 gives a numerical examples for the comparison of the projection method with the maximum likelihood method or other methods. In Section 4, we discuss the performance of the projection method.

2. The projection method

2.1 Estimation of failure rates

A bivariate exponential random variable (X, Y) has a density function

$$f(x, y) = \begin{cases} \lambda_0 \bar{F}(x, y) & \text{on} \quad L = \{x = y\}\,, \\ \lambda_2(\lambda_1 + \lambda_0)\bar{F}(x, y) & \text{on} \quad D = \{x > y\}\,, \\ \lambda_1(\lambda_2 + \lambda_0)\bar{F}(x, y) & \text{on} \quad U = \{x < y\} \end{cases}$$

with respect to the carrier measure μ defined by $\mu(B) = \mu_2(B) + \mu_1(B \cap L)/\sqrt{2}$, where μ_k denotes the k-dimensional Lebesgue measure. For a random sample $(x_1, y_1), \ldots, (x_n, y_n)$ from a distribution with density $f(x, y)$, we write

$$\bar{p}_1 = \frac{1}{n}\sum 1_D(x_i, y_i)\,, \qquad \bar{p}_2 = \frac{1}{n}\sum 1_U(x_i, y_i)\,, \qquad \bar{x} = \frac{1}{n}\sum x_i$$

$$\bar{y} = \frac{1}{n}\sum y_i \qquad \text{and} \qquad \bar{z} = \frac{1}{n}\sum \max(x_i, y_i)\,,$$

where 1_S denotes the indicator function of a set S. Then the log-likelihood function is written as

$$l(\lambda) = n\{\bar{p}_0 \log \lambda_0 + \bar{p}_1 \log \lambda_2(\lambda_1 + \lambda_0) + \bar{p}_2 \log \lambda_1(\lambda_2 + \lambda_0) - \lambda_1 \bar{x} - \lambda_2 \bar{y} - \lambda_0 \bar{z}\} \tag{2.1}$$

with $\bar{p}_0 = 1 - \bar{p}_1 - \bar{p}_2$ and $\lambda = (\lambda_1, \lambda_2, \lambda_0)^T$. The estimating equation system probably cannot be solved in the form of elementary functions. It follows from (2.1) that the statistic vector $\boldsymbol{t} = (\bar{p}_1, \bar{p}_2, \bar{x}, \bar{y}, \bar{z})^T$ is minimal sufficient with mean vector

$$\boldsymbol{m} = (\lambda_1/\lambda_*, \lambda_2/\lambda_*, 1/(\lambda_1 + \lambda_0), 1/(\lambda_2 + \lambda_0), 1/(\lambda_1 + \lambda_0) + 1/(\lambda_2 + \lambda_0) - \lambda_*)$$

with $\lambda_* = \lambda_0 + \lambda_1 + \lambda_2$.

We extend the family of *BVED*'s to a 5-dimensional exponential family equipping with the sufficient statistic $\boldsymbol{t}$. The density is defined as

$$f^*(x, y) = \begin{cases} \alpha_0 \bar{F}(x, y) & \text{on} \quad L = \{x = y\}\,, \\ \alpha_1 \bar{F}(x, y) & \text{on} \quad D = \{x > y\}\,, \\ \alpha_2 \bar{F}(x, y) & \text{on} \quad U = \{x < y\}\,, \end{cases}$$

where α_0 is determined as

$$\alpha_0 = \lambda_0 + \lambda_1 + \lambda_2 - \alpha_1/(\lambda_0 + \lambda_1) - \alpha_2/(\lambda_0 + \lambda_2)$$

by the condition of total mass one. Note that the restriction of

$$\alpha_1 = \lambda_2(\lambda_1 + \lambda_0) \quad \text{and} \quad \alpha_2 = \lambda_1(\lambda_2 + \lambda_0) \tag{2.2}$$

reduces the extended family to the original family. If the sample follows from a distribution with density $f^*(x, y)$, then the likelihood is

