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Periodic zeta functions for rank 1 space forms of symmetric spaces

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1. Introduction

For the modular group $\Gamma = PSL(2, Z)$ and a positive number α, A. Fujii [5], [6] has studied a periodic zeta function

$$Z_\alpha(s) = \sum_{r_j>0} \frac{\sin \alpha r_j}{r_j^s} \qquad \text{Re } s > 1 \tag{1.1}$$

associated with the discrete spectrum $0 = \lambda_0 \le \lambda_1 \le \cdots$ of the Laplace-Beltrami operator acting on $L^2(\Pi^+/\Gamma)$ where Π^+ is the upper half-plane. Here, as usual, r_j is given by $\lambda_j = \frac{1}{4} + r_j^2$. Using the Selberg trace formula Fujii proves that Z_α has an analytic continuation Z_α to the whole plane—ie. Z_α is an entire function. Among other results he also proves that

$$\lim_{\alpha \to \log N(P_1)} (\alpha - \log N(P_1)) Z_\alpha(0) = \frac{1}{2\pi} \sum_{\{P\},\, N(P)=N(P_1)} \tilde{\Lambda}(P)/\sqrt{N(P)} \tag{1.2}$$

where $\{P_1\}$ is any hyperbolic conjugacy class, N is the norm function and $\tilde{\Lambda}$ is the von Mangoldt function for the Selberg zeta function. Some related work appears in [2], [4], [10], [14].

It seems natural to replace Π^+ by a general rank one symmetric space G/K where G is a connected non-compact semisimple Lie group with finite center and K is a maximal compact subgroup of G. A suitable version of the trace formula is available in this context for Γ a discrete subgroup of G. In this paper we consider indeed a corresponding zeta function Z_α, as in (1.1), and prove that Z_α extends to an entire function on the complex plane at least when G is simple and Γ is without torsion and is co-compact. Actually we construct an infinite family $\{Z_{\alpha,b}\}_{b\ge 0}$ of zeta function with $Z_{\alpha,0} = Z_\alpha$. Each $Z_{\alpha,b}$ is entire; see Theorems 5.17 and 6.10.

For the modular group Γ one has the well known fact that $\lambda_1 > \frac{1}{4}$; ie. no complementary series representations of $PSL(2, R)$ occur in the discrete spectrum of $L^2(\Gamma|PSL(2, R))$. However, in the case at hand complementary

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series indeed can occur in $L^2(\Gamma|G)$[18]. Extra care therefore must be taken to analytically continue the $Z_{\alpha,b}$. We consider an appropriate version of a von Mangoldt function $\tilde{\Lambda}$ for the space form $X_\Gamma = \Gamma|G/K$, and we formulate the analogue of (1.2). As in [6] this requires a formula for the special value $Z_\alpha(0)$. .

2. Normalization of measures

Let $\mathfrak{g}_0$, $\mathfrak{k}_0$ denote the Lie algebras of G, K and let $(,)$ denote the Killing form of $\mathfrak{g}_0$. Then for $\mathfrak{p}_0 = \{x \in \mathfrak{g}_0 | (x, \mathfrak{k}_0) = 0\}$, $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ is a Cartan decomposition of $\mathfrak{g}_0$. Let θ be the corresponding Cartan involution and let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ denote the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$. Fix an Iwasawa decomposition $G = KA_\mathfrak{p}N$ of G where $A_\mathfrak{p} = \exp \mathfrak{a}_\mathfrak{p}$, $N = \exp \mathfrak{n}_0$ for $\mathfrak{a}_\mathfrak{p}$ maximal abelian in $\mathfrak{p}_0$ and $\mathfrak{n}_0$ is the sum over a positive system Σ^+ of restricted root spaces. Let $\mathfrak{a}^C$ be the complixification of a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}_0$ which contains $\mathfrak{a}_\mathfrak{p}$. Then $\mathfrak{a}^C$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$. The set of non-zero roots of $(\mathfrak{g}, \mathfrak{a}^C)$ is denoted by Φ. Choose in Φ an $\mathfrak{a}_\mathfrak{p}$-compatible system of positive roots Φ^+ and set

$$
\begin{aligned}
P^+ &= \{\alpha \in \Phi^+ | \alpha \neq 0 \text{ on } \mathfrak{a}_\mathfrak{p}\} \\
2\rho &= \langle P^+ \rangle
\end{aligned}
\tag{2.1}
$$

where $\langle Q \rangle = \sum_{\alpha \in Q} \alpha$ for $Q \subset \Phi$. Then in fact we can take $\Sigma^+ = \{\alpha|_{\mathfrak{a}_\mathfrak{p}} | \alpha \in P^+\}$. We will assume that the R-rank of G is 1 (ie. $\dim \mathfrak{a}_\mathfrak{p} = 1$) so that Σ^+ has the form $\Sigma^+ = \{\beta\}$ or $\Sigma^+ = \{\beta, 2\beta\}$. The Iwasawa decomposition of G gives rise to a smooth map $H: G \to \mathfrak{a}_\mathfrak{p}$ for each $x \in G$, $x = k(x) \exp H(x) \in KA_\mathfrak{p}N$. We fix the choice of basis element H_0 of $\mathfrak{a}_\mathfrak{p}$ by

$$\beta(H_0) = 1 \tag{2.2}$$

Haar measures da, dn, dx, dv on $A_\mathfrak{p}$, N, G, $\mathfrak{a}_\mathfrak{p}^*$ (dual space of $\mathfrak{a}_\mathfrak{p}$) respectively will be normalized by the equations

$$
\begin{aligned}
&\int_{A_\mathfrak{p}} h(a)\, da = \int_R h(\exp tH_0)\, dt \\
&\int_N e^{-2\rho(H(\theta n))}\, dn = 1 \\
&\int_G f(x)\, dx = \int_N \int_{A_\mathfrak{p}} \int_K f(kan) e^{2\rho(\log a)}\, dk da dn \\
&\int_{\mathfrak{a}_\mathfrak{p}^*} \omega(v)\, dv = \frac{1}{2\pi} \int_R \omega(t\beta)\, dt
\end{aligned}
\tag{2.3}
$$

for $h \in C_c(A_p)$, $f \in C_c(G)$. $\omega \in C_c(\mathfrak{a}_{\mathfrak{p}}^*)$ where dt denotes Lebesgue measure on R. dk = normalized Haar measure on K. For Γ a discrete subgroup of G let m_Γ be the unique G-invariant measure on $\Gamma \backslash G$ such that

$$\int_G f(x)\,dx = \int_{\Gamma\backslash G} \left(\textstyle\sum_{\gamma\in\Gamma} f(\gamma x)\right) dm_\Gamma(\Gamma x) \tag{2.4}$$

Let G be one of the following Lie groups: $SO_1(2n, 1)$, $SO_1(2n+1, 1)$ $(n \geq 1)$, $SU(n, 1)(n \geq 2)$, $Sp(n, 1)(n \geq 2)$, or $F_{4(-20)}$, up to a local isomorphism. Let c denote Harish-Chandra's c-function for the spherical Plancherel measure of G/K. Given the normalization of measures in (2.3) Miatello's computation [13] of $|c(\cdot)|^{-2}$ takes the form

$$|c(r)|^{-2} = \begin{cases} C_G \pi r P(r) \tanh \pi r & \textit{for} \quad G = SO_1(2n, 1) \\ C_G \pi P(r) & \textit{for} \quad G = SO_1(2n+1, 1) \\ C_G \pi r P(r) \tanh^{\varepsilon} \dfrac{\pi r}{2} & \textit{for} \quad G = SU(n, 1) \\ C_G \pi r P(r) \tanh \dfrac{\pi r}{2} & \textit{for} \quad G = Sp(n, 1), F_{4(-20)}, \end{cases} \tag{2.5}$$

where C_G, $P(r)$, ε are given in the following table. Here $\Gamma(\cdot)$ is the classical gamma function

TABLE 1

G (*local isomorphism*)	C_G	$P(r)$	ε	ρ_0
$SO_1(2n, 1)$ $n \geq 1$	$\dfrac{1}{2^{4n-4}\Gamma(n)^2}$	$\prod_{j=2}^{n} (r^2 + (n - j + \frac{1}{2})^2)$		$n - \frac{1}{2}$
$SO_1(2n+1, 1)$ $n \geq 1$	$\dfrac{1}{2^{4n-2}\Gamma(n+\frac{1}{2})^2}$	$\prod_{j=1}^{n} (r^2 + (n-j)^2)$		n
$SU(n, 1)$ $n \geq 2$	$\dfrac{1}{2^{2n-2}\Gamma(n)^2 2}$	$\prod_{j=1}^{n-1} \left(\left(\dfrac{r}{2}\right)^2 + \dfrac{(n-2j)^2}{4}\right)$	$(-1)^{n+1}$	n
$Sp(n, 1)$ $n \geq 2$	$\dfrac{1}{2^{4n}\Gamma(2n)^2 2}$	see (2.6)		$2n+1$
$F_{4(-20)}$	$\dfrac{1}{2^{20}\Gamma(8)^2 2}$	see (2.7)		11

For $Sp(n, 1)$, $F_{4(-20)}$, $P(r)$ is given respectively by

(2.6) $$P(r)=\prod_{j=3}^{n+1}\left(\left(\frac{r}{2}\right)^2+\left(n-j+\frac{3}{2}\right)^2\right)\left(\left(\frac{r}{2}\right)^2+\left(n-j+\frac{5}{2}\right)^2\right)\left(\left(\frac{r}{2}\right)^2+\left(\frac{1}{2}\right)^2\right)$$

(2.7) $$P(r)=\left(\left(\frac{r}{2}\right)^2+\left(\frac{1}{2}\right)^2\right)^2\left(\left(\frac{r}{2}\right)^2+\left(\frac{3}{2}\right)^2\right)^2\left(\left(\frac{r}{2}\right)^2+\left(\frac{5}{2}\right)^2\right)\times \left(\left(\frac{r}{2}\right)^2+\left(\frac{7}{2}\right)^2\right)\left(\left(\frac{r}{2}\right)^2+\left(\frac{9}{2}\right)^2\right)$$

Thus for $G \neq SO_1(2n+1, 1)$, $P(r)$ is an even polynomial of degree $d-2$ where $d = \dim G/K$. In these cases we write

(2.8) $$P(r) = a_0 + a_2 r^2 + a_4 r^4 + \cdots + a_{2(d/2-1)} r^{2(d/2-1)}$$

For $G = SO_1(2n+1, 1)$, $P(r)$ is also an even polynomial but of degree $d-1 = 2n$ which we write as

(2.9) $$P(r) = a_0 + a_2 r^2 + a_4 r^4 + \cdots + a_{2n} r^{2n}$$

Note that the normalization of Haar measures in [13] differs from that given in (2.3).

