Hiroshima Math. J.
23 (1993), 79–153

Continuity properties of potentials and Beppo-Levi-Deny functions

Dedicated to Professor M. Ohtsuka on the occasion of his seventieth birthday

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(Received November 11, 1991)

1. Introduction

In this paper we first study the behavior of Riesz potentials of functions near a given point, which may be assumed, without loss of generality, to be the origin. For $0 < \alpha < n$ and a nonnegative measurable function f on R^n, we define $U_\alpha f$ by

$$U_\alpha f(x) = \int_{R^n} |x - y|^{\alpha - n} f(y)\, dy.$$

It is easy to see that $U_\alpha f \not\equiv \infty$ if and only if

$$\int_{R^n} (1 + |y|)^{\alpha - n} f(y)\, dy < \infty. \tag{1.1}$$

By Sobolev's imbedding theorem, we know that if f is a nonnegative function in $L^p(R^n)$ satisfying (1.1), and if $\alpha p > n$, then $U_\alpha f$ is continuous at the origin (in fact, on R^n); however, in case $\alpha p \leq n$, $U_\alpha f$ may fail to be continuous at the origin. Thus, our main concern in this paper is the bordering case $p = n/\alpha$, and one of our aims is to find a condition on f, which is stronger than the condition that $f \in L^p(R^n)$ with $p = n/\alpha$ but assures the continuity at 0 of $U_\alpha f$.

For this purpose, we assume that f satisfies a condition of the form:

$$\int_{R^n} \Phi_p(f(y))\omega(|y|)\, dy < \infty. \tag{1.2}$$

Here $\Phi_p(r)$ and $\omega(r)$ are positive monotone functions on the interval $(0, \infty)$ with the following properties:

(φ1) $\Phi_p(r)$ is of the form $r^p \varphi(r)$, where $1 \leq p < \infty$ and φ is a positive nondecreasing function on the interval $[0, \infty)$.

(φ2) φ is of logarithmic type, that is, there exists $A_1 > 0$ such that

$$A_1^{-1}\varphi(r) \le \varphi(r^2) \le A_1\varphi(r) \qquad \text{whenever} \quad r > 0.$$

($\omega 1$) ω satisfies the (Δ_2) condition; that is, there exists $A_2 > 0$ such that

$$A_2^{-1}\omega(r) \le \omega(2r) \le A_2\omega(r) \qquad \text{whenever} \quad r > 0.$$

For example, $\varphi(r) = [\log(2+r)]^{\delta}$, $\delta \ge 0$, and $\omega(r) = r^{\beta}$ satisfy all the conditions. We know in [18] that if $\omega \equiv 1$, $p > 1$ and

$$\int_0^1 [\varphi(r^{-1})]^{-1/(p-1)} \frac{dr}{r} < \infty, \tag{1.3}$$

then $U_\alpha f$ is continuous on R^n. Thus we aim to find a more general condition relating to both φ and ω, under which $U_\alpha f$ is continuous at the origin. Further, if $U_\alpha f$ is not continuous at 0, then we shall find a function κ for which $[\kappa(|x|)]^{-1} U_\alpha f(x)$ tends to zero as $x \to 0$, possibly avoiding an exceptional set. As an application of the existence of such fine limits, the radial limit theorems can be derived. Our results will give generalizations of those in [5] and [11], where $\varphi(r) \equiv 1$ and $\omega(r)$ is of the form r^{β}.

We also deal with the limit of q-th means of $U_\alpha f$ over the spheres $\partial B(0, r)$, where $\partial B(x, r)$ denotes the boundary of the open ball $B(x, r)$ with center at x and radius r. In case $p = 1$, our results imply Gardiner's results in [4].

If α is a positive integer, then $U_\alpha f$ is a Beppo-Levi-Deny function on R^n (cf. Mizuta [8]); for the definition of Beppo-Levi-Deny functions, we refer the reader to Deny-Lions [3] and Mizuta [8]. Conversely, Beppo-Levi-Deny functions are represented as Riesz type potentials in [8], [16] and [19], as an extension of a result by Wallin [26]. In this paper, we give another integral representation, as a generalization of the sobolev integral representation for infinitely differentiable functions with compact support.