$$l(\alpha, \lambda) = n(\bar{p}_0 \log \alpha_0 + \bar{p}_1 \log \alpha_2 + \bar{p}_2 \log \alpha_1 - \lambda_1 \bar{x} - \lambda_2 \bar{y} - \lambda_0 \bar{z})\,, \tag{2.3}$$

from which $\boldsymbol{t}$ is still sufficient. The *MLE* is easily obtained as

$$\hat{\alpha}_1 = \frac{\bar{p}_1^2}{\bar{w}(\bar{z} - \bar{y})}\,, \qquad \hat{\alpha}_2 = \frac{\bar{p}_2^2}{\bar{w}(\bar{z} - \bar{y})}\,, \qquad \hat{\lambda}_1 = \frac{1}{\bar{w}} - \frac{\bar{p}_2}{\bar{z} - \bar{x}}\,,$$

$$\hat{\lambda}_2 = \frac{1}{\bar{w}} - \frac{\bar{p}_1}{\bar{w} - \bar{y}} \qquad \text{and} \qquad \hat{\lambda}_0 = \frac{\bar{p}_2}{\bar{z} - \bar{x}} + \frac{\bar{p}_1}{\bar{z} - \bar{y}} - \frac{1}{\bar{w}}\,,$$

where $\bar{w} = \bar{x} + \bar{y} - \bar{z}$. In view of the extension we introduce a transformation $\boldsymbol{s} = \varphi(\boldsymbol{t})$ by

$$\boldsymbol{s} = \varphi(\boldsymbol{t}) = (\bar{p}_1/\bar{w}, \bar{p}_2/\bar{w}, \bar{p}_0/\bar{w}, \bar{p}_1/(\bar{z} - \bar{y}), \bar{p}_2/(\bar{z} - \bar{x}))^T\,. \tag{2.4}$$

Note that $\boldsymbol{s}$ is also sufficient since φ is one-to-one. Furthermore $\boldsymbol{s}$ is the joint *MLE* of a parameter vector

$$\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5)^T = (\alpha_1/(\lambda_0 + \lambda_1), \alpha_2/(\lambda_0 + \lambda_2), \alpha_0, \lambda_1 + \lambda_0, \lambda_2 + \lambda_0)^T$$

since it is satisfied that

$$\boldsymbol{s} = (s_1, s_2, s_3, s_4, s_5)^T = (\hat{\alpha}_1/(\hat{\lambda}_0 + \hat{\lambda}_1),\ \hat{\alpha}_2/(\hat{\lambda}_0 + \hat{\lambda}_2),\ \hat{\alpha}_0,\ \hat{\lambda}_1 + \hat{\lambda}_0,\ \hat{\lambda}_2 + \hat{\lambda}_0)^T\,.$$

By substitution of the inverse transformation

$$(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \lambda_0)$$
$$= (\sigma_1\sigma_5, \sigma_2\sigma_4, \sigma_1+\sigma_2+\sigma_3-\sigma_4, \sigma_1+\sigma_2+\sigma_3-\sigma_5, \sigma_1+\sigma_2+\sigma_3-\sigma_4-\sigma_5)$$

into (2.3) it holds that the log-likelihood function for σ is

$$l^*(\sigma) = n\{\bar{p}_0 \log \sigma_3 + \bar{p}_1 \log \sigma_1\sigma_5 + \bar{p}_2 \log \sigma_2\sigma_4 - (\sigma_1 + \sigma_2 + \sigma_3 - \sigma_4)\bar{x}$$
$$- (\sigma_1 + \sigma_2 + \sigma_3 - \sigma_5)\bar{y} - (\sigma_1 + \sigma_2 + \sigma_3 - \sigma_4 - \sigma_5)\bar{z}\}\,.$$

Thus the observed information matrix is given by

$$\text{(2.5)}\quad \mathscr{I}_\sigma(s) = -\frac{1}{n}\frac{\partial^2}{\partial\sigma\,\partial\sigma^T} l^*(\sigma)\bigg|_{\sigma=s} = \operatorname{diag}(p_1/s_1^2, \bar{p}_2/s_2^2, \bar{p}_0/s_3^2, \bar{p}_1/s_4^2, \bar{p}_2/s_5^2)\,.$$

