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Asymptotic and oscillatory properties of differential equations with deviating argument

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We consider the n-th order differential equation with deviating argument of the form

$$L_n u(t) + p(t)|u(g(t))|^\alpha \operatorname{sgn} u(g(t)) = 0, \tag{1}$$

where $\alpha > 0$,

$$L_2 u(t) = (r(t)u'(t))',$$

and

$$L_n u(t) = (r(t) \cdots (r(t)(r(t)u'(t))')' \cdots)',$$

for $n \geq 2$.

We always assume that $p, g \in C([t_0, \infty))$, $r \in C^{n-1}([t_0, \infty))$, $r(t) > 0$, and $g(t) \to \infty$ as $t \to \infty$.

We consider only nontrivial solutions of (1). Such solution is called oscillatory if the set of its zeros is unbounded. Otherwise, it is called nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory, otherwise it is said to be nonoscillatory.

The oscillatory behavior of equation (1) has recently been studied by many authors (see e.g. [3], [5] and [6]) in the case

$$\int_{t_0}^{\infty} \frac{ds}{r(s)} = \infty. \tag{2}$$

In this paper we present a technique that enables to tranfer many oscillatory results from equation

$$y^{(n)}(t) + p(t)|y(g(t))|^\alpha \operatorname{sgn} y(g(t)) = 0 \tag{3}$$

to equation (1) provided that

$$\int_{t_0}^{\infty} \frac{ds}{r(s)} < \infty. \tag{4}$$

This technique is related to v-transformation of an equation (see [5]), which permits to tranfer oscillatory and asymptotic results from (3) to (1) if condition (2) holds.

Now we define the function

$$\rho(t) = \int_t^\infty \frac{ds}{r(s)}, \qquad t \geq t_0,$$

which belongs to the class $C^1([t_0, \infty))$, is decreasing and maps the interval $[t_0, \infty)$ onto the interval $(0, \rho_0]$, where $\rho_0 = \rho(t_0)$. Let δ be the inverse function to ρ. Then a composite function $\delta(1/s)$ belongs to the class $C^1[s_0, \infty)$, where $s_0 = 1/\rho_0$, is increasing on this interval and maps the interval $[s_0, \infty)$ onto $[t_0, \infty)$.

Let us denote

(5) $$p_1(s) = (1/s)^{n+1} p[\delta(1/s)]\, r[\delta(1/s)]\, \rho^{\alpha(n-1)}(g[\delta(1/s)])$$

and

(6) $$w(s) = 1/\rho(g[\delta(1/s)]),$$

then we have the following:

THEOREM 1. Assume that (4) holds. Let $u(t)$ be a solution of (1) on the interval $[t_0, \infty)$. Then the function

(7) $$y(s) = s^{n-1} u[\delta(1/s)], \qquad s \geq s_0$$

is a solution of

(8) $$y^{(n)}(s) + p_1(s)\,|y(w(s))|^\alpha \operatorname{sgn} y(w(s)) = 0$$

on the interval $[s_0, \infty)$.

Conversely, if $y(s)$ is a solution of (8) on $[s_0, \infty)$, then the function $u(t)$ determined by relation (7), i.e.

(9) $$u(t) = \rho^{n-1}(t)\, y[1/\rho(t)], \qquad t \in [t_0, \infty),$$

satisfies (1) on $[t_0, \infty)$.

PROOF. We will use the notation:

$$L_0 u(t) = u(t),$$
$$L_k u(t) = r(t)[L_{k-1} u(t)]', \qquad 1 \leq k \leq n-1,$$
$$L_n u(t) = [L_{n-1} u(t)]'.$$

Differentiating the relation (9) and considering $\rho'(t) = -1/r(t)$ we easily verify (by induction on i), that for every $i \in \{1, 2, \ldots, n-1\}$

$$(10) \qquad L_i u(t) = \sum_{j=0}^{i} (-1)^{i+j} \binom{i}{j} \frac{(n-1-j)!}{(n-1-i)!} y^{(j)}[1/\rho(t)] (\rho(t))^{n-1-i-j}.$$

Differentiation of relation (10), where $i = n - 1$ leads to

$$r(t) L_n u(t) = [1/\rho(t)]^{n+1} y^{(n)}[1/\rho(t)].$$

From the last equality using $t = \delta(1/s)$ and (5) and (6) it follows

$$\begin{aligned} r(t)\,[L_n u(t) + p(t)|u(g(t))|^{\alpha} \operatorname{sgn} u(g(t))] \\ = s^{n+1}[y^{(n)}(s) + p_1(s)|y(w(s))|^{\alpha} \operatorname{sgn} y(w(s))], \end{aligned}$$

for $t \in [t_0, \infty)$ and $s \in [s_0, \infty)$.