3. The zeta functions Z_α, $Z_{\alpha,b}$

From now on Γ will denote a discrete torsion free co-compact subgroup of G. Let $\hat{G}$ be the unitary dual space of G—the set of equivalence classes of irreducible unitary representations (π, H_π) of G where H_π is the Hilbert space of π. π is called class 1 if $\pi|_K$ contains the trivial representation of K. That is, there is a $\pi(K)$-fixed unit v in H_π. The latter gives rise to the corresponding positive definite spherical function of ϕ_π which in fact determines π:

(3.1) $\phi_\pi(x) = \langle v, \pi(x)v \rangle$ for $x \in G$ where $\langle , \rangle$ is the inner product on H_π.

We let $\{\pi_j\}_{j\geq 0} \subset \hat{G}$ be a representative set of all the class 1 representations of G which occur as subrepresentations of the right regular representation of G on $L^2(\Gamma\backslash G)$ (ie. where G acts by right translation). This L^2-space is formed with respect to the measure m_Γ in (2.4). Let n_j be the multiplicity $m_{\pi_j}(\Gamma)$ with which π_j occurs in $L^2(\Gamma\backslash G)$. One knows that each n_j is finite [18]. We arrange the labeling so that $\pi_0 = 1$, the trivial representation of G; then $n_0 = 1$. As a spherical function each ϕ_{π_j} has the form $\phi_{\pi_j} = \phi_{\nu_j}$ for some ν_j in the complexification $\mathfrak{a}_\mathfrak{p}^{*C}$ of $\mathfrak{a}_\mathfrak{p}^*$, by a theorem of Harish-Chandra [11], where for any $\nu \in \mathfrak{a}_\mathfrak{p}^{*C}$,

(3.2) $$\phi_\nu(x) \overset{\text{def}}{=} \int_K e^{(i\nu-\rho)(H(xk))}\, dk$$

for $x \in G$. If M, M' are the centralizer, normalizer of $A_{\mathfrak{p}}$ in K, respectively, so that $W = M'/M$ is the Weyl group of $(\mathfrak{g}_0, \mathfrak{a}_{\mathfrak{p}})$ then the v_j are determined up to the action of W. For the sake of specificity we normalize the choice of the v_j by

$$(3.3)\qquad \begin{aligned} &v_j(H_0) \geq 0 && \textit{if } v_j(H_0) \in R \\ &iv_j(H_0) < 0 && \textit{if } v_j(H_0) \in iR - \{0\}\,. \end{aligned}$$

Then $v_0 = i\rho - ie.$ $\phi_{i\rho} = 1$. We set

$$(3.4)\qquad \lambda_j = \rho_0^2 + v_j(H_0)^2$$

Relative to a suitable Riemannian metric on G/K (and thus on X_Γ) one may regard the λ_j as the spectrum $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ of $-\Delta$ on X_Γ, where Δ is the Laplace-Beltrami operator. Then n_j is the multiplicity of the eigenvalue λ_j on $C^\infty(X_\Gamma)$. Note that for $G = PSL(2, R)$, $\rho_0^2 = \frac{1}{4}$ and the $v_j(H_0)^2$ correspond to the r_j^2 above; compare the remarks accompaning (1.1). Given $\alpha > 0$ we therefore define Z_α by

$$(3.5)\qquad Z_\alpha(s) = \sum\nolimits_{j, r_j > 0} \frac{n_j \sin \alpha r_j}{r_j^s}$$

for $s \in C$ with Re s sufficiently large where we set $r_j \overset{\text{def}}{=} v_j(H_0)$. More generally for $b \geq 0$ we set

$$(3.6)\qquad Z_{\alpha,b}(s) = \sum\nolimits_{j, r_j > 0} \frac{r_j n_j \sin \alpha r_j}{(b + r_j^2)^{(s+1)/2}}$$

Thus $Z_{\alpha,0} = Z_\alpha$.

THEOREM 3.7. *Let $b \geq 0$, $\sigma \in R$. Then $\sum_{j, r_j > 0} \frac{n_j r_j}{(b + r_j^2)^{(\sigma+1)/2}}$ converges for $\sigma > d \overset{\text{def.}}{=} \dim\ G/K$. In particular $Z_{\alpha,b}(s)$ in* (3.6) *converges absolutely for* Re $s > d$.

To prove this we use

THEOREM 3.8 [8]. $\sum_{j \geq 0} \frac{n_j}{[1 + r_j^2 + \rho_0^2]^\sigma}$ *converges for* $\sigma > \frac{d}{2}$.

PROOF OF THEOREM 3.7. Take $\sigma > d$ and $r_j > 0$. Then $\frac{r_j(1 + r_j^2 + \rho_0^2)^{\sigma/2}}{(b + r_j^2)^{(\sigma+1)/2}} \leq \frac{r_j(1 + r_j^2 + \rho_0^2)^{\sigma/2}}{(r_j^2)^{(\sigma+1)/2}} = \frac{(1 + r_j^2 + \rho_0^2)^{\sigma/2}}{(r_j^2)^{\sigma/2}} = (1 + (1 + \rho_0^2) r_j^{-2})^\sigma \to 1$ as $j \to \infty$. Thus for $j >$ some j_0 sufficiently large $\frac{r_j(1 + r_j^2 + \rho_0^2)^{\sigma/2}}{(b + r_j^2)^{(\sigma+1)/2}} < 2 \Rightarrow \frac{n_j r_j}{(b + r_j^2)^{(\sigma+1)/2}} < \frac{2n_j}{(1 + r_j^2 + \rho_0^2)^{\sigma/2}}$ for $j > j_0$, so Theorem 3.8 $\Rightarrow$ Theorem 3.7.

One knows that only finitely many of the r_j satisfy $r_j^2 < 0$; recall that $v_0 = i\rho$ so that $r_0 = i\rho_0 \Rightarrow r_0^2 < 0$. We assume that $r_0, r_1, \ldots, r_l$ only satisfy $r_j^2 < 0$; $r_0^2 < r_1^2 < \cdots < r_l^2 < r_{l+1}^2 < \cdots$, $r_j^2 \to \infty$ in accordance with $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$

In sections 5, 6 we shall study the analytic continuation of the $Z_{\alpha,b}$, using the Selberg trace formula. To state this formula, in a form convenient for our purpose, we first introduce additional notation. Let $A_{\mathfrak{p}}^+ = \exp\{tH_0 | t > 0\}$. As Γ is torsion free and co-compact any $\gamma \in \Gamma - \{1\}$ is conjugate in G to an element of $MA_{\mathfrak{p}}^+$ (using that γ is semisimple and acts freely on G/K [15], and that as $\dim \mathfrak{a}_{\mathfrak{p}} = 1$, G has at most 2 Cartan subgroups, up to conjugacy). Thus we can choose $x \in G$ such that $x\gamma x^{-1} = m_\gamma(x) \exp t_\gamma(x) H_0$, where $m_\gamma(x) \in M$, $t_\gamma(x) > 0$. By Lemma 6.6 of [16], $t_\gamma(x)$ is independent of the particular choice x in G, and up to conjugation in M so is $m_\gamma(x)$. We therefore write $t_\gamma = t_\gamma(x)$, $m_\gamma = m_\gamma(x)$. $\delta \in \Gamma - \{1\}$ is called primitive if it cannot be written in the form γ_1^j for some γ_1 in Γ and j some integer > 1. According to [7] each $\gamma \in \Gamma - \{1\}$ can be written $\gamma = \delta^{j(\gamma)}$ for a unique primitive element δ in $\Gamma - \{1\}$ and a unique positive integer $j(\gamma)$. Let C_Γ be a complete set of representatives in Γ of its conjugacy classes, and let

$$C(\gamma)^{-1} = e^{t_\gamma \rho_0} |\det_{\mathfrak{n}_0}(\mathrm{Ad}(m_\gamma \exp t_\gamma H_0)^{-1} - 1)| \tag{3.9}$$

for $\gamma \in \Gamma - \{1\}$. Given the normalization of measures in (2.3) the trace formula can be stated as follows [7], [8], [16], [18]

$$\sum_{j \geq 0} n_j F^*(v_j(H_0)) = \frac{vol(\Gamma \backslash G)}{4\pi} \int_R F^*(r) |c(r)|^{-2} \, dr + \sum_{\gamma \in C_\Gamma - \{1\}} t_\gamma j(\gamma)^{-1} C(\gamma) F(t_\gamma) \tag{3.10}$$

where F^* is an even, holomorphic function of suitable growth at infinity and

$$F(u) = \frac{1}{2\pi} \int_R F^*(r) e^{-iru} \, dr \tag{3.11}$$

(3.10) holds in particular for all F^* which arise as the spherical Fourier transform of a K-biinvariant function in the Harish-Chandra-Schwartz space $\mathscr{C}_1(G)$ [18]. Such a function is $F^*: r \to re^{-(\alpha^2 + r^2)x} \sin \alpha r$ where x, $a > 0$ are fixed with α real.