By theory of parametric estimation it holds that $\mathscr{I}_\sigma(s)$ is strongly a consistent estimator of the Fisher information matrix $\mathscr{I}_\sigma = \mathscr{I}_\sigma(\sigma)$ of σ and that $\sqrt{n}(s - \sigma)$ has asymptotically a 5-variate normal law with mean $\mathbf{0}$ and variance $\mathscr{I}_\sigma^{-1}$. In view of this $\mathscr{I}_\sigma^{-1}(s)$ is a consistent estimator of the variance matrix of s.

We now back to the case of *BVED*. It follows from (2.2) that $\varphi(\boldsymbol{m}) = A\lambda$, where $\lambda = (\lambda_1, \lambda_2, \lambda_0)^T$ and

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Thus s is a Fisher-consistent estimator of $A\lambda$. A weighted sum of squares is introduced as

$$D_0^2 = n(s - A\lambda)^T \mathscr{I}_\sigma(s)(s - A\lambda)\,,$$

where $\mathscr{I}_\sigma(s)$ is defined in (2.5). The minimizer of D_0^2 with respect to λ is given by

$$\text{(2.6)}\qquad \hat{\lambda} = (A^T\hat{\mathscr{I}}_\sigma A)^{-1}A^T\hat{\mathscr{I}}_\sigma s$$

and the minimum is

$$\text{(2.7)}\qquad \hat{D}_0^2 = ns^T\{\hat{\mathscr{I}}_\sigma - \hat{\mathscr{I}}_\sigma A(A^T\hat{\mathscr{I}}_\sigma A)^{-1}A^T\hat{\mathscr{I}}_\sigma\}s\,,$$

where $\hat{\mathscr{I}}_\sigma = \mathscr{I}_\sigma(s)$. In consequence we can regard $\hat{\lambda}$ as a generalized least-square estimator (*GLSE*) and $\hat{D}_0^2$ as the residual sum of squares (*RSS*). We write

$$\hat{\lambda}^* = (A^T\mathscr{I}_\sigma A)^{-1}\mathscr{I}_\sigma s\,.$$

Since $\hat{\mathscr{I}}_\sigma - \mathscr{I}_\sigma = O_P(n^{-1/2})$ and

$$\hat{\lambda}^* - \lambda = (A^T \mathscr{I}_\sigma A)^{-1} A^T \mathscr{I}_\sigma (s - A\lambda),$$

we see that $\sqrt{n}(\hat{\lambda} - \lambda)$ and $\sqrt{n}(\hat{\lambda}^* - \lambda)$ both have asymptotically a trivariate normal distribution with mean $\mathbf{0}$ and variance $(A^T \mathscr{I}_\sigma A)^{-1}$, which is the inverse matrix of Fisher information. Accordingly we conclude the asymptotic efficiency of the estimator $\hat{\lambda}$.

2.2 Accelerated life test model

We now investigate the case of bivariate accelerated life test model in Introduction. Let (x_{j1}, y_{j1}), ..., (x_{jn_j}, y_{jn_j}) be a random sample from (X_j, Y_j) for $j = 1, \ldots, J$, where (X_j, Y_j)'s have failure rates λ_{ij}'s satisfying (1.1). The *MLE* of (C_0, C_1, C_2, P) can be only numerically sought by some iteration method since it is too complicated to obtain an explicit solution of the system of likelihood equations, cf. Ebrahimi [3].