Now we see that $u(t)$ is a solution of (1) on $[t_0, \infty)$ if and only if $y(s) = s^{n-1} u[\delta(1/s)]$ is a solution of (8) on $[s_0, \infty)$. The proof is complete.

COROLLARY 1. Let (4) hold. Then a function u is an oscillatory (nonoscillatory) solution of (1) if and only if the function y given in (7) or (9) is an oscillatory (nonoscillatory) solution of (8).

For the linear differential equation

$$(11) \qquad (r(t)u'(t))' + p(t)u(t) = 0, \qquad t \in [t_0, \infty),$$

which is a special case of (1) we have:

COROLLARY 2. Let (4) hold. Then equation (11) is oscillatory (nonoscillatory) if and only if the equation

$$(12) \qquad y''(t) + (1/t^4) r[\delta(1/t)] p[\delta(1/t)]\, y(t) = 0, \qquad t \in [1/\rho_0, \infty)$$

is oscillatory (nonoscillatory).

Now using sufficient conditions for equation (12) to be oscillatory (nonoscillatory) we can obtain sufficient conditions for equation (11) to be oscillatory (nonoscillatory). We present an example of that.

THEOREM 2. Let (4) hold. Let p is of constant sign.

(i) Equation (11) is oscillatory if

$$(13) \qquad \liminf_{t \to \infty} \frac{1}{\rho(t)} \int_t^{\infty} \rho^2(s) p(s)\, ds > \frac{1}{4}.$$

(ii) Equation (11) is nonoscillatory if

$$(14) \qquad \limsup_{t \to \infty} \frac{1}{\rho(t)} \int_t^{\infty} \rho^2(s) p(s)\, ds < \frac{1}{4}.$$

PROOF. Condition (13) ((14)) is equivalent to

$$\liminf_{t\to\infty} t\int_t^\infty (1/x^4)r[\delta(1/x)]p[\delta(1/x)]\,dx > \frac{1}{4},$$

$$\left(\limsup_{t\to\infty} t\int_t^\infty (1/x^4)r[\delta(1/x)]p[\delta(1/x)]\,dx < \frac{1}{4}\right),$$

which is sufficient condition for equation (12) to be oscillatory (nonoscillatory) as it follows from [1] or [2].

For our next consideration we need the concept of a principal system for the operator L_n. By a principal system for L_n we mean a set of n solutions $X_0(t),\dots,X_{n-1}(t)$ of the equation

$$L_n u(t) = 0, \tag{15}$$

which are eventually positive and satisfy the relation

$$\lim_{t\to\infty} \frac{X_m(t)}{X_k(t)} = 0, \qquad \text{for } 0 \le m < k \le n-1.$$

A basic property of a principal system is involved in the following:

LEMMA 1. If both $\{X_0(t),\dots,X_{n-1}(t)\}$ and $\{Y_0(t),\dots,Y_{n-1}(t)\}$ are principal systems for L_n, then for each k, $0 \le k \le n-1$, $X_k(t)$ and $Y_k(t)$ have the same order of growth (or decay) as $t\to\infty$, that is, the limits

$$\lim_{t\to\infty} \frac{X_k(t)}{Y_k(t)} > 0, \qquad 0 \le k \le n-1,$$

exist and are finite.

For the proof see [7]. Now we can modify well-known properties (A), (B) and some other properties.

Let $X_0(t),\dots,X_{n-1}(t)$ be a principal system for L_n. Let k be an integer number, $1 \le k \le n-1$. We say that a nonoscillatory solution $u(t)$ of (1) satisfies condition (P_k), if

$$\lim_{t\to\infty} \frac{|u(t)|}{X_{k-1}(t)} = \infty, \quad \text{and} \quad \lim_{t\to\infty} \frac{|u(t)|}{X_k(t)} = 0.$$

DEFINITION 1. Equation (1) is said to have property (A) if for even n equation (1) is oscillatory and for odd n every nonoscillatory solution $u(t)$ of (1) satisfies

$$(P_0) \qquad \lim_{t\to\infty} \frac{|u(t)|}{X_0(t)} = 0.$$

Definition 2. Equation (1) is said to have property (B) if for even n every nonoscillatory solution $u(t)$ of (1) satisfies either (P_0) or

$$(P_n) \qquad \lim_{t\to\infty} \frac{|u(t)|}{X_{n-1}(t)} = \infty$$

and for odd n every nonoscillatory solution $u(t)$ of (1) satisfies (P_n).