4. Some integral formulas

In addition to the trace formula the analytic continuation of the $Z_{\alpha,b}$ will be based on some integral formulas. It seems convenient to consider these now as an effort to maintain the flow of ideas of the next section. Let α, a, $b > 0$ and let $n = 0, 1, 2, 3, \ldots$, be a non-negative integer. Since

$r \to \dfrac{r^n}{(b+r^2)^s} \in L^1(R)$ for Re $s > \dfrac{n+1}{2}$, the functions $I_n = I_{n,\alpha,a,b}$ given by

$$I_n(s) = \int_R \frac{r^{2n}(\sin \alpha r) \tanh ar\, dr}{(b+r^2)^s} \tag{4.1}$$

are well-defined for Re $s > \dfrac{2n+1}{2} = n + \dfrac{1}{2}$.

We study the integral $I_0(s)$. Write

$$\tanh ar = \frac{e^{ar} - e^{-ar}}{e^{ar} + e^{-ar}} = \frac{e^{2ar} - 1}{e^{2ar} + 1} = 1 - \frac{2}{e^{2ar} + 1}$$

to obtain $I_0(s) = 2\displaystyle\int_0^\infty \frac{(\sin \alpha r) \tanh ar\, dr}{(b+r^2)^s} =$

$$2\int_0^\infty \frac{\sin \alpha r}{(b+r^2)^s} dr - 4\int_0^\infty \frac{\sin \alpha r\, dr}{(e^{2ar}+1)(b+r^2)^s} \tag{4.2}$$

The modified Struve Functions L_ν and Bessel functions of an imaginary argument I_ν are defined by

$$L_\nu(z) = \sum_{m=0}^\infty \frac{(z/2)^{2m+\nu+1}}{\Gamma(m+3/2)\Gamma(\nu+m+3/2)} \tag{4.3}$$

$$I_\nu(z) = e^{-\pi/2\nu i} J_\nu(e^{\pi/2 i} z) \qquad -\pi < \arg z \le \frac{\pi}{2} \tag{4.4}$$

where

$$J_\nu(z) = \frac{z^\nu}{2^\nu} \sum_{m=0}^\infty (-1)^m \frac{z^{2m}}{2^{2m} m!\, \Gamma(\nu+m+1)} \qquad |\arg z| < \pi\,. \tag{4.5}$$

That is, the J_ν are Bessel functions of the first kind. From page 426 of [9]

$$\int_0^\infty (\beta^2 + r^2)^{\nu - (1/2)} \sin \alpha r\, dr = \frac{\sqrt{\pi}}{2} \left(\frac{2\beta}{\alpha}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right)[I_{-\nu}(\alpha\beta) - L_\nu(\alpha\beta)] \tag{4.6}$$

for $\alpha > 0$, Re $\beta > 0$, Re $\nu < \frac{1}{2}$, $\nu \neq -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$. Therefore by (4.2)

$$\begin{aligned} I_0(s) = {} & -4\int_0^\infty \frac{\sin \alpha r\, dr}{(e^{2ar}+1)(b+r^2)^s} \\ & + \sqrt{\pi}\left(\frac{2\sqrt{b}}{\alpha}\right)^{(1/2)-s} \Gamma(1-s)[I_{-((1/2)-s)}(\alpha\sqrt{b}) - L_{(1/2)-s}(\alpha\sqrt{b})] \end{aligned} \tag{4.7}$$

for $s \neq 1, 2, 3, 4, \ldots$, Re $s > \frac{1}{2}$.

If H_ν are the Struve functions, ie.

$$H_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{2m+\nu+1}}{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(\nu+m+\frac{3}{2}\right)} \tag{4.8}$$

then from page 38 of [3] $\left(\frac{z}{2}\right)^{-\nu} H_\nu(z) = \frac{z}{\sqrt{\pi}}\,{}_1F_2\left(1; \frac{3}{2}+\nu, \frac{3}{2}; -\frac{z^2}{4}\right)\Big/\Gamma\left(\nu+\frac{3}{2}\right)$ is an entire function of z and of ν (where ${}_1F_2$ is a generalized hypergeometric series). Replace z by iz to obtain in particular that $\nu \to \frac{iz}{\sqrt{\pi}}\,{}_1F_2\left(1; \frac{3}{2}+\nu, \frac{3}{2}; \frac{z^2}{4}\right)\Big/\Gamma\left(\nu+\frac{3}{2}\right)$ is an entire function Ψ_z of ν. But $\psi_z(\nu) = i\left(\frac{z}{2}\right)^{-\nu} L_\nu(z)$ as $L_\nu(z) = -ie^{-i\nu\pi/2} H_\nu(ze^{i\pi/2})$. Thus we see that in particular $\nu \to L_\nu(\alpha\sqrt{b})$ is an entire function; i.e., in (4.11) $s \to L_{1/2-s}(\alpha\sqrt{b})$ is an entire function. Similarly $s \to I_{-(1/2-s)}(\alpha\sqrt{b})$ is an entire function since in fact $\nu \to J_\nu(z)$ is entire. Now $s \to \Gamma(1-s)$ is meromorphic with simple poles at $s = 1, 2, 3, \ldots$, and the residue at $1+k$ is $-(-1)^k/k!$ for $k = 0, 1, 2, \ldots$ from the identity $L_{-(k+1/2)}(z) = I_{k+1/2}(z)$, $k = 0, 1, 2, \ldots$, page 39 of [3], we see that

$$\lim_{s\to 1+k} [s-(1+k)]\Gamma(1-s)[I_{-((1/2)-s)}(\alpha\sqrt{b}) - L_{(1/2)-s}(\alpha\sqrt{b})] = \frac{(-1)^k}{k!}0 = 0$$

and we therefore conclude that each of the points $s = 1, 2, 3, \ldots$ is a removable singularity of $s \to \Gamma(1-s)[I_{-((1/2)-s)}(\alpha\sqrt{b}) - L_{(1/2)-s}(\alpha\sqrt{b})]$ is entire; ie.

PROPOSITION 4.9. *In* (4.7) *the function* $s \to \Gamma(1-s)[I_{-((1/2)-s)}(\alpha\sqrt{b}) - L_{(1/2)-s}(\alpha\sqrt{b})]$ *is entire.*

On the other hand it is easy to check that $\int_1^\infty \frac{\sin \alpha r\, dr}{(e^{2ar}+1)(b+r^2)^s}$ converges uniformly on compact subsets of the plane and thus is an entire function of s. $\int_0^1 \frac{\sin \alpha r\, dr}{(e^{2ar}+1)(b+r^2)^s}$ is also an entire function of s. Given Proposition 4.9 we therefore have

THEOREM 4.10. *The right hand side of equation* (4.7) *defines an analytic continuation of* I_0 *as an entire function.*

We should observe in general that the I_n are holomorphic functions on $\operatorname{Re} s > n + \frac{1}{2}$. Namely $I_n(s) = 2\int_0^1 \frac{r^{2n}(\sin \alpha r)\tanh ar\, dr}{(b+r^2)^s} + 2\int_1^\infty \frac{r^{2n}(\sin \alpha r)\tanh ar\, dr}{(b+r^2)^s}$, where the 1st integral is an entire function of s and 2nd one converges uniformly on compact subsets of $\operatorname{Re} s > n + \frac{1}{2}$.

Let $\Delta_n(s) = \int_0^\infty \frac{r^{2n}(\sin \alpha r)\, dr}{(b + r^2)^s}$ for $\operatorname{Re} s > n + \frac{1}{2}$. Then as in (4.2) $I_n(s) = 2\Delta_n(s) - 4\int_0^\infty \frac{r^{2n}(\sin \alpha r)\, dr}{(e^{2ar} + 1)(b + r^2)^s}$ where the latter integral is entire in s. Now $\frac{r^{2n}(\sin \alpha r)}{(b+r^2)^s} = r^{2(n-1)}\frac{(r^2+b)\sin \alpha r}{(r^2+b)^s} - b\frac{r^{2(n-1)}\sin \alpha r}{(b+r^2)^s} = \frac{r^{2(n-1)}\sin \alpha r}{(b+r^2)^{s-1}} - b\frac{r^{2(n-1)}\sin \alpha r}{(b+r^2)^s} \Rightarrow$ $\Delta_n(s) \overset{\#}{=} \Delta_{n-1}(s-1) - b\Delta_{n-1}(s)$. We have observed that Δ_0 extends to an entire function, by (4.2), (4.7) and Proposition 4.9. By #, inductively, each Δ_n extends to an entire function and thus each I_n extends to an entire function; ie.

THEOREM 4.11. *The functions $I_n = I_{n,\alpha,a,b}$ defined in* (4.1) *are holomorphic on* $\operatorname{Re} s > n + \frac{1}{2}$ *and extend to entire functions.*

Similar to the definition of I_n in (4.1) we define $K_n = K_{n,\alpha,a,b}$ for α, a, $b > 0$, $n = 0, 1, 2, 3, \ldots$, by

$$K_n(s) = \int_R \frac{r^{2n}(\sin \alpha r) \coth ar\, dr}{(b + r^2)^s} \tag{4.12}$$

For $\operatorname{Re} s > n + \frac{1}{2}$. Using that $\coth x - \tanh x = (\tanh x)\operatorname{csch}^2 x$ we get

$$K_n(s) - I_n(s) = \int_R \frac{r^{2n}(\sin \alpha r)(\tanh ar)\operatorname{csch}^2 ar}{(b + r^2)^s} dr \tag{4.13}$$

for $\operatorname{Re} s > n + \frac{1}{2}$ where the integral in (4.13) is an entire function of s, as $r \to r^{2n} \operatorname{csch}^2 ar$ has exponential decay at ∞. Because of Theorem 4.11 we may conclude

THEOREM 4.14. *For $n \geq 1$ the function $s \to K_n(s)$, which is holomorphic on* $\operatorname{Re} s > n + \frac{1}{2}$, *extends to an entire function.*