Moreover, we are concerned with Beppo-Levi-Deny functions u on the half space $D = \{x = (x_1, \ldots, x_n) \in R^n;\ x_n > 0\}$ satisfying

$$\sum_{|\lambda| = m} \int_G \Phi_p(|(\partial/\partial x)^{\lambda} u(x)|)\omega(x_n)\, dx < \infty \tag{1.4}$$

for any bounded open set $G \subset D$, and study the existence of limits along curves or sets tangential to the boundary ∂D, where $n \ge 2$ and $(\partial/\partial x)^{\lambda} = (\partial/\partial x_1)^{\lambda_1} \cdots (\partial/\partial x_n)^{\lambda_n}$ for a point $x = (x_1, \ldots, x_n)$ and a multi-index $\lambda = (\lambda_1, \ldots, \lambda_n)$ with length $|\lambda| = \lambda_1 + \cdots + \lambda_n$. If φ satisfies condition (1.3), then u is continuous on D as shown in [18]. We show that u has limits along the sets

$$T_{\psi}(\xi, a) = \{x \in D;\ \psi(|x - \xi|) < a x_n\},$$

where $\xi \in \partial D$, $a > 0$ and ψ is a positive nondecreasing function on the interval

$(0, \infty)$. In case $\psi(r) = r$, such limits are called nontangential limits; in case $\psi(r) = r^{\beta}$, $\beta > 1$, they are called tangential limits. First we prepare some results concerning the existence of limits at points of ∂D for Riesz potentials $U_\alpha f$ with nonnegative measurable functions f satisfying (1.1) and

$$\int_G \Phi_p(f(y))\omega(|y_n|)\,dy < \infty \qquad \text{for any bounded open set} \quad G \subset R^n,$$

and then apply the same discussions to the study of boundary limits of Beppo-Levi-Deny functions u on D satisfying condition (1.4), with the aid of the integral representations. Nagel, Rudin and Shapiro [20] proved the existence of (non) tangential limits of harmonic functions represented as Poisson integrals in D. Their results will correspond to ours in the case where $\alpha p > n$ or condition (1.3) holds. The size of the exceptional sets of ξ, at which $U_\alpha f$ or u fails to have a boundary limit under consideration, will be evaluated by Hausdorff measures and Bessel type capacities.

Our arguments are applicable to the study of boundary limits of Green potentials $G_\alpha f$ defined by

$$G_\alpha f(x) = \begin{cases} \displaystyle\int_D \{|x-y|^{\alpha-n} - |\bar{x}-y|^{\alpha-n}\} f(y)\,dy & \text{in case} \quad \alpha < n, \\ \displaystyle\int_D \log(|\bar{x}-y|/|x-y|) f(y)\,dy & \text{in case} \quad \alpha = n, \end{cases}$$

where $\bar{x} = (x_1, \dots, x_{n-1}, -x_n)$ for $x = (x_1, \dots, x_{n-1}, x_n)$ and f is a nonnegative measurable function on D satisfying

$$\int_{D'} \Phi_p(f(y))\omega(y_n)\,dy < \infty \qquad \text{for any bounded open set} \quad D' \subset D.$$

We try to give generalizations of results in Aikawa [1], Mizuta [14], Rippon [23] and Wu [27].

In the last section, we investigate continuity properties for logarithmic potentials Lf in R^n, which is defined by

$$Lf(x) = \int \log \frac{1}{|x-y|} f(y)\,dy;$$

here it is natural to assume

$$\int \log(2 + |y|)|f(y)|\,dy < \infty.$$

We note that if $f \in L^p(R^n)$ with $p > 1$, then Lf is continuous on R^n. Thus we deal mainly with functions f satisfying

$$\int \Phi_1(|f(y)|)\omega(|y|)\,dy < \infty,$$

and give extensions of the results in [15].

The author is grateful to Professor Fumi-Yuki Maeda for a number of useful suggestions and improvements.