Remark 1. If function $r(t)$ satisfies (2), then Definitions 1 and 2 are equivalent to well-known definitions of properties (A) and (B) (see e.g. [3]).

Let $k_1, \ldots, k_m$ be all mutually different integers, such that $0 < k_i < n$ for $i = 1, 2, \ldots, m$, where m is an integer.

Definition 3. Equation (1) is said to have property $A_{k_1,\ldots,k_m}$ if for even n every nonoscillatory solution $u(t)$ of (1) satisfies condition (P_{k_i}), $1 \le i \le m$ and for odd n every nonoscillatory solution $u(t)$ of (1) satisfies either condition (P_0) or (P_{k_i}), for some i.

Definition 4. Equation (1) is said to have property $B_{k_1,\ldots,k_m}$ if for even n every nonoscillatory solution $u(t)$ of (1) satisfies either (P_0) or (P_n) or (P_{k_i}), $1 \le i \le m$ and for odd n every nonoscillatory solution $u(t)$ of (1) satisfies either condition (P_n) or condition (P_{k_i}) for some i.

Similarly as in [1] we use the following notation. Let $\{M_i\}_{i=1}^{\tau}$ be decreasing sequence of all mutually different local maxima of the polynomial

$$P_n(k) = -k(k-1)\cdots(k-n+1).$$

Let $(N_i)_{i=1}^{\lambda}$ be increasing sequence of all mutually different local minima of the polynomial $P_n(k)$. Now, if

$n = 4j$ then $\tau = \lambda = j$;
$n = 4j + 1$ then $\tau = \lambda = 2j$;
$n = 4j + 2$ then $\tau = j + 1$, $\lambda = j$;
$n = 4j + 3$ then $\tau = \lambda = 2j + 1$,

where $j = 0, 1, \ldots$ (and $n \ge 2$).

Let us denote

$$(16) \qquad \beta = (n-1) \liminf_{t\to\infty} [1/\rho(g(t))]^{n-1} \int_t^\infty \rho^{n-1}(s)\rho^{n-1}[g(s)]\, p(s)\, ds$$

and

(17) $$\gamma = (n-1) \limsup_{t\to\infty} [1/\rho(g(t))]^{n-1} \int_t^{\infty} \rho^{n-1}(s)\rho^{n-1}[g(s)]p(s)\,ds.$$

Suppose that function g satisfies the following conditions

(18) $$g(t) \in C^1([t_0, \infty)),\ g'(t) > 0,\ g(t) \le t.$$

It is useful to notice that if (18) holds then function $w(t)$ given in (6) satisfies:

$$w(s) \in C^1([s_0, \infty)),\ w'(s) > 0,\ w(s) \le s.$$

Now we are prepared to present several results concerning of the equation

(19) $$L_n u(t) + p(t)u(g(t)) = 0,$$

which is a special case of (1) for $\alpha = 1$. In the sequel we will assume that function p is of constant sign.

THEOREM 3. Suppose (4) and (18) hold.
(i) If $\beta > M_1$, then equation (19) has property (A).
(ii) If $\gamma < N_1$ and $n > 2$, then equation (19) has property (B).

THEOREM 4. Suppose (4) and (18) hold. Let n be even.
(i) If $M_k > \beta > M_{k+1}$, for some $k \in \{1, 2, \dots, \tau - 1\}$, then equation (19) has property $A_{1,3,\dots,2k-1,n-2k+1,\dots,n-1}$.
(ii) If $N_k < \gamma < N_{k+1}$, for some $k \in \{1, 2, \dots, \lambda - 1\}$, then equation (19) has property $B_{2,4,\dots,2k,n-2k,\dots,n-2}$.

THEOREM 5. Suppose (4) and (18) hold. Let n be odd.
(i) If $M_1 > \beta > M_2$, then equation (19) has property (A_{n-1}).
(ii) If $N_1 < \gamma < N_2$, then equation (19) has property (B_1)
(iii) If $M_k > \beta > M_{k+1}$, for some $k \in \{2, 3, \dots, \tau - 1\}$, and if k is even (odd), then equation (19) has property $A_{2,4,\dots,k,n-k+1,\dots,n-1}(A_{2,4,\dots,k-1,n-k,\dots,n-1})$.
(iv) If $N_k < \gamma < N_{k+1}$, for some $k \in \{2, 3, \dots, \lambda - 1\}$ and if k is even (odd), then equation (19) has property $B_{1,3,\dots,k-1,n-k,\dots,n-2}(B_{1,3,\dots,k,n-k+1,\dots,n-2})$.

THEOREM 6. Let (4) and (18) hold.
(i) If $M_\tau > \beta > 0$ and n is even (odd), then equation (19) has property $A_{1,3,\dots,n-1}(A_{2,4,\dots,n-1})$.
(ii) If $N_\lambda < \gamma < 0$ and n is even (odd), then equation (19) has property $B_{2,4,\dots,n-2}(B_{1,3,\dots,n-2})$.

PROOF OF THEOREM 3(i). Let $u(t)$ be a nonoscillatory solution of (19) on $[t_0, \infty)$. We may assume that $u(t)$ is positive. By direct computation we can easily verify that functions $\rho^{n-1}(t)$, $\rho^{n-2}(t),\dots,\rho(t)$, 1 form a principal system for operator L_n. Hence, with regard to Definition 1 and Lemma 1 we have to show that

$$\lim_{t\to\infty}\frac{u(t)}{\rho^{n-1}(t)}=0. \tag{20}$$

Theorem 1 implies that function $y(s)$ given in (7) is a positive solution of the equation

$$y^{(n)}(s)+p_1(s)y(w(s))=0, \qquad s\in[s_0,\infty), \tag{21}$$

where functions p_1 and w are given in (5) and (6) with $\alpha=1$. We know (see Theorem 5 in [1]), that equation (21) has property (A) if

$$(n-1)\liminf_{s\to\infty}[w(s)]^{n-1}\int_s^\infty p_1(x)\,dx>M_1. \tag{22}$$

One can see that (22) is equivalent to condition $\beta>M_1$ for β determined by relation (16). Therefore equation (21) has property (A) and generalization of well-known lemma of Kiguradze implies that n must be odd (for n even we have a contradiction because of existence of nonoscillatory solution $y(t)$ of (21)).

Since functions $1, s, s^2,\dots,s^{n-1}$ form a principal system for the operator D_n $(D_ny(t)=y^{(n)}(t))$ of the equation (21) we obtain

$$\lim_{s\to\infty}y(s)=0.$$

Now, from relation (9) we have

$$\lim_{t\to\infty}\frac{u(t)}{\rho^{n-1}(t)}=\lim_{t\to\infty}y[1/\rho(t)]=0.$$

(ii) Let $u(t)$ be a positive solution of (19) on $[t_0,\infty)$. We shall show that $u(t)$ satisfies either (20) or

$$\lim_{t\to\infty}u(t)=\infty.$$

Again, Theorem 1 implies that function $y(s)$ given in (7) is a positive solution of (21). From Theorem 5 in [1] we see that equation (21) has property (B) if

$$(n-1)\limsup_{s\to\infty}[w(s)]^{n-1}\int_s^\infty p_1(x)<N_1,$$

which is equivalent to condition $\gamma < N_1$. Hence, taking definition of property (B) into account we see that $y(s)$ satisfies either

$$\lim_{s\to\infty} y(s) = 0 \text{ or } \lim_{s\to\infty} y(s)/s^{n-1} = \infty.$$

In the first case arguing as in the proof of part (i), it follows that $u(t)$ satisfies (20). In the second case taking relation (9) into account it is easy to see that

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty} \rho^{n-1}(t)\, y(1/\rho(t)) = \infty.$$

The proof of Theorem 3 is complete.

PROOF OF THEOREM 5(i). Let $u(t)$ be a positive solution of (19) on $[t_0, \infty)$. Then $y(s)$ given in (7) is a positive solution of (21). Conditions $M_1 > \beta > M_2$ and n is odd imply (see [1]), that (21) has property A_{n-1} and so we have that either $\lim_{s\to\infty} y(s) = 0$ or y satisfies (P_{n-1}), i.e.

$$\lim_{s\to\infty} y(s)/s^{n-2} = \infty \quad \text{and} \quad \lim_{s\to\infty} y(s)/s^{n-1} = 0.$$

First case leads to the condition (20) and so we may suppose that y satisfies (P_{n-1}). But then from (9) we have

$$\lim_{t\to\infty} u(t)/\rho(t) = \lim_{t\to\infty} \rho^{n-2}(t)\, y(1/\rho(t)) = \infty$$

and

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty} \rho^{n-1}(t)\, y(1/\rho(t)) = 0.$$

Hence, equation (19) has property A_{n-1}.