For $n = 0, 1, 2, 3, \ldots$, α, $b > 0$ define $S_n = S_{n,\alpha,b}$ by

$$S_n(s) = \int_R \frac{r^{2n+1}(\sin \alpha r)\, dr}{(b + r^2)^s} \tag{4.15}$$

for $\operatorname{Re} s > n + 1$. Similar to the argument which led to equation # preceding Theorem 4.11 we have $\frac{r^{2n+1}(\sin \alpha r)}{(b + r^2)^s} = \frac{r^{2(n-1)+1}\sin \alpha r}{(r^2 + b)^{s-1}} - \frac{br^{2(n-1)+1}\sin \alpha r}{(b + r^2)^s} \Rightarrow$ $S_n(s) = S_{n-1}(s-1) - bS_{n-1}(s)$. By induction (again) each S_n, $n \geq 1$, will extend to an entire if only S_0 does. By page 427 of [9], $S_0(s)/2 =$

$$(4.16)\quad \int_0^\infty \frac{r(\sin \alpha r)\,dr}{(b+r^2)^s} = \sqrt{\frac{b}{\pi}}\left(\frac{2\sqrt{b}}{\alpha}\right)^{-s+(1/2)}\left[\cos \pi\left(\frac{1}{2}-s\right)\right]\Gamma(1-s)K_{-s+3/2}(\alpha\sqrt{b})$$

for $\operatorname{Re} s > 1$, $s \neq 1, 2, 3, \ldots$, where K_ν is the K-Bessel function: For $\nu, z \in \boldsymbol{C}$

$$(4.17)\qquad K_\nu(z) = \frac{1}{2}\int_0^\infty e^{-z/2(t+1/t)}t^{-\nu-1}\,dt\,.$$

$s \to K_s(\alpha\sqrt{b})$ is entire in s and $s \to \cos \pi(\frac{1}{2} - s)$ vanishes at the poles $s = 1$, $2, 3, \ldots$, of $s \to \Gamma(1 - s)$. That is, $s \to [\cos \pi(\frac{1}{2} - s)]\Gamma(1 - s)$ is entire ($s = 1$, $2, 3, \ldots$ are removable singularities) and thus by (4.16) S_0 extends to an entire function. That is

PROPOSITION 4.18. *The holomorphic function $S_{n,\alpha,b}$ defined in* (4.15) *extends to an entire function.*

For application of the trace formula, (3.10) we shall need the Fourier transform of the function $r \to e^{-r^2x}r \sin \alpha r$. Namely

PROPOSITION 4.19. *For $u \in R$, x, $\alpha > 0$,*

$$\int_R e^{-iru}e^{-r^2x}r \sin \alpha r\,dr = \frac{\sqrt{\pi}(\alpha - u)e^{-(u-\alpha)^2/4x}}{4x^{3/2}} + \frac{\sqrt{\pi}(\alpha + u)e^{-(u+\alpha)^2/4x}}{4x^{3/2}}$$

PROOF. We assume the known formula (4.20) $\int_R e^{-irc}e^{-r^2x}\,dr = \sqrt{\frac{\pi}{x}}\; e^{-c^2/4x}$ for the Fourier transform of $r \to e^{-r^2x}$, $x > 0$; $c \in R$. Let $I(u) = \int_R e^{-iru}e^{-r^2x}r \sin \alpha r\,dr$, $H(u) = \int_R e^{-iru}e^{-r^2x} \cos \alpha r\,dr$, $J(u) = \int_R e^{-iru}e^{-r^2x} \sin \alpha r\,dr$ for $u \in R$. Write the integrand of $I(u)$ as $f(r)g'(r)$ where $f(r) = e^{-iru} \sin \alpha r$, $g'(r) = e^{-r^2x}r$. Integrating by parts one therefore obtains $I(u) \overset{(i)}{=} \frac{\alpha}{2x}H(u) - \frac{iu}{2x}J(u)$. On the other hand one can write $2 \cos \alpha r = e^{\alpha ri} + e^{-\alpha ri}$, $2i \sin \alpha r = e^{\alpha ri} - e^{-\alpha ri}$ and use (4.20) to obtain

$$(4.21)\qquad \begin{aligned} H(u) &= \frac{1}{2}\sqrt{\frac{\pi}{x}}e^{-(u-\alpha)^2/4x} + \frac{1}{2}\sqrt{\frac{\pi}{x}}e^{-(u+\alpha)^2/4x} \\ J(u) &= \frac{1}{2i}\sqrt{\frac{\pi}{x}}e^{-(u-\alpha)^2/4x} - \frac{1}{2i}\sqrt{\frac{\pi}{x}}e^{-(u+\alpha)^2/4x} \end{aligned}$$

Then Proposition 4.19 follows from (i).

For $t \in R$, α, $x > 0$, $k = 0, 1, 2, 3, \ldots$, define

$$F_k(t) = \int_0^\infty e^{-r^2} r^{2k} \sin tr\, dr$$

(4.22)
$$I_k(x;\alpha) = \int_0^\infty e^{r^2 x} r^{2k} (\sin \alpha r)\, dr$$

$$J_k(x;\alpha) = \int_0^\infty e^{-r^2 x} r^{2k+1} (\cos \alpha r)\, dr\,.$$

The integrand of the second integral is $f(r)g'(r)$ for $f(r) = r^{2k-1}(\sin \alpha r)$, $g'(r) = re^{-r^2 x}$ so that integration by parts gives

(4.23)
$$I_k(x;\alpha) = \frac{2k-1}{2x} I_{k-1}(x;\alpha) + \frac{\alpha}{2x} J_{k-1}(x;\alpha) \qquad \text{for} \quad k \geq 1.$$

The change of variables $r \to r\sqrt{x}$ also provides the relations $I_k(x;\alpha) = x^{-k-1/2} F_k\left(\frac{\alpha}{\sqrt{x}}\right)$,

(4.24)
$$J_k(x;\alpha) = x^{-k-1} \int_0^\infty e^{-r^2} r^{2k+1} \left(\cos \frac{\alpha r}{\sqrt{x}}\right) dr.$$

We define $\phi_k(s;\alpha)$ by

(4.25)
$$\phi_k(s;\alpha) = \int_0^1 x^{s-1} I_k(x;\alpha)\, dx \qquad \text{for} \quad s \in \boldsymbol{C},$$

Re s sufficiently large. Namely, using

(4.24) $\phi_k(s;\alpha) = \int_1^\infty x^{-s-1} I_k\left(\frac{1}{x};\alpha\right) dx = \int_1^\infty \frac{F_k(\alpha\sqrt{x})}{x^{s-k+1/2}}\, dx.$ Since F_k is clearly bounded we see therefore that $\phi_k(s;\alpha)$ is defined if and only if $\operatorname{Re} s > k + \frac{1}{2}$ and moreover we see that $\phi_k(;\alpha)$ is holomorphic on $\operatorname{Re} s > k + \frac{1}{2}$, by uniform convergence of the integral on compact subsets of the latter domain.

PROPOSITION 4.26. *For $k \geq 1$, $\operatorname{Re} s > k + \frac{1}{2}$, $\phi_k(s;\alpha) = -F_{k-1}(\alpha) + (s-1)\int_0^1 x^{s-2} I_{k-1}(x;\alpha)\, dx$.*

PROOF. By (4.23), $\phi_k(s;\alpha) = \left(k - \frac{1}{2}\right) \int_0^1 x^{s-2} I_{k-1}(x;\alpha)\, dx + \frac{\alpha}{2} \int_0^1 x^{s-2} J_{k-1}(x;\alpha)\, dx$, for $k \geq 1$. Let $\Psi(s;\alpha) = \frac{\alpha}{2} \int_0^1 x^{s-2} J_{k-1}(x;\alpha)\, dx$ be the 2nd integral. By (4.24) and the preceding argument $\Psi(s;\alpha)$ is well defined for $\operatorname{Re} s > k+1$, which we assume. Note that $\Psi(s;\alpha)=\int_0^1 f_1(x)g_1'(x)\,dx$ for $f_1(x)=-x^{s-k-1/2}$, $g_1(x)=F_{k-1}\left(\frac{\alpha}{\sqrt{x}}\right)$. Since F_{k-1} is bounded and $\operatorname{Re} s>k+\frac{1}{2}$ one has $f_1(x)g_1(x)|_0^1=-F_{k-1}(\alpha)$. Thus integration by parts yields $\Psi(s;\alpha)=-F_{k-1}(\alpha)+\left(s-k-\frac{1}{2}\right)\times\int_0^1 x^{s-k-3/2}F_{k-1}\left(\frac{\alpha}{\sqrt{x}}\right)dx=-F_{k-1}(\alpha)+\left(s-k-\frac{1}{2}\right)\int_0^1 x^{s-2}I_{k-1}(x;\alpha)\,dx$, by (4.24) again. Therefore $\phi_k(s;\alpha)\overset{\#}{=}-F_{k-1}(\alpha)+(s-1)\int_0^1 x^{s-2}I_{k-1}(x;\alpha)\,dx$ for $\operatorname{Re} s>k+1$, where the r.h.s. is $-F_{k-1}(\alpha)+(s-1)\phi_{k-1}(s-1,\alpha)$ by (4.25). On the other hand we have seen that both sides of equation # are holomorphic on $\operatorname{Re} s>k+\frac{1}{2}$. Therefore # holds for $\operatorname{Re} s>k+\frac{1}{2}$, as desired.