We denote $\boldsymbol{s}_j$ the corresponding statistic based on the j-th observation to (2.4) and denote $\hat{\lambda}_j$ and $\hat{D}^2_{0j}$ the *GLSE* and the *RSS* based on $\boldsymbol{s}_j$ by a similar way to (2.6) and (2.7), respectively, for $j = 1, \ldots, J$. Writing

$$\boldsymbol{u}_j = (\log \hat{\lambda}_{j0}, \log \hat{\lambda}_{j1}, \log \hat{\lambda}_{j2})^T \tag{2.8}$$

and $\boldsymbol{u} = (\boldsymbol{u}_1^T, \ldots, \boldsymbol{u}_J^T)^T$, we see that the statistic $\boldsymbol{u}$ is sufficient and Fisher-consistent for $\xi = (\xi_1, \ldots, \xi_J)$ with

$$\xi_j = (\log \lambda_{j0}, \log \lambda_{j1}, \log \lambda_{j2})^T.$$

From the assumption (1.1) it is written as that $\xi = X\theta$, where

$$\theta = (\log C_0, \log C_1, \log C_2, P) \tag{2.9}$$

and $X^T = (X_1, \ldots, X_J)$ with

$$X_j = \begin{bmatrix} 1 & 0 & 0 & \log V_j \\ 0 & 1 & 0 & \log V_j \\ 0 & 0 & 1 & \log V_j \end{bmatrix}, \qquad j = 1, \ldots, J.$$

By a similar way of defining D_0^2, we introduce a weighted sum of squares

$$D^2 = (\boldsymbol{u} - X\theta)^T \hat{\mathscr{I}}_\xi (\boldsymbol{u} - X\theta),$$

Here $\hat{\mathscr{I}}_\xi = \mathrm{diag}(\hat{\mathscr{I}}_1, \ldots, \hat{\mathscr{I}}_J)$ with $\hat{\mathscr{I}}_j = \mathrm{diag}(\hat{\lambda}) A^T \mathscr{I}_\sigma(\boldsymbol{s}_j) A \ \mathrm{diag}(\hat{\lambda})$ for $j = 1, \ldots, J$. The minimizer of D^2 with respect to θ or the *GLSE* is given by

$$\hat{\theta} = (X^T \hat{\mathscr{I}}_\xi X)^{-1} X^T \hat{\mathscr{I}}_\xi \boldsymbol{u}$$

and the minimum, or the *RSS* is

(2.10) $$\hat{D}^2 = \boldsymbol{u}^T(\hat{\mathscr{I}}_\xi - \hat{\mathscr{I}}_\xi X(X^T\hat{\mathscr{I}}_\xi X)^{-1}X^T\hat{\mathscr{I}}_\xi)\boldsymbol{u}\,.$$

We note that

$$(\hat{C}_0, \hat{C}_1, \hat{C}_2, \hat{P}) = (\exp(\hat{\theta}_1),\ \exp(\hat{\theta}_2),\ \exp(\hat{\theta}_3), \hat{\theta}_4)$$

minimizes D^2 with respect to (C_0, C_1, C_2, P) when D^2 is regarded as a function in the parameters through the transformation (2.9).

Consider the goodness-of-fit test of the accelerated life test model. The null hypothesis is composite with nuisance parameter vector (C_0, C_1, C_2, P). We propose $\hat{D}^2$ as a test statistic. On the other hand, the usual likelihood ratio is defined $2(l_1 - l_2)$. Here $l_1 = \sum_{j=1}^J \max_{\lambda_j} l(\lambda_j)$ and

$$l_2 = \max\left\{\sum_{j=1}^J l(\lambda_j)\,\middle|\,\lambda_1, \ldots, \lambda_J \text{ are subject to (1.1)}\right\}.$$

By a similar argument in the section 2.1, the limiting distribution of $\sqrt{n}(\hat{\theta} - \theta)$ is 4-variate normal distribution with mean $\mathbf{0}$ and covariance $(X^T\mathscr{I}_\xi X)^{-1}$ which is the inverse of Fisher information matrix of θ. Since it follows that

$$\begin{aligned}(\hat{\theta} - \theta)^T = \operatorname{diag}&\left(\frac{1}{C_0}, \frac{1}{C_1}, \frac{1}{C_2}, 1\right)\\ &\times (\hat{C}_0 - C_0, \hat{C}_1 - C_1, \hat{C}_2 - C_2, \hat{P} - P) + o_P(n^{-1/2})\end{aligned}$$

