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Nonoscillatory solutions of systems of neutral differential equations

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1. Introduction

We consider the following system of neutral differential equations of the form

$$(1_\mu) \qquad \frac{d^n}{dt^n}[x_i(t) + (-1)^\mu a_i(t)x_i(h_i(t))] = \sum_{j=1}^{N} P_{ij}(t)f_{ij}(x_j(g_{ij}(t))),$$

$i = 1, 2,\dots, N,$ $N \geq 2,$ $n \geq 1,$ $\mu \in \{0, 1\},$ $t_0 \geq 0,$ where

(a) $a_i\colon [t_0, \infty) \to (0, \beta_i],$ $0 < \beta_i < 1,$ $h_i, g_{ij}, P_{ij}\colon [t_0, \infty) \to R,$ and $f_{ij}\colon R \to R,$ $i, j = 1, 2,\dots, N$ are continuous functions

(b) $h_i(t) \leq t,$ $t \geq t_0,$ $\lim_{t\to\infty} h_i(t) = \infty,$ $\lim_{t\to\infty} g_{ij}(t) = \infty,$ $i, j = 1,\dots, N;$

(c) $uf_{ij}(u) > 0$ for $u \neq 0,$ $i, j = 1, 2,\dots, N;$

(d) $\lim_{t\to\infty} a_i(t) = a_{i0} \in [0, \beta_i],$ $i = 1, 2,\dots, N.$

Let $t_1 > t_0$. Denote

$$t_2 = \min\{\inf_{t\geq t_1} h_i(t),\ \inf_{t\geq t_1} g_{ij}(t);\ i, j = 1,\dots, N\}.$$

A function $X = (x_1,\dots, x_N)$ is a solution of (1_μ), if there exists a $t_1 \geq t_0$ such that $X(t)$ is continuous on $[t_2, \infty)$, $x_i(t)+(-1)^\mu a_i(t)x_i(h_i(t))$, $(i = 1,\dots, N)$ are n-times continuously differentiable on $[t_1, \infty)$ and X satisfies (1_μ) on $[t_1, \infty)$.

A solution $X = (x_1,\dots, x_N)$ of (1_μ) is nonoscillatory if there exists an $a \geq t_0$ such that every its component is different from zero for all large $t \geq a$.

The asymptotic properties of nonoscillatory solutions of neutral differential equations with variable coeficients and systems of nonlinear differential equations with deviating arguments have been studied for examle in [1–3, 4, 6, 7].

In this paper we prove the existence of nonoscillatory solutions of the system (1_μ) which approach to nonzero constant vectors as $t \to \infty$.

Denote

$$(2) \qquad H_i(0, t) \equiv t,\ H_i(k, t) = H_i(k-1, h_i(t)),\ i = 1,\dots, N,\ k = 1, 2,\dots,$$

(3) $$A_i(0, t) \equiv 1,\ A_i(k, t) = \prod_{j=0}^{k-1} a_i(H_i(j, t)),\ i = 1,\dots, N,$$

2. Main results

THEOREM 1. Let the conditions (a)-(d) hold and

(4) $$\int_{t_0}^{\infty} t^{n-1} \sum_{j=1}^{N} |P_{ij}(t)|\,dt < \infty, \qquad i = 1,\dots, N.$$

Then for any $(b_1,\dots,b_N)$ $(b_i > 0, i = 1,\dots, N)$ there exists a nonoscillatory solution $X = (x_1,\dots, x_N)$ of the system (1_μ) such that $\lim_{t\to\infty} x_i(t) = b_i$ $(i = 1,\dots, N)$.

PROOF. Let $c_i > 0$, $i = 1,\dots, N$, be given constants.

(I) Let $\mu = 0$. Choose $\delta_i > 0$, $M_i > 0$, $i = 1,\dots,.N$, $T \geq t_0$ such that

$$0 < \delta_i < (1 - \beta_i)/(1 + \beta_i),$$

(5) $$M_i = \max\{f_{ij}(z);\, z \in (c_i(1 - \beta_i) - \delta_i(1 + \beta_i),\, c_i + \delta_i),\ j = 1,\dots, N\},\quad i = 1,\dots, N,$$

(6) $$\int_{T}^{\infty} (t - T)^{n-1} \sum_{j=1}^{N} |P_{ij}(t)| \leq \delta_i/M_i,\ i = 1,\dots, N$$

and

(7) $$T_0 = \min\{\inf_{t\geq T} h_i(t),\ \inf_{t\geq T} g_{ij}(t);\ i, j = 1,\dots, N\} > t_0.$$

We denote $C[t_0, \infty)$ the locally convex space of all vector continuous functions $X(t) = (x_1(t),\dots, x_N(t))$ defined on $[T_0, \infty)$, which are constant on $[T_0, T]$ with the topology of uniform convergence on any compact subinterval of $[T_0, \infty)$. Thus $C[T_0, \infty)$ is a Fréchet space.