2. Preliminary lemmas

First we give several properties which follow from conditions ($\varphi 1$) and ($\varphi 2$):

($\varphi 3$) φ satisfies the (Δ_2) condition, that is, there exists $A_3 > 1$ such that

$$\varphi(2r) \le A_3 \varphi(r) \qquad \text{whenever} \quad r > 0.$$

($\varphi 4$) For any $\gamma > 0$, there exists $A(\gamma) > 1$ such that

$$A(\gamma)^{-1}\varphi(r) \le \varphi(r^\gamma) \le A(\gamma)\varphi(r) \qquad \text{whenever} \quad r > 0.$$

($\varphi 5$) If $\gamma > 0$, then

$$s^\gamma \varphi(s^{-1}) \le A_1 t^\gamma \varphi(t^{-1}) \qquad \text{whenever} \quad 0 < s < t < A_1^{-1/\gamma}.$$

Throughout this paper, let $M, M_1, M_2, \ldots,$ denote various constants independent of the variables in question.

For $x \in R^n - \{0\}$, the Riesz potential $U_\alpha f$ of f satisfying (1.1) will be written as $U_1 + U_2 + U_3$, where

$$U_1(x) = \int_{R^n - B(0,2|x|)} |x-y|^{\alpha-n} f(y)\,dy,$$

$$U_2(x) = \int_{B(0,2|x|) - B(x,|x|/2)} |x-y|^{\alpha-n} f(y)\,dy,$$

$$U_3(x) = \int_{B(x,|x|/2)} |x-y|^{\alpha-n} f(y)\,dy.$$

Then we can easily find a positive constant M such that

$$U_1(x) \le M \int_{R^n - B(0,2|x|)} |y|^{\alpha-n} f(y)\,dy \tag{2.1}$$

and

(2.2) $$U_2(x) \le M|x|^{\alpha-n}\int_{B(0,2|x|)} f(y)\,dy.$$

LEMMA 2.1. *Let $p > 1$, $0 < \delta < \beta \le n$ and f be a nonnegative measurable function on R^n. If $0 \le 2r < a < 1$, then*

$$\int_{R^n - B(0,r)} |y|^{\beta-n} f(y)\,dy \le \int_{R^n - B(0,a)} |y|^{\beta-n} f(y)\,dy + Ma^{\beta-\delta}$$
$$+ M\left(\int_r^a [t^{n-\beta p}\eta(t)]^{1/(1-p)} t^{-1}\,dt\right)^{1-1/p} \left(\int_{B(0,a)} \Phi_p(f(y))\omega(|y|)\,dy\right)^{1/p},$$

where $\eta(t) = \varphi(t^{-1})\omega(t)$ and M is a positive constant independent of x and a.

PROOF. Let $0 < a < 1$ and assume that $f = 0$ outside $B(0, a)$. We write

$$\int_{R^n - B(0,r)} |y|^{\beta-n} f(y)\,dy = \int_{\{y\in R^n - B(0,r);\, f(y) > |y|^{-\delta}\}} |y|^{\beta-n} f(y)\,dy$$
$$+ \int_{\{y\in R^n - B(0,r);\, 0 < f(y) \le |y|^{-\delta}\}} |y|^{\beta-n} f(y)\,dy$$
$$= U_{11}(x) + U_{12}(x).$$

From Hölder's inequality, we obtain

$$U_{11}(x) \le \left(\int_{\{y\in R^n - B(0,r);\, f(y) > |y|^{-\delta}\}} f(y)^p \varphi(f(y))\omega(|y|)\,dy\right)^{1/p}$$
$$\times \left(\int_{\{y\in R^n - B(0,r);\, f(y) > |y| - \delta\}} |y|^{\beta-n}[\varphi(f(y))\omega(|y|)]^{-p'/p}\,dy\right)^{1/p'},$$

where $1/p + 1/p' = 1$. By condition ($\varphi 4$), we see that

$$\varphi(f(y)) \ge \varphi(|y|^{-\delta}) \ge M_1\varphi(|y|^{-1})$$

whenever $f(y) > |y|^{-\delta}$. Hence

$$U_{11}(x) \le M_2\left(\int_r^a [t^{n-\beta p}\eta(t)]^{-p'/p} t^{-1}\,dt\right)^{1/p'} \left(\int_{R^n - B(0,r)} \Phi_p(f(y))\omega(|y|)\,dy\right)^{1/p}.$$

On the other hand,

$$U_{12}(x) \le M_3 \int_{B(0,a) - B(0,r)} |y|^{\beta-\delta-n}\,dy \le M_3 a^{\beta-\delta}.$$

Thus Lemma 2.1 is proved.