and the Fisher information matrix of (C_0, C_1, C_2, P) is

$$\operatorname{diag}(C_0, C_1, C_2, 1)X^T\mathscr{I}_\xi X \operatorname{diag}(C_0, C_1, C_2, 1)\,,$$

we can see that the estimator $(\hat{C}_0, \hat{C}_1, \hat{C}_2, \hat{P})$ is asymptotically efficient. Alternatively the goodness-of-fit test $\hat{D}^2$ is asymptotically a chi-square distribution with $3J - 4$ degrees of freedom because

$$P(\xi) = I - \mathscr{I}_\xi^{1/2}X(X^T\mathscr{I}_\xi X)^{-1}\mathscr{I}_\xi^{1/2}$$

is a symmetric idempotent matrix of rank $3J - 4$ and

$$\boldsymbol{u}(\xi) = \mathscr{I}_\xi^{1/2}(\boldsymbol{u} - X\theta)$$

has asymptotically the standard ($3J$-variate) normal distribution. For we have that

$$\hat{D}^2 = \boldsymbol{u}(\hat{\xi})^T P(\hat{\xi})\boldsymbol{u}(\hat{\xi})\,,$$

and that the effect of substitution of ξ into $\hat{\xi}$ is asymptotically negligible. Here $\mathscr{I}_\xi^{1/2}$ denotes the grammian square root of $\mathscr{I}_\xi$. Hence the estimate and test statistic are shown to be equivalent to the *MLE* and the likelihood ratio. Thus we observe that the methods of projection and maximum likelihood are equal in the asymptotical sense.

3. Numerical examples

We investigate the finite sample behavior of the estimates and test statistics via projection method in a numerical study. We give a numerical experiments with 4 stress levels $V_1 = 1.$, $V_2 = 2.$, $V_3 = 4.$ and $V_4 = 8.$, assuming the sample sizes as $n_1 = n_2 = n_3 = n_4 = 50$. In the following table, by determining appropriately true values of parameters (C_0, C_1, C_2, P), we first estimates the 12 failure rates λ_{ij}'s by two methods of projection and maximum likelihood, say $\hat{\lambda}_{ij}$'s and $\tilde{\lambda}_{ij}$'s, respectively. Secondly based on $\hat{\lambda}_{ij}$'s the estimates and goodness-of-fit test by the second projection method are given in addition to the *MLEs* and the log-likelihood ratios.

TABLE The estimates and goodness-of-fit tests

Case 1

True value	$C_0 = 1.0225$	$C_1 = 1.4294$	$C_2 = 1.8383$	$P = 1.2518$
Projection	.7868	1.4358	2.0093	1.2714
MLE	.7474	1.2901	1.8402	1.3271
Projection test	5.3734 (2.95, 0.64, 1.41, 0.37)			
Likelihood Ratio	6.7049 (3.49, 1.35, 1.55, 0.31)			

	λ_{01}	λ_{11}	λ_{21}	λ_{02}	λ_{12}	λ_{22}	λ_{03}	λ_{13}	λ_{23}	λ_{04}	λ_{14}	λ_{24}
True	1.0	1.4	1.8	2.4	3.4	4.4	5.8	8.1	10.4	13.8	19.3	24.8
Proj.	.88	.86	2.0	1.7	3.8	5.3	3.8	8.9	9.9	12.3	21.8	27.1
MLE	.89	.87	2.0	1.7	3.8	5.4	3.9	9.2	10.0	12.3	21.8	27.1

Case 2

True value	$C_0 = .4314$	$C_1 = 2.9621$	$C_2 = 1.6939$	$P = 0.3802$
Projection	.3928	3.1699	2.0093	0.3582
MLE	.3818	3.0460	1.8837	0.3852
Projection test	11.8964 (1.42, 2.07, 5.06, 3.35)			
Likelihood Ratio	11.1535 (1.04, 1.37, 4.69, 4.06)			