We put

(8) $$x_i(t) + a_i(t)x_i(h_i(t)) = u_i(t),\ t \geq T,\ i = 1,\dots, N.$$

We consider the closed, convex subset S of $C[T_0, \infty)$ defined by

(9) $$S = \{U = (u_1,\dots,u_N) \in C[T_0, \infty),\ U(t) = U(T) \text{ on } [T_0, T],\ |u_i(t) - c_i| \leq \delta_i \text{ for } t \geq T,\ i = 1,\dots, N\}.$$

From (9) in view of $u_i(t) = u_i(T)$ for $t \in [T_0, T]$, $i = 1,\dots, N$, (2) and (3) we obtain for $i = 1,\dots, N$:

$$(10)\qquad x_i(t) = \begin{cases} \dfrac{u_i(T)}{1 + a_i(T)}, \ t\varepsilon[T_0, T], \\ \displaystyle\sum_{k=0}^{n_i(t)-1} (-1)^k A_i(k, t) u_i(H_i(k, t)) + (-1)^{n_i(t)}(A_i(n_i(t), t)) \\ \qquad \dfrac{u_i(T)}{1 + a_i(T)}, \ t \geq T, \end{cases}$$

where $n_i(t)$, $(i = 1,\dots, N)$ are the last positive integers such that $T_0 < H_i(n_i(t), t) \leq T$. It is easy to see that $x_i(t)$, $i = 1,\dots, N$ are continuous on $[T, \infty)$. The functions in (10) are adaptation of the function introduced in [3, 5].

We now prove if $U = (u_1,\dots, u_N) \in S$ for $t \geq T$ then

$$(11)\qquad 0 \leq \bar{c}_i - \bar{\delta}_i \leq x_i(t) \leq u_i(t) \leq c_i + \delta_i,$$

where $\bar{c}_i = c_i(1 - \beta_i)$, $\bar{\delta}_i = \delta_i(1 + \beta_i)$, $i = 1,\dots, N$. The inequalities $x_i(t) \leq u_i(t) \leq c_i + \delta_i$, $i = 1,\dots, N$ follow from (8) with regard to (9). From (10) in view of the observation $c_i - \delta_i \leq u_i(t) \leq c_i + \delta_i$ for $t \geq T$ $(i = 1,\dots, N)$, and (3) we get $x_i(t) \geq c_i - \delta_i - a_i(t)(c_i + \delta_i) + A_i(2, t)[c_i - \delta_i - a_i(H_i(2, t))(c_i + \delta_i)] + \cdots + A_i(2m_i, t)[c_i - \delta_i - a_i(H_i(2m_i, t))(c_i + \delta_i)]$ for $n_i(t) = 2m_i + 1$ or $n_i(t) = 2m_i + 2$, $m_i = 0, 1,\dots,$, $i = 1,\dots, N$. From the last inequality with regard to the assumption (a) we have

$$(12)\qquad \begin{aligned} x_i(t) &\geq [c_i - \delta_i - \beta_i(c_i + \delta_i)][1 + A_i(2, t) + \cdots + A_i(2m_i, t)] \\ &\geq c_i(1 - \beta_i) - \delta_i(1 + \beta_i), \ i = 1,\dots, N. \end{aligned}$$

We define the operator $F = (F_1,\dots, F_N)\colon S \to C[T_0, \infty)$ by

$$(13)\qquad (F_i U)(t) = \begin{cases} c_i + (-1)^n \displaystyle\int_t^\infty \frac{(s - t)^{n-1}}{(n-1)!} \sum_{j=1}^N P_{ij}(s) f_{ij}(x_j(g_{ij}(s)))\, ds, \quad t \geq T, \\ c_i + (-1)^n \displaystyle\int_T^\infty \frac{(s - T)^{n-1}}{(n-1)!} \sum_{j=1}^N P_{ij}(s) f_{ij}(x_j(g_{ij}(s)))\, ds, \\ \qquad\qquad T_0 \leq t \leq T, \quad i = 1,\dots, N. \end{cases}$$

We shall show that F maps S into inself. Let $U = (u_1,\dots, u_N) \in S$. Then using (5), (6), (11) and the assumption (c), we get for $t \geq T$ and $i = 1,\dots, N$:

$$\begin{aligned} (F_i U)(t) &\leq c_i + \int_T^\infty (s - T)^{n-1} \sum_{j=1}^N |P_{ij}(s)| f_{ij}(x_j(g_{ij}(s)))\, ds \\ &\leq c_i + M_i \int_T^\infty (s - T)^{n-1} \sum_{j=1}^N |P_{ij}(s)|\, ds \leq c_i + \delta_i, \end{aligned}$$

$$(F_iU)(t) \geq c_i - \int_T^\infty (s-T)^{n-1} \sum_{j=1}^N |P_{ij}(s)| f_{ij}(x_j(g_{ij}(s))\, ds$$

$$\geq c_i - M_i \int_T^\infty (s-T)^{n-1} \sum_{j=1}^N |P_{ij}(s)|\, ds \geq c_i - \delta_i.$$