For $\eta(r) = \varphi(r^{-1})\omega(r)$, set

$$\kappa_1(r) = \begin{cases} \left(\int_r^1 [t^{n-\alpha p}\eta(t)]^{-p'/p} t^{-1}\, dt \right)^{1/p'}, & \text{in case} \quad p > 1, \\ \sup_{r \le t < 1} t^{\alpha - n} [\eta(t)]^{-1}, & \text{in case} \quad p = 1, \end{cases}$$

where $0 < r \le 1/2$; further, set $\kappa_1(r) = \kappa_1(1/2)$ when $r > 1/2$.

COROLLARY 2.1. *Let $0 < \delta < \alpha$ and f be a nonnegative measurable function on R^n. If $0 < 2|x| < a < 1$, then*

$$U_1(x) \le \int_{R^n - B(0,a)} |x-y|^{\alpha - n} f(y)\, dy + M a^{\alpha - \delta}$$

$$+ M\kappa_1(|x|) \left(\int_{B(0,a)} \Phi_p(f(y))\omega(|y|)\, dy \right)^{1/p},$$

where M is a positive constant independent of x and a.

The case $p > 1$ follows readily from (2.1) and Lemma 2.1 with $\beta = \alpha$ and $r = |x|$, and the case $p = 1$ is trivial.

By using (2.2) and the case $\beta = n$ in Lemma 2.1, we can establish the following result.

COROLLARY 2.2. *If $0 < \delta < \alpha$, then there exists a positive constant M such that*

$$U_2(x) \le M\kappa_2(|x|) \left(\int_{B(0,2|x|)} \Phi_p(f(y))\omega(|y|)\, dy \right)^{1/p} + M|x|^{\alpha - \delta}$$

for any $x \in B(0, 1/2) - \{0\}$, where

$$\kappa_2(r) = \begin{cases} r^{\alpha - n} \left(\int_0^r [\eta(t)]^{-p'/p} t^{n-1}\, dt \right)^{1/p'}, & \textit{in case} \quad p > 1, \\ r^{\alpha - n} \sup_{0 < t \le r} [\eta(t)]^{-1}, & \textit{in case} \quad p = 1. \end{cases}$$

For a set $E \subset R^n$ and an open set $G \subset R^n$, we define

$$C_{\alpha, \Phi_p}(E; G) = \inf_g \int_G \Phi_p(g(y))\, dy,$$

where the infinum is taken over all nonnegative measurable functions g on R^n such that g vanishes outside G and $U_\alpha g(x) \ge 1$ for every $x \in E$.

The following results can be proved easily by the definition of C_{α, Φ_p} (cf. [11, Lemmas 1 and 2]).

LEMMA 2.2. *Let G and G' be bounded open sets in R^n.*

(i) $C_{\alpha,\Phi_p}(\cdot\,; G)$ *is countably subadditive.*

(ii) *If F is a compact subset of $G \cap G'$, then there exists $M > 0$ such that*

$$C_{\alpha,\Phi_p}(E; G) \le M C_{\alpha,\Phi_p}(E; G') \qquad \textit{for any} \quad E \subset F.$$

(iii) *If $C_{\alpha,\Phi_p}(E; G) = 0$, then $C_{\alpha,\Phi_p}(E \cap G'; G') = 0$.*

(iv) *If $C_{\alpha,\Phi_p}(E; G) = 0$, $E \subset G$, then, for any positive nonincreasing function ω on $(0, \infty)$, there exists a nonnegative measurable function f on G such that $U_\alpha f \not\equiv \infty$, $U_\alpha f = \infty$ on E and $\int_G \Phi_p(f(y))\omega(\rho(y))\,dy < \infty$, where $\rho(y)$ denotes the distance of y from the boundary ∂G.*

For the reader's convenience, we give a proof for (iv). Let $\{a_j\}$ be a sequence of positive numbers. If we define $G_j = \{x \in G; \rho(x) > j^{-1}\}$ for each positive integer j, then $C_{\alpha,\Phi_p}(E \cap G_j; G_j) = 0$ by (iii). Hence, for each j, we can find a nonnegative measurable function f_j on G_j such that $U_\alpha f_j \ge 1$ on $E \cap G_j$ and $\int_{G_j} \Phi_p