	λ_{01}	λ_{11}	λ_{21}	λ_{02}	λ_{12}	λ_{22}	λ_{03}	λ_{13}	λ_{23}	λ_{04}	λ_{14}	λ_{24}
True	.43	3.0	1.7	.56	3.9	2.2	.73	5.0	2.9	.95	6.5	3.7
Proj.	.26	3.3	1.7	.60	4.6	2.6	.76	4.2	2.4	.79	6.6	5.8
MLE	.27	3.4	1.8	.60	4.6	2.6	.76	4.2	2.4	.79	6.6	5.9

Case 3

True value	$C_0 = 1.0065$	$C_1 = 2.5276$	$C_2 = 2.1743$	$P = 2.2799$
Projection	1.0192	2.7342	2.5607	2.1507
MLE	1.0225	2.8114	2.4255	2.1256
Projection test	9.5049 (.63, 4.38, 1.42, 3.08)			
Likelihood Ratio	9.3885 (.53, 5.17, 1.00, 2.69)			

	λ_{01}	λ_{11}	λ_{21}	λ_{02}	λ_{12}	λ_{22}	λ_{03}	λ_{13}	λ_{23}	λ_{04}	λ_{14}	λ_{24}
True	1.0	2.5	2.2	4.9	12.2	10.6	23.7	59.6	51.3	115.3	289.5	249.0
Proj.	.88	2.5	2.4	3.9	11.9	15.7	18.9	55.7	39.2	109.4	245.4	150.7
MLE	.92	2.6	2.5	3.9	11.9	15.8	19.2	57.1	39.2	111.8	253.2	149.8

Case 4

True value	$C_0 = 1.8380$	$C_1 = .2326$	$C_2 = .5258$	$P = 2.2799$
Projection	1.7610	.2971	.5809	2.1507
MLE	1.7986	.2920	.6134	2.1256
Projection test	9.1999 (1.64, 1.46, 2.23, 3.88)			
Likelihood Ratio	9.7205 (1.90, 1.50, 2.00, 4.32)			

	λ_{01}	λ_{11}	λ_{21}	λ_{02}	λ_{12}	λ_{22}	λ_{03}	λ_{13}	λ_{23}	λ_{04}	λ_{14}	λ_{24}
True	1.8	.23	.53	6.0	.75	1.7	19.3	2.4	5.6	62.6	7.9	18.0
Proj.	1.9	.43	.65	4.7	.88	1.8	14.5	2.4	6.8	73.3	7.6	13.6
MLE	2.0	.44	.67	4.8	.88	1.9	15.1	2.3	7.3	74.9	7.6	14.0

4. Discussion

We consider a situation where some j_0-th sample violates the condition (1.1). A way of detecting the j-th sample is proposed by the statistic $\hat{D}$. By the definition (2.10), $\hat{D}^2$ is decomposed into $\hat{D}^2 = \sum_{j=1}^{J} \hat{D}_j^2$, where

$$\hat{D}_j^2 = \boldsymbol{u}_j^T(\hat{\mathscr{I}}_j - \hat{\mathscr{I}}_j X_j (X^T \hat{\mathscr{I}}_\xi X)^{-1} \hat{\mathscr{I}}_j)\boldsymbol{u}_j \,.$$

If the j-th component $\hat{D}_j^2$ follows from the accelerated life test model, then it has asymptotically a moment

$$\boldsymbol{m}_j(C_0, C_1, C_2, P) = 3 - \delta_j$$

with $\delta_j = tr\{X_j^T \mathscr{I}_j X_j (X^T \mathscr{I}_\xi X)^{-1}\}$. The term δ_j is a nonnegative function of (C_0, C_1, C_2, P) satisfying $\sum_{j=1}^{J} \delta_j = 4$. In view of this we determine a j_0 such that

$$|\hat{D}_{j_0}^2 - \hat{\boldsymbol{m}}_{j_0}| = \max_j |\hat{D}_j^2 - \hat{\boldsymbol{m}}_j| \,,$$

where $\hat{\boldsymbol{m}}_j = \boldsymbol{m}_j(\hat{C}_0, \hat{C}_1, \hat{C}_2, \hat{P})$.

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