On page 172 of [5], Fujii defines the sum of two integrals I_{16}, I_{17} by $6(I_{16}+I_{17})=\int_0^1 x^{s-1}\left(\int_0^1+\int_1^\infty\right)e^{-r^2x}r^2(\sin\alpha r)\,dr\,dx$. By our notation $6(I_{16}+I_{17})\equiv\phi_1(s;\alpha)$. $I_{16}/\Gamma(\cdot)$ extends to an entire function and Fujii shows that $I_{17}/\Gamma(\cdot)$ extends to an entire function. That is $\phi_1(;\alpha)/\Gamma(\cdot)$ extends to an entire function. Inductively we have $\square$

PROPOSITION 4.27. *For $k\geq 1$, $\phi_k(;\alpha)/\Gamma(\cdot)$ extends to an entire function.*

PROOF. We have observed the result to true for $k=1$. By Proposition 4.26.

$$\frac{\phi_k(s;\alpha)}{\Gamma(s)}=-\frac{F_{k-1}(\alpha)}{\Gamma(s)}+\frac{s-1}{\Gamma(s)}\phi_{k-1}(s-1;\alpha)$$

for $\operatorname{Re} s>k+\frac{1}{2}$. The induction is completed by this equation as $\Gamma(s)=(s-1)\Gamma(s-1)$. $\square$

5. Analytic continuation of $Z_{\alpha,b}$ $b\neq 0$.

For $x,\alpha>0$ define F^* by $F^*(r)=re^{-(\rho_0^2+r^2)x}\sin\alpha r$, $r\in C$. We have observed that F^* plugs into the trace formula. Moreover $F(u)\overset{\text{def}}{=}\frac{1}{2\pi}\int_R F^*(r)e^{-iru}\,dr=\frac{\sqrt{\pi}e^{-\rho_0^2x}}{2\pi 4x^{3/2}}(\alpha-u)e^{-(u-\alpha)^2/4x}+\frac{\sqrt{\pi}e^{-\rho_0^2x}}{2\pi 4x^{3/2}}(\alpha+u)e^{-(u+\alpha)^2/4x}$ for $u\in R$ by Proposition 4.19. The trace formula (3.10) therefore provides

$$\text{(5.1)}\qquad \sum\nolimits_{j\geq 0}n_jr_je^{-(\rho_0^2+r_j^2)x}\sin\alpha r_j$$
$$=\frac{vol(\Gamma\backslash G)}{4\pi}\int_R re^{-(\rho_0^2+r^2)x}(\sin\alpha r)|c(r)|^{-2}\,dr$$

$$+\sum_{\gamma\in C_\Gamma-\{1\}} \frac{e^{-\rho_0^2 x}}{8\sqrt{\pi}x^{3/2}} t_\gamma j(\gamma)^{-1}C(\gamma)[(\alpha-t_\gamma)e^{-(t_\gamma-\alpha)^2/4x}$$
$$+ (\alpha + t_\gamma)e^{-(t_\gamma+\alpha)^2/4x}]$$

Multiply both sides of (5.1) by $e^{\rho_0^2 x}e^{-bx}$ for $b \geq 0$ to obtain

(5.2) $$\sum_{j\geq 0} n_j r_j e^{-(b+r_j^2)x} \sin \alpha r_j$$
$$=\frac{vol(\Gamma\backslash G)}{4\pi}\int_R re^{-(b+r^2)x}(\sin \alpha r)|c(r)|^{-2}\,dr$$
$$+\frac{e^{-bx}}{8\sqrt{\pi}x^{3/2}}\sum_{\gamma\in C_\Gamma-\{1\}} t_\gamma j(\gamma)^{-1}C(\gamma)[(\alpha-t_\gamma)e^{-(t_\gamma-\alpha)^2/4x}$$
$$+ (\alpha + t_\gamma)e^{-(t_\gamma+\alpha)^2/4x}]$$

Consider the sum on the l.h.s. of (5.2). As $r_0 = i\rho_0$ and $n_0 = 1$ the summand corresponding to $j = 0$ is $i\rho_0 e^{-(b-\rho_0^2)x} \sin \alpha i\rho_0 = e^{-(b-\rho_0^2)x}(\rho_0/2)[e^{-\alpha\rho_0} - e^{\alpha\rho_0}]$. Similarly, by earlier notation, we may have $r_1, r_2, \ldots, r_l \in iR - \{0\}$, say $r_j = it_j$ with $t_j > 0$ by (3.3). Then $[n_j r_j e^{-(b+r_j^2)x} \sin \alpha r_j = n_j e^{-(b-t_j^2)x}\frac{t_j}{2}[e^{-\alpha t_j} - e^{\alpha t_j}]$, $1 \leq j \leq l$. If $r_j \in R$ then $r_j \geq 0$ by (3.3). Thus we can write (5.2) as

(5.3) $$2\sum_{j,r_j>0} n_j r_j e^{-(b+r_j^2)x} \sin \alpha r_j$$
$$=\sum_{j=0}^{l} n_j e^{-(b-t_j^2)x} t_j[e^{\alpha t_j}-e^{-\alpha t_j}]+\frac{vol(\Gamma\backslash G)}{2\pi}\int_R re^{-(b+r^2)x}(\sin \alpha r)|c(r)|^{-2}\,dr$$
$$+\frac{e^{-bx}}{4\sqrt{\pi}x^{3/2}}\sum_{\gamma\in C_\Gamma-\{1\}} t_\gamma j(\gamma)^{-1}C(\gamma)[(\alpha-t_\gamma)e^{-(t_\gamma-\alpha)^2/4x}$$
$$+ (\alpha + t_\gamma)e^{-(t_\gamma+\alpha)^2/4x}]$$

where we write $t_0 = \rho_0$; $n_0 = 1$. We note also that

$$j > 0 \Rightarrow \lambda_j \overset{\text{def}}{=} r_j^2 + \rho_0^2 > 0 \Rightarrow t_j < \rho_0 .$$

Consider

$$I(s) = \int_0^\infty x^{s-1}\sum_{j,r_j>0} n_j r_j e^{-(b+r_j^2)x} \sin \alpha r_j\,dx .$$

For $\sigma = \mathrm{Re}\, s$, $r_j > 0$

$$|x^{s-1}n_j r_j e^{-(b+r_j^2)x} \sin \alpha r_j| \leq x^{\sigma-1}n_j r_j e^{-(b+r_j^2)x}$$

where

$$\sum_{j,r_j>0}\int_0^\infty x^{\sigma-1}n_jr_je^{-(b+r_j^2)x}\,dx=\sum_{j,r_j>0}\frac{\Gamma(\sigma)n_jr_j}{(b+r_j^2)^\sigma}$$

$$=\Gamma(\sigma)\sum_{j,r_j>0}\frac{n_jr_j}{(b+r_j^2)^{((2\sigma-1)+1)/2}}<\infty$$

for $2\sigma-1>d$ by Theorem 3.7. Hence by Fubini's theorem

$$I(s)=\Gamma(s)\sum_{j,r_j>0}\frac{n_jr_j}{(b+r_j^2)^{((2s-1)+1)/2}}=\Gamma(s)Z_{\alpha,b}(2s-1) \tag{5.4}$$

for $\operatorname{Re} s>\dfrac{d+1}{2}$. Let

(5.5)

$$\theta_0(x)=\sum_{j=0}^{l}n_je^{-(b-t_j^2)x}t_j[e^{\alpha t_j}-e^{-\alpha t_j}]$$

$$\theta_1(x)=\frac{vol(\Gamma\backslash G)}{2\pi}\int_R re^{-(b+r^2)x}(\sin\alpha r)|c(r)|^{-2}\,dr$$

$$\theta_2(x)=\frac{e^{-bx}}{4\sqrt{\pi}x^{3/2}}\sum_{\gamma\in C_\Gamma-\{1\}}t_\gamma j(\gamma)^{-1}C(\gamma)[(\alpha-t_\gamma)e^{-(t_\gamma-\alpha)^2/4x}+(\alpha+t_\gamma)e^{-(t_\gamma+\alpha)^2/4x}]$$

LEMMA 5.6. *Let j_0 be the smallest j for which $r_j>0$; thus $j_0\geq l+1$. There is a constant $B>0$ such that $\sum n_jr_je^{-r_j^2x}\leq Be^{-r_{j_0}^2x}$ for $x\geq 1$.*

PROOF. We adapt the proof of Lemma 4.23 of [17] to the present situation. Let $M(x)=e^{r_{j_0}^2x}\sum_{j\geq j_0+1}n_jr_je^{-r_j^2x}$ for $x>0$. For $j\geq j_0+1$, $r_j^2>r_{j_0}^2\Rightarrow e^{-(r_j^2-r_{j_0}^2)x}\leq e^{-(r_j^2-r_{j_0}^2)}$ for $x\geq 1$; ie. $M(x)=\sum_{j\geq j_0+1}n_jr_je^{-(r_j^2-r_{j_0}^2)x}\leq\sum_{j\geq j_0+1}n_jr_je^{-(r_j^2-r_{j_0}^2)}=M(1)$ for $x\geq 1$. We set $B=n_{j_0}r_{j_0}+M(1)$ and obtain $\sum_{j,r_j>0}n_jr_je^{-r_j^2x}=n_{j_0}r_{j_0}e^{-r_{j_0}^2x}+\sum_{j\geq j_0+1}n_jr_je^{-r_j^2x}=e^{-r_{j_0}^2x}(n_{j_0}r_{j_0}+M(x))\leq e^{-r_{j_0}^2x}B$ for $x\geq 1$.

COROLLARY 5.7. $\int_1^\infty x^{s-1}[\sum_{j,r_j>0}n_jr_je^{-(b+r_j^2)x}\sin\alpha r_j]\,dx$ converges uniformly on compact subsets of the plane and thus defines an entire function $I_{(1)}$ of s, for $b\geq 0$.