To prove the other parts of Theorem 5 and also other mentioned theorems we can proceed similarly as above. The only one thing we should realise is following. A solution $y(s)$ of (21) satisfies (P_k) for some $k \in \{1, 2, \dots, n-1\}$ i.e.

$$\lim_{s\to\infty} y(s)/s^{k-1} = \infty \quad \text{and} \quad \lim_{s\to\infty} y(s)/s^{k} = 0$$

if and only if the solution $u(t)$ of (19) given in (9) satisfies

$$\lim_{t\to\infty} u(t)/\rho^{n-k}(t) = \lim_{t\to\infty} \rho^{k-1}(t)\, y(1/\rho(t)) = \infty$$

and

$$\lim_{t\to\infty} u(t)/\rho^{n-1-k}(t) = \lim_{t\to\infty} \rho^k(t)\, y(1/\rho(t)) = 0.$$

Therefore $u(t)$ satisfies (P_k) too.

Now we consider another special case of equation (1), i.e.

$$y^{(n)}(t) + p(t)\, y(g(t)) = 0, \qquad t \in [t_0, \infty), \tag{23}$$

for which moreover we assume that

$$p(t) > 0 \quad \text{and} \quad n \text{ is even.} \tag{24}$$

Recently M. Naito [4] has established the following results:

THEOREM A. Let (18) and (24) hold. Let $\liminf_{t\to\infty} g(t)/t > 0$.

(i) The equation (23) is strongly oscillatory if and only if either

$$\int_{t_0}^{\infty} s^{n-2} p(s)\, ds = \infty \quad \text{or} \quad \limsup_{t\to\infty} t \int_t^{\infty} s^{n-2} p(s) = \infty.$$

(ii) The equation (23) is strongly nonoscillatory if and only if

$$\int_{t_0}^{\infty} s^{n-2} p(s)\, ds < \infty \quad \text{and} \quad \lim_{t\to\infty} t \int_t^{\infty} s^{n-2} p(s)\, ds = 0.$$

The notions of strong oscillation and strong nonoscillation mentioned above are defined as follows: An equation of the form (19) is said to be strongly oscillatory (strongly nonoscillatory) if the related equation

$$(r(t)\cdots(r(t)(r(t)u'(t))')'\cdots)' + \theta p(t) u(g(t)) = 0$$

is oscillatory (nonoscillatory) for all positive values of θ.

In the paper [5] the above theorem is extended to the equation (19) provided that condition (2) holds. The purpose of the following theorem is to extend Theorem A to the equation (21) if condition (4) is satisfied.

THEOREM 7. Suppose (4), (18) and (24) are satisfied. Let

$$\liminf_{t\to\infty} \rho(t)/\rho(g(t)) > 0. \tag{25}$$

(i) The equation (19) is strongly oscilatory if and only if it satisfies either

$$\int_{t_0}^{\infty} \rho(s)\, [\rho(g(s))]^{n-1} p(s)\, ds = \infty \tag{26}$$

or

$$\limsup_{t\to\infty}\,[1/\rho(t)]\int_t^\infty \rho(s)\,[\rho(g(s))]^{n-1}p(s)\,ds = \infty. \tag{27}$$

(ii) The equation (19) is strongly nonoscillatory if and only if

$$\int_{t_0}^\infty \rho(s)\,[\rho(g(s))]^{n-1}p(s)\,ds < \infty \tag{28}$$

and

$$\lim_{t\to\infty}\,[1/\rho(t)]\int_t^\infty \rho(s)\,[\rho(g(s))]^{n-1}p(s) < \infty. \tag{29}$$

PROOF. If we use functions p_1 and w from Theorem 1, with $\alpha = 1$ we can easily verify that

$$\liminf_{t\to\infty}\rho(t)/\rho(g(t)) = \liminf_{s\to\infty} w(s)/s$$

and moreover

$$\int_t^\infty \rho(s)\,[\rho(g(s))]^{n-1}p(s)\,ds = \int_{1/\rho(t)}^\infty s^{n-2}p_1(s)\,ds.$$

That is why

$$\limsup_{t\to\infty}\,[1/\rho(t)]\int_t^\infty \rho(s)\,[\rho(g(s))]^{n-1}p(s)\,ds = \limsup_{t\to\infty}\, t\int_t^\infty s^{n-2}p_1(s)\,ds$$

and

$$\lim_{t\to\infty}\,[1/\rho(t)]\int_t^\infty \rho(s)\,[\rho(g(s))]^{n-1}p(s)\,ds = \lim_{t\to\infty}\, t\int_t^\infty s^{n-2}p_1(s)\,ds.$$

From this according to Theorem A we see that conditions (25) and (26) or (27) are necessary and sufficient for strong oscillation of equation (21) and similarly conditions (25) and (28) and (29) are necessary and sufficient for equation (21) to be strongly nonoscillatory. Now by application of Theorem 1 on equations (19) and (21) we find out that Theorem 7 holds and the proof is complete.

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