We prove that F is continuous. Let $U_r = (u_{1r}, \dots, u_{Nr}) \in S$ for $r = 1, 2, \dots$, and $u_{ir} \to u_i$ for $r \to \infty$, $i = 1, \dots, N$ in the space $C[T_0, \infty)$. Denote

$$x_{ir}(t) = \sum_{k=0}^{n_i(t)-1} (-1)^k A_i(k, t)\, u_{ir}(H_i(k, t)) + (-1)^{n_i(t)} A_i(n_i(t), t)$$

$$u_{ir}(T)/(1 + a_i(T)),\ t \geq T,\ i = 1, \dots, N,\ r = 1, 2, \dots.$$

From (13) we obtain for $i = 1, \dots, N$:

$$|(F_iU_r)(t) - (F_iU)(t)| \leq \int_T^\infty (s-T)^{n-1} \sum_{j=1}^N |P_{ij}(s)| \times$$

$$|f_{ij}(x_{jr}(g_{ij}(s))) - f_{ij}(x_j(g_{ij}(s)))|\, ds \leq \int_T^\infty s^{n-1} P_i^r(s)\, ds,$$

where

$$P_i^r(t) = \sum_{j=1}^N |P_{ij}(s)|\, |f_{ij}(x_{jr}(g_{ij}(t)) - f_{ij}(x_j(g_{ij}(t)))|,\ t \geq T.$$

It is easy to see that $\lim_{r \to \infty} P_i^r(t) = 0$ and

$$P_i^r(t) \leq 2M_i \sum_{j=1}^N |P_{ij}(t)|,\ t \geq T,\ i = 1, \dots, N.$$

With regard to (4) and the Lebesque's dominant convergence theorem we get $(F_iU_r)(t) \to (F_iU)(t)$ uniformly in $C[T_0, 0)$ for $r \to \infty$, $i = 1, \dots, N$. This implies the continuity of F.

Using the Arzela-Ascoli theorem we can prove in a routine manner that $F(S)$ is relative compact in the topology of $C[T_0, \infty)$. Therefore by the Schauder-Tychonov fixed point theorem, there exists a $\bar{U} = (\bar{u}_1, \dots, \bar{u}_N) \in S$ such that $F\bar{U} = \bar{U}$. The components of $\bar{U}$ satisfy the following system:

$$\bar{u}_i(t) = c_i - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \sum_{j=1}^N P_{ij}(s) f_{ij}(\bar{x}_j(g_{ij}(s)))\, ds,\quad t \geq T,$$

(14)

$$i = 1, \dots, N,$$

for which

$$\lim_{t\to\infty} \bar{u}_i(t) = c_i > 0, \qquad i = 1,\dots,N.$$

The system (14) in view of (8) can be rewritten as

$$\bar{x}_i(t) + a_i(t)\bar{x}_i(h_i(t)) = c_i - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \sum_{j=1}^N P_{ij}(s) \times \tag{15}$$
$$f_{ij}(\bar{x}_j(g_{ij}(s)))\,ds,\ t \ge T,\ i = 1,\dots,N.$$

From (10) with regard to (a), (d) and (12) there exist $b_i > 0$, $i = 1,\dots,N$ such that $\lim_{t\to\infty} x_i(t) = b_i$. Differentiating (15) we get $\bar{X}(t) = (\bar{x}_1(t),\dots,\bar{x}_N(t))$ is a nonoscillatory solution of the system (1_0) on $[T, \infty)$ such that $\lim_{t\to\infty} \bar{x}_i(t) = b_i$. $i = 1,\dots,N$.

(II) Let $\mu = 1$. Choose $d_i > 0$, $\bar{M}_i > 0$ $(i = 1,\dots,N)$, $T \ge t_0$ such that

$$0 < d_i < c_i,\ \bar{M}_i = \max\{f_{ij}(z)\colon z \in (c_i - d_i, (c_i + d_i)/(1-\beta_i)),\ 1 \le i \le N\},$$
$$i = 1,\dots,N,$$

$$\int_T^\infty (t-T)^{n-1} \sum_{j=1}^N |P_{ij}(t)|\,dt < d_i/\bar{M}_i, \qquad i = 1,\dots,N, \tag{16}$$

and (7) hold.