We have $I(s)=I_{(0)}(s)+I_{(1)}(s)$ where

$$I_{(0)}(s)\stackrel{\text{def}}{=}\int_0^1 x^{s-1}[\sum_{j,r_j>0}n_jr_je^{-(b+r_j^2)x}\sin\alpha r_j]\,dx\,.$$

Given Corollary 5.7. we focus our study on $I_{(0)}$. By (5.3) and (5.5)

(5.8) $$2\sum_{j,r_j>0} n_j r_j e^{-(b+r_j^2)x} \sin \alpha r_j = (\theta_0+\theta_1+\theta_2)(x)$$

so that

(5.9) $$2I_{(0)}(s) = \int_0^1 x^{s-1}(\theta_0+\theta_1+\theta_2)(x)\,dx$$

To study $\int_0^1 x^{s-1}\theta_1(x)\,dx$ we first consider $\int_0^\infty x^{s-1}\theta_1(x)\,dx$. Assume that $b>0$. Then

$$\begin{aligned}\int_R\int_0^\infty |x^{s-1}re^{-(b+r^2)x}(\sin \alpha r)|c(r)|^{-2}|\,dx\,dr &\le \int_R |r||c(r)|^{-2}\int_0^\infty x^{\mathrm{Re}\, s-1}e^{-(b+r^2)x}\,dx\,dr\\ &= \int_R |r||c(r)|^{-2}\frac{\Gamma(\mathrm{Re}\, s)}{(b+r^2)^{\mathrm{Re}\, s}}\,dr\\ &= \Gamma(\mathrm{Re}\, s)C_G\pi\int_R \frac{|r|rP(r)\tanh^\varepsilon ar\,dr}{(b+r^2)^{\mathrm{Re}\, s}}\end{aligned}$$

for $G\neq SO_1(2n+1,1)$ by (2.5) where $\varepsilon=\pm 1$, $a=\pi$ or $\frac{\pi}{2}$, and $P(r)$ is a polynomial of degree $d-2$, $d=\dim G/K$; cf. (2.8). Assume for now $G\neq SO_1(2n+1,1)$. We see that the latter integral is finite if $\mathrm{Re}\, s>\frac{d+1}{2}$. Therefore by Fubini's Theorem

$$\begin{aligned}\int_0^\infty x^{s-1}\theta_1(x)\,dx &= \frac{vol(\Gamma\backslash G)}{2\pi}\int_R\int_0^\infty x^{s-1}re^{-(b+r^2)x}(\sin \alpha r)|c(r)|^{-2}\,dx\,dr\\ &= \frac{vol(\Gamma\backslash G)}{2\pi}\int_R r(\sin \alpha r)|c(r)|^{-2}\int_0^\infty x^{s-1}e^{-(b+r^2)x}\,dx\,dr\\ &= \frac{vol(\Gamma\backslash G)}{2\pi}\Gamma(s)\int_R \frac{r(\sin \alpha r)|c(r)|^{-2}\,dr}{(b+r^2)^s}\\ &= \frac{vol(\Gamma\backslash G)}{2\pi}\Gamma(s)C_G\pi\int_R \frac{r^2P(r)\sin \alpha r\tanh^\varepsilon ar\,dr}{(b+r^2)^s}\\ &= \sum_{k=0}^{d/2-1} a_{2k}\frac{vol(\Gamma\backslash G)}{2}\Gamma(s)C_G\int_R \frac{r^{2(k+1)}(\sin \alpha r)\tanh^\varepsilon ar\,dr}{(b+r^2)^s}\end{aligned}$$

(by (2.8)). In case $\varepsilon=1$ we use (4.1) to write

$$\int_0^\infty x^{s-1}\theta_1(x)\,dx = \sum_{k=0}^{d/2-1} a_{2k}\frac{vol(\Gamma\backslash G)}{2}\Gamma(s)C_G I_{k+1,\alpha,a,b}(s)\,,$$

for

$$\operatorname{Re} s > \left(\frac{d}{2} - 1\right) + 1 + \frac{1}{2} = \frac{d+1}{2}$$

in which case $\operatorname{Re} s >$ each $(k+1) + \frac{1}{2}$, $0 \leq k \leq \left(\frac{d}{2} - 1\right)$.

By Theorem 4.11 we see that, in case $\varepsilon = 1$, $\frac{1}{\Gamma(s)} \int_0^\infty x^{s-1}\theta_1(x)\,dx$ extends to an entire function. On the other hand for $x \geq 1$, $e^{-(b+r^2)x} \leq e^{-bx}e^{-r^2}$ $\forall r \in R \Rightarrow$ $|\theta_1(x)| \leq \frac{vol(\Gamma\backslash G)e^{-br}}{2\pi} A$, by (5.5), where $A = \int_R re^{-r^2}|c(r)|^{-2}\,dr$. This means that $\int_1^\infty x^{s-1}\theta_1(x)\,dx$ converges uniformly on compact subsets of the plane; ie.

LEMMA 5.10. *$\int_1^\infty x^{s-1}\theta_1(x)\,dx$ is an entire function of s.*

As $\int_0^1 x^{s-1}\theta_1(x)\,dx = \int_0^\infty x^{s-1}\theta_1(x)\,dx - \int_1^\infty x^{s-1}\theta_1(x)\,dx$ we have therefore established that $\frac{1}{\Gamma(s)} \int_0^1 x^{s-1}\theta_1(x)\,dx$ as a function of s extends meromorphically to $\boldsymbol{C}$ (at least when $\varepsilon = 1$) with possibly simple poles at $s = 1, 2, \ldots, d$. In case $\varepsilon = -1$ we argue pretty much the same.

Namely

$$\begin{aligned}\int_R &\frac{r^{2(k+1)}(\sin \alpha r)\tanh^{-1} ar\,dr}{(b+r^2)^s} \\ &= K_{k+1,\alpha,a,b}(s) \Rightarrow \int_0^\infty x^{s-1}\theta_1(x)\,dx \\ &= \textstyle\sum_{k=0}^{d/2-1} a_{2k} \frac{vol(\Gamma\backslash G)}{2} \Gamma(s) C_G K_{k+1,\alpha,a,b}(s) \qquad \text{for} \quad \operatorname{Re} s > \frac{d+1}{2}\,.\end{aligned}$$

In place of Theorem 4.11 we now appeal to Theorem 4.14 to conclude that $\frac{1}{\Gamma(s)} \int_0^1 x^{s-1}\theta_1(x)\,dx$ still extends to an entire function.

The final case to consider in studying $\int_0^1 x^{s-1}\theta_1(x)\,dx$ is the case $G = SO_1(2n+1, 1)$ (or G locally isomorphic to $SO_1(2n+1, 1)$). Then by (2.5), (2.9)

$$|c(r)|^{-2} = C_G \pi \textstyle\sum_{j=0}^{n} a_{2j} r^{2j}\,,$$

with

$$(5.11)\quad d-1=2n, \Rightarrow \int_0^\infty x^{s-1}\theta_1(x)\,dx \overset{\text{again}}{=} \frac{vol(\Gamma\backslash G)}{2\pi}\Gamma(s)\int_R \frac{r(\sin\alpha r)|c(r)|^{-2}\,dr}{(b+r^2)^s}$$

$$= C_G\frac{vol(\Gamma\backslash G)}{2}\Gamma(s)\textstyle\sum_{j=0}^n a_{2j}\displaystyle\int_R \frac{r^{2j+1}(\sin\alpha r)\,dr}{(b+r^2)^s}$$

$$= C_G\frac{vol(\Gamma\backslash G)}{2}\Gamma(s)\textstyle\sum_{j=0}^n a_{2j}S_j(s)$$

by (4.15), for Re $s > \dfrac{d+1}{2} = n+1$. Thus by Proposition 4.18, $\int_0^\infty x^{s-1}\theta_1(x)\,dx$ extends to an entire function.

We turn attention now to the study of the term $\int_0^1 x^{s-1}\theta_2(x)\,dx$ in (5.9) which we write as $T(s) = \int_0^\infty \theta_2\left(\dfrac{1}{t}\right)t^{-s-1}\,dt$ using the transformation $x = \dfrac{1}{t}$. We shall argue as in [12], [17] and rely on the following result of DeGeorge. For $x \geq 0$ let $E(x) = |\{\gamma \in C_\Gamma - \{1\}\,|\,t_\gamma \leq x\}|$, $\tilde{E}(x) = |\{\gamma \in C_\Gamma - \{1\}\,|\,x \leq t_\gamma < x+1\}|$, where $|s|$ denotes the cardinality of a set S. Then by [1], for some $\beta > 0$ it is true that $\lim_{x\to\infty} \beta x e^{-\beta x}E(x) = 1$. From this it follows that there is an integer j_0 sufficiently large and a positive number δ such that

$$(5.12)\qquad \tilde{E}(x) \leq \frac{\delta}{x}e^{\beta x} \qquad \text{for} \quad x \geq j_0\,.$$

Now by definition of $\tilde{E}(x)$ one has

$$\begin{aligned}
&\textstyle\sum_{x\leq t_\gamma < x+1} t_\gamma(\alpha + t_\gamma)e^{-t_\gamma^2 t/4}e^{t_\gamma \alpha t/2} \\
&\quad\leq \textstyle\sum_{x\leq t_\gamma < x+1}(x+1)(\alpha+x+1)e^{-x^2t/4}e^{(x+1)\alpha t/2} \\
&\quad= (x+1)(\alpha+x+1)e^{-x^2t/4}e^{(x+1)\alpha t/2}\tilde{E}(x) \\
&\quad\overset{\#}{\leq} \delta\frac{x+1}{x}(\alpha+x+1)e^{-x^2t/4}e^{(x+1)\alpha t/2}e^{\beta x} \qquad \text{for} \quad x \geq j_0\,.
\end{aligned}$$

Taking j_0 a bit larger, if necessary, we assume $j_0 > 2\alpha + 1$. Then if $f_n(t) \overset{\text{def}}{=} -(j_0+n-1)^2t/4 + (j_0+n)\alpha t/2$ for $t \in R$, we clearly have

$$4\frac{df_n}{dt}\bigg/(j_0+n) = -\frac{(j_0+n-1)^2 + (j_0+n)2\alpha}{(j_0+n)}$$

$$= -\frac{[(j_0+n)^2 - 2(j_0+n) + 1] + (j_0+n)2\alpha}{(j_0+n)}$$

$$= -j_0 - n + 2 - \frac{1}{j_0+n} + 2\alpha < -j_0 - 1 + 2(\alpha+1) < 0\,;$$

ie. $f_n'(t) < 0\ \forall t \Rightarrow f_n$ is decreasing:

$$(5.13)\qquad -(j_0+n-1)^2 t/4 + (j_0+n)\alpha t/2$$
$$\le -(j_0+n-1)^2/4 + (j_0+n)\alpha/2 \qquad \forall n, \qquad \text{for} \quad t \ge 1\,.$$

LEMMA 5.14. *Let*

$$S(t) = \textstyle\sum_{t_\gamma > j_0} t_\gamma(\alpha + t_\gamma) e^{-t_\gamma^2 t/4} e^{t_\gamma \alpha t/2} \qquad \textit{for} \quad t \in R\,.$$

Then $S(t)$ *converges for every* $t > 0$, *and is bounded for* $t \ge 1$.

PROOF.

$$S(t) = \textstyle\sum_{t_\gamma > j_0} \le \sum_{j_0 \le t_\gamma < j_0+1} + \sum_{j_0+1 \le t_\gamma < j_0+2} + \sum_{j_0+2 \le t_\gamma < j_0+3} + \cdots$$
$$\le \delta \frac{j_0+1}{j_0}(\alpha + j_0 + 1) e^{-j_0^2 t/4} e^{(j_0+1)\alpha t/2} e^{\beta j_0}$$
$$+ \delta \frac{j_0+2}{j_0+1}(\alpha + j_0 + 2) e^{-(j_0+1)^2 t/4} e^{(j_0+2)\alpha t/2} e^{\beta(j_0+1)}$$
$$+ \delta \frac{j_0+3}{j_0+2}(\alpha + j_0 + 3) e^{-(j_0+2)^2 t/4} e^{(j_0+3)\alpha t/2} e^{\beta(j_0+2)}$$
$$+ \cdots \quad (\text{by } \#) =$$
$$= \delta \textstyle\sum_{n=1}^{\infty} \dfrac{(j_0+n)}{j_0+n-1}(\alpha + j_0 + n) e^{-(j_0+n-1)^2 t/4} e^{(j_0+n)\alpha t/2} e^{\beta(j_0+n-1)}$$

which converges for every $t > 0$ by the ratio test. In particular for $t \ge 1$ we have

$$S(t) \le \delta \textstyle\sum_{n=1}^{\infty} \dfrac{j_0+n}{j_0+n-1}(\alpha + j_0 + n) e^{-(j_0+n-1)^2/4} e^{(j_0+n)\alpha/2} e^{\beta(j_0+n-1)}$$

Going back to (5.5) we have

$$(5.15)\quad \theta_2\left(\frac{1}{t}\right) = \frac{e^{-b/t}}{4\sqrt{\pi}} t^{3/2} \textstyle\sum_{t_\gamma \le j_0} t_\gamma j(\gamma)^{-1} C(\gamma)(\alpha - t_\gamma) e^{-(t_\gamma - \alpha)^2 t/4}$$
$$+ e^{-\alpha^2 t/4} \frac{e^{-b/t}}{4\sqrt{\pi}} t^{3/2} \textstyle\sum_{t_\gamma > j_0} t_\gamma j(\gamma)^{-1} C(\gamma)(\alpha - t_\gamma) e^{-t_\gamma^2 t/4} e^{t_\gamma \alpha t/2}$$

$$+ e^{-\alpha^2 t/4}\frac{e^{-b/t}}{4\sqrt{\pi}} t^{3/2} \sum_{\gamma\in C_\Gamma-\{1\}} t_\gamma j(\gamma)^{-1} C(\gamma)(\alpha + t_\gamma) e^{-t_\gamma^2 t/4} e^{-t_\gamma \alpha t/2}$$

for $t > 0$.

We denote the 3 terms in (5.15) by $T_1(t)$, $T_2(t)$, $T_3(t)$ respectively. Therefore $T(s) \stackrel{\text{def}}{=} \int_1^\infty \theta_2\left(\frac{1}{t}\right) t^{-s-1}\, dt = \int_1^\infty T_1(t) t^{-s-1}\, dt + \int_1^\infty T_2(t) t^{-s-1}\, dt + \int_1^\infty T_3(t) t^{-s-1}\, dt$. We claim 1st that $\int_1^\infty T_2(t) t^{-s-1}\, dt$ is an entire function of s. As in [12] there is a bound M_0 for the numbers $C(\gamma)$. If $M = \frac{M_0}{4\sqrt{\pi}}$ we have for $t \geq 1$, $|T_2(t)| \leq e^{-\alpha^2 t/4} M t^{3/2} \sum_{t_\gamma > j_0} t_\gamma(\alpha + t_\gamma) e^{-t_\gamma^2 t/4} e^{t_\gamma \alpha t/2} = e^{-\alpha^2 t/4} M t^{3/2} S(t) \leq e^{-\alpha^2 t/4} t^{3/2} MC$, for some constant C, by Lemma 5.14. It follows that $\int_1^\infty T_2(t) t^{-s-1}\, dt$ converges uniformly on compact subsets of the plane and thus defines an entire function of s. $T_1(t)$ is a finite sum with each term $\frac{e^{-b/t} t^{3/2}}{4\sqrt{\pi}} t_\gamma j(\gamma) C(\gamma)(\alpha - t_\gamma) e^{-(t_\gamma - \alpha)^2 t/4}$, $t_\gamma \leq j_0$, bounded by $t^{3/2} M t_\gamma(\alpha + t_\gamma)$. $e^{-(t_\gamma-\alpha)^2 t/4}$; ie. $\int_1^\infty T_1(t) t^{-s-1}\, dt$ similarly is entire in s, being a finite sum of functions entire in s. We have $|T_3(t)| \leq e^{-\alpha^2 t/4} M t^{3/2} \tilde{S}(t)$, where

$$\tilde{S}(t) \stackrel{\text{def}}{=} \sum_{\gamma \in C_\Gamma - \{1\}} t_\gamma(\alpha + t_\gamma) e^{-t_\gamma^2 t/4} = \tilde{S}_1(t) + \tilde{S}_2(t)$$

where

$$\tilde{S}_1(t) = \sum_{t_\gamma \leq j_0} t_\gamma(\alpha + t_\gamma) e^{-t_\gamma^2 t/4},$$

$$\tilde{S}_2(t) = \sum_{t_\gamma > j_0} t_\gamma(\alpha + t_\gamma) e^{-t_\gamma^2 t/4} \leq S(t) \Rightarrow \tilde{S}_2(t) \leq C \qquad \text{for} \quad t \geq 1,$$

again by Lemma 5.14. Recalling the definition of $E(x)$ one has $\tilde{S}_1(t) \leq \sum_{t_\gamma \leq j_0} j_0(\alpha + j_0) = j_0(\alpha + j_0) E(j_0)$ for $t > 0$. Thus we see that, similarly, $\int_1^\infty T_3(t) t^{-s-1}\, dt$ converges uniformly on compact subsets of the plane and therefore also is an entire function of s. In conclusion, we have that $T(s) = \int_0^1 x^{s-1} \theta_2(x)\, dx = \int_1^\infty \theta_2\left(\frac{1}{t}\right) t^{-s-1}\, dt$ is an entire function of s; here we allow $b \geq 0$.

The one remaining term $\int_0^1 x^{s-1} \theta_0(x)\, dx$ in (5.9) is the easiest to analyze. From (5.5), $\int_0^1 x^{s-1} \theta_0(x)\, dx = \sum_{j=0} n_j t_j [e^{\alpha t_j} - e^{-\alpha t_j}] \Psi_j(s)$, where $\Psi_j(s) \stackrel{\text{def}}{=}$

$\int_0^1 e^{-(b-t_j^2)x}x^{s-1}\,dx$. Let γ_2 be the incomplete gamma function:

$$\gamma_2(s,t)=\int_0^t e^{-x}x^{s-1}\,dx \tag{5.16}$$

where $\mathrm{Re}\, s>0$, $t\in R$. For $b-t_j^2\neq 0$ $\dfrac{\Psi_j(s)}{\Gamma(s)}=\dfrac{(b-t_j^2)^{-s}\gamma_2(s,b-t_j^2)}{\Gamma(s)}$, which is known to be entire in s. If $b=t_j^2$, $\Psi_j(s)=\dfrac{1}{s}$ and clearly $\dfrac{\Psi_j(s)}{\Gamma(s)}=\dfrac{1}{s\Gamma(s)}$ (defined to be 1 for $s=0$) is entire. Thus $\dfrac{1}{\Gamma(s)}\displaystyle\int_0^1 x^{s-1}\theta_0(x)\,dx$ is entire in s for $b\geq 0$.

In conclusion we deduce from (5.4), Corollary 5.7, (5.9), and the definition $I=I_{(0)}+I_{(1)}$ of $I_{(0)}$, $I_{(1)}$ the following key theorem.

THEOREM 5.17. *$Z_{\alpha,b}$ as defined in* (3.6) *indeed extends to an entire function, for every $b>0$.*

REMARK. We shall see in the next section that for $b=0$, $Z_{\alpha,0}$ also extends to an entire function.

6. Analytic continuation of Z_α

Because of the term $\int_0^1 x^{s-1}\theta_1(x)\,dx$ in (5.9) we had to assume $b>0$ to analytically continue $Z_{\alpha,b}$ as we did in section 5. There we saw that $I_{(1)}$ was entire for $b\geq 0$ (Corollary 5.7), $\int_0^1 x^{s-1}\theta_2(x)\,dx$ was entire for $b\geq 0$ and $\dfrac{1}{\Gamma(s)}\displaystyle\int_0^1 x^{s-1}\theta_0(x)\,dx$ was entire for $b\geq 0$. Thus to handle the analytic continuation of $Z_\alpha=Z_{\alpha,0}$ we need only to analytically continue $\int_0^1 x^{s-1}\theta_1(x)\,dx$ (by some different means) in case $b=0$. We address this matter in this section. For $b=0$,

$$\int_0^1 x^{s-1}\theta_1(x)\,dx=\frac{vol(\Gamma\backslash G)}{\pi}\int_0^1\int_0^\infty x^{s-1}re^{-r^2x}(\sin\alpha r)|c(r)|^{-2}\,dr\,dx \tag{6.1}$$

$$=\frac{vol(\Gamma\backslash G)}{C_G}\sum_{j=0}^{(d/2)-1}a_{2j}\int_0^1\int_0^\infty x^{s-1}e^{-r^2x}(\sin\alpha r)r^{2(j+1)}\tanh^\varepsilon ar\,dr\,dx\,,$$

say for $G\neq SO_1(2n+1,1)$, where $\varepsilon=\pm 1$, $a=\pi$ or $\pi/2$. Consider the case

$\varepsilon = 1$; write $\tanh ar = 1 - \dfrac{2}{e^{2ar}+1}$ so that the double integral in (6.1) which we denote by $\Phi_j(s)$ is

$$\Phi_j(s) = \int_0^1 \int_0^\infty x^{s-1} e^{-r^2 x} (\sin \alpha r) r^{2(j+1)}\, dr\, dx \tag{6.2}$$
$$- 2 \int_0^1 \int_0^\infty \frac{x^{s-1} e^{-r^2 x} (\sin \alpha r) r^{2(j+1)}\, dr\, dx}{e^{2ar}+1}$$

The 1st integral in (6.2) is by (4.22), (4.25) exactly $\phi_{j+1}(s; \alpha)$, which we know is well-defined and holomorphic in s for $\operatorname{Re} s > j + 1 + \frac{1}{2}$; this inequality is satisfied for $\operatorname{Re} s > \frac{d+1}{2}$ (see (5.4)), as $j \le \frac{d}{2} - 1$. Moreover by Proposition 4.27, $\phi_{j+1}(; \alpha)/\Gamma(\cdot)$ extends to an entire function. Thus we concentrate on the 2nd integral in (6.2), which we denote by $\Psi_j(s)$. For $\operatorname{Re} s > 0$, Fubini's Theorem applies:

$$\Psi_j(s) = \int_0^\infty \frac{(\sin \alpha r) r^{2(j+1)}}{e^{2ar}+1} \left[\int_0^1 x^{s-1} e^{-r^2 x}\, dx \right] dr \tag{6.3}$$

where $\int_0^1 x^{s-1} e^{-r^2 x}\, dx = r^{-2s} \int_0^{r^2} e^{-u} u^{s-1}\, du = r^{-2s} \gamma_2(s, r^2)$, by (5.16). Define γ_2^* by $\gamma_2^*(s, t) = t^{-s} \gamma_2(s, t)/\Gamma(s)$, say for $s \in C$, $t \in R$. Then $\gamma_2^*(s, t)$ is an entire function of s, a fact already used, following (5.16). We therefore have

$$\Psi_j(s)/\Gamma(s) \overset{\#}{=} \int_0^\infty \frac{(\sin \alpha r) r^{2(j+1)}}{e^{2ar}+1} \gamma_2^*(s, r^2)\, dr\,.$$

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$$\gamma_2^*(s, t) = t^{-s} - \frac{t^{-1} e^{-t}}{\Gamma(s)} [1 + O(|t|)^{-1}] \quad \text{as} \quad |t| \to \infty\,. \tag{6.4}$$

There are positive constants C, M therefore such that for $r \ge M$

$$|\gamma_2^*(s, r^2)| \le r^{-2\operatorname{Re} s} + \frac{e^{-r^2}}{r^2 |\Gamma(s)|} + \frac{C e^{-r^2}}{r^4 |\Gamma(s)|}\,; \tag{6.5}$$

that is

$$|\gamma_2^*(s, r^2)| \le r^{-2\operatorname{Re} s} + \frac{C_1 e^{-r^2}}{|\Gamma(s)|} \tag{6.6}$$

for $r \ge M$, where $C_1 = 1 + C$. Accordingly we have

$$\Psi_j(s)/\Gamma(s) = \int_0^M \frac{(\sin \alpha r)}{e^{2ar}+1} r^{2(j+1)} \gamma_2^*(s, r^2)\, dr + \int_M^\infty \frac{(\sin \alpha r)}{e^{2ar}+1} r^{2(j+1)} \gamma_2^*(s, r^2)\, dr\,,$$

where the 1st term is entire in s. The 2nd term is also entire in s as the integral converges uniformly on compact subsets K of the plane, by (6.6):

$$\left|\frac{(\sin \alpha r)}{e^{2ar}+1} r^{2(j+1)} \gamma_2^*(s, r^2)\right| \le r^{2(j+1)-2\eta} e^{-2ar} + C_1 C_2 r^{2(j+1)} e^{-2ar}\,,$$

where $\operatorname{Re} s > \eta$, $\dfrac{1}{|\Gamma(s)|} \le C_2$ for $s \in K$. We now have that $s \to \dfrac{1}{\Gamma(s)} \displaystyle\int_0^1 x^{s-1} \theta_1(x)\, dx$ extends to an entire function in case $\varepsilon = 1$. To handle the case $\varepsilon = -1$ we use the idea preceding Theorem 4.4. Namely, write $\coth ar = \tanh ar + (\tanh ar) \operatorname{csch}^2 ar$, so that the double integral in (6.1) is now $\Phi_j(s) + \displaystyle\int_0^1 \int_0^\infty x^{s-1} e^{-r^2 x} (\sin \alpha r) r^{2(j+1)} (\tanh ar) \operatorname{csch}^2 ar\, dr\, dx$, where $s \to \dfrac{1}{\Gamma(s)} \Phi_j(s)$ extends to an entire function, as we have just shown, and where again Fubini's Theorem applies to the 2nd term—call it $T_j(s)$:

$$\frac{T_j(s)}{\Gamma(s)} = \int_0^\infty (\sin \alpha r) r^{2(j+1)} (\tanh ar)(\operatorname{csch}^2 ar) \gamma_2^*(s, r^2)\, dr\,, \tag{6.7}$$

exactly as in equation #. Using (6.6) again we therefore see that $s \to T_j(s)/\Gamma(s)$ extends to an entire function—noting that $r \to r^2 \operatorname{csch}^2 ar$ (defined to be $\dfrac{1}{a^2}$ at $r = 0$) is continuous on R, and that for some $C > 0$, $r^2 \operatorname{csch}^2 ar < Ce^{-ar}\ \forall r \ge 0$. This gives the analytic continuation of $s \to \dfrac{1}{\Gamma(s)} \displaystyle\int_0^1 x^{s-1} \theta_1(x)\, dx$ to an entire function in the case $\varepsilon = -1$.

In case G is locally isomorphis to $SO_1(2n+1, 1)$ (the final case to consider).

$$\int_0^1 x^{s-1} \theta_1(x)\, dx = C_G vol(\Gamma \backslash G) \textstyle\sum_{j=0}^n a_{2j} \displaystyle\int_0^1 \int_0^\infty x^{s-1} e^{-r^2 x} (\sin \alpha r) r^{2j+1}\, dr\, dx \tag{6.8}$$

by (2.5), (2.9), and the 1st double integral in (6.1). By page 496 of [9]

$$\int_0^\infty e^{-r^2 x} (\sin \alpha r) r^{2j+1}\, dr = (-1)^j \frac{\sqrt{\pi}}{(2\sqrt{x})^{2j+2}} e^{-(\alpha^2/4x)} H_{2j+1}\left(\frac{\alpha}{2\sqrt{x}}\right) \tag{6.9}$$

where H_n is the n^{th} Hermite polynomial. Therefore

$$\int_0^1 x^{s-1}\theta_1(x)\,dx$$
$$= C_G vol(\Gamma\backslash G)\sqrt{\pi}\sum_{j=0}^{n}\frac{a_{2j}(-1)^j}{2^{2j+2}}\int_1^\infty x^{-s-1}x^{j+1}H_{2j+1}\left(\frac{\alpha}{2}\sqrt{x}\right)e^{-\alpha^2 x/4}\,dx\,,$$

which is entire in s. This concludes the proof of

THEOREM 6.10. *The zeta function* $Z_\alpha \stackrel{\text{def}}{=} Z_{\alpha,0}$ *defined in* (3.6) *for* Re $s > \dim G/K$ *admits an extension to the whole plane, which in all cases of* G *is an entire function.*

Compare Theorem 5.17

7. A limit formula

As in [6] we can compute the special value $Z_\alpha(0)$ (given Theorem 6.10). The result, which is rather long and technical, will not be stated here. Using this result we can prove the following theorem. Recall the notation of section 3; see (3.9), (3.10) in particular.

THEOREM 7.1. *For any* $\gamma_1 \in \Gamma - \{1\}$

$$\lim_{\alpha\to t_{\gamma_1}}(\alpha - t_{\gamma_1})Z_\alpha(0) = \frac{1}{2}\sum_{\gamma\in C_\Gamma - \{1\}} j(\gamma)^{-1}t_\gamma C(\gamma).$$

The proof of Theorem 7.1 will appear elsewhere. We note in closing that Theorem 7.1 coincides with statement (1.2). Namely we define first of all the von Mangoldt function $\tilde{\Lambda}$ by

$$\tilde{\Lambda}(\gamma) = e^{t_\gamma/2}j(\gamma)^{-1}t_\gamma C(\gamma) \qquad \text{for} \quad \gamma\in\Gamma - \{1\}\,. \tag{7.2}$$

Define the *norm* $N(\gamma)$ of $\gamma \in \Gamma - \{1\}$ by $N(\gamma) = e^{t_\gamma}$. For $G = SL(2, R)$, $N(\gamma) =$ maximum of $\{|c|^2 | c = \text{an eigenvalue of } \gamma\}$ is the usual definition of the norm. Also in this case $C(\gamma) = \dfrac{1}{e^{t_\gamma/2} - e^{-t_\gamma/2}}$ so that $\tilde{\Lambda}(\gamma) = \dfrac{j(\gamma)^{-1}t_\gamma}{1 - e^{-t_\gamma}} = \dfrac{\log N(\delta)}{1 - N(\gamma)^{-1}}$ for $\gamma = \delta^{j(\gamma)}$ with δ a primitive element as section 3. That is $\tilde{\Lambda}$ in (7.2) reduces to the usual von Mangoldt function for the Selberg zeta function.

Theorem 7.1 can now be written as

$$\lim_{\alpha\to\log N(\gamma_1)}(\alpha - \log N(\gamma_1))Z_\alpha(0) = \frac{1}{2}\sum_{\substack{\gamma\in C_\Gamma - \{1\}\\ N(\gamma)=N(\gamma_1)}}\tilde{\Lambda}(\gamma)/\sqrt{N(\gamma)}\,,$$

which is (1.2) for our normalization of Haar measures.

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