We put

$$x_i(t) - a_i(t)x_i(h_i(t)) = v_i(t) \qquad \text{for } t \ge T,\ i = 1,\dots,N. \tag{17}$$

Let $S_1 = \{V = (v_1,\dots,v_n) \in C[T_0, \infty),\ V(t) = V(T) \text{ on } [T_0, T],\ |v_i(t) - c_i| \le d_i \text{ for } t \ge T,\ i = 1,\dots,N\}$.

From (17) in view of $v_i(t) = v_i(T)$ for $t \in [T_0, T]$, $i = 1,\dots,N$, (2) and (3) we obtain for $i = 1,\dots,N$:

$$x_i(t) = \begin{cases} \dfrac{v_i(T)}{1 - a_i(T)}, & t \in [T_0, T], \\ \displaystyle\sum_{k=0}^{n_i(t)-1} A_i(k, t)v_i(H_i(k, t)) + A_i(n_i(t), t)\frac{v_i(T)}{1 - a_i(T)}, & t \ge T, \end{cases} \tag{18}$$

where $n_i(t)$, $i = 1,\dots,n$ are as in the case (I). From (18) with regard to (17), $V = (v_1,\dots,v_n) \in S_1$ and the assumption (a) we get $c_i - d_i \le v_i(t) \le x_i(t) \le (c_i + d_i)/(1 - \beta_i)$ for $i = 1,\dots,N$.

Define the operator $F = (F_1,\dots,F_N)\colon S_1 \to C[T_0, \infty)$ by (12), in which $u_i(t)$ we replace by $v_i(t)$, $i = 1,\dots,N$.

Proceeding in the same way as in the case (I) we show that there exists a fixed point $\bar{V}=(\bar{v}_1,\dots,\bar{v}_N)\in S_1$, $F\bar{V}=\bar{V}$ and for its components the following holds: $\lim_{t\to\infty}\bar{v}_i(t)=\lim_{t\to\infty}(\bar{x}_i(t)-a_i(t)\bar{x}_i(h_i(t)))=c_i>0$, $i=1,\dots,N$. Then it is easy to see that $(\bar{x}_1(t),\dots,\bar{x}_N(t))$ satisfies the system (1_1) on $[T,\infty)$ and $\lim_{t\to\infty}\bar{x}_i(t)=\bar{b}_i$ for some $\bar{b}_i\in[c_i,(c_i+d_i)/(1-\beta_i)]$, $i=1,\dots,N$.

The proof of theorem is complete.

Let now

$$p_{ij}(t)=\sigma_i q_{ij}(t),\ \sigma_i\in\{-1,1\},\ q_{ij}\colon[t_0,\infty)\longrightarrow(0,\infty)\quad\text{for all } i,j=1,\dots,N. \tag{19}$$

THEOREM 2. Let the assumptions (a)-(d), (19) hold. System (1_μ) has a nonoscillatory solution $(x_1,\dots,x_N)$ with the property

$$\lim_{t\to\infty}x_i(t)=c_i>0,\qquad i=1,\dots,N \tag{20}$$

if and only if

$$\int_{t_0}^{\infty}t^{n-1}\sum_{j=1}^{N}q_{ij}(t)\,dt<\infty,\qquad i=1,\dots.N. \tag{21}$$

PROOF. (i) Let $c_i>0$, $i=1,\dots,N$, be fixed constants and $X(t)=(x_1,\dots,x_N)$ be a nonoscillatory solution of (1_μ), which satisfies (20). If we put $y_i(t)=x_i(t)+(-1)^\mu a_i(t)x_i(h_i(t))$, then with regard to (a) and (20) we obtain

$$\lim_{t\to\infty}y_i^{(k)}(t)=0,\ k=1,\dots,n-1,\ i=1,\dots,N. \tag{22}$$

Integrating $(1_\mu)(n-1)$-times from $t(\geq t_0)$ to $\tau\to\infty$ and using (22) we have for $i=1,\dots,N$:

$$\sigma_i(-1)^{n-1}y_i'(t)=\int_t^{\infty}\frac{(s-t)^{n-2}}{(n-2)!}\sum_{j=1}^{N}q_{ij}(s)f_{ij}(x_j(g_{ij}(s)))\,ds. \tag{23}$$

In view of (a), (c) and (20) there exist $\delta>0$, $T_1>t_0$ such that

$$f_{ij}(x_j(g_{ij}(t)))\geq\delta\qquad\text{for } t\geq T_1,\ i,j=1,\dots,N. \tag{24}$$

Then integrating (23) from T_1 to $\tau\to\infty$, using (20), (24) and (d) we get for $i=1,\dots,N$:

$$(-1)^n\sigma_i[c_i(1+(-1)^\mu a_{i0})-y_i(T)]\geq\delta\int_{T_1}^{\infty}\frac{(s-T_1)^{n-1}}{(n-1)!}\sum_{j=1}^{N}q_{ij}(s)\,ds.$$

From the last inequalities after modification we get (21).

(ii) The "if" part follows from Theorem 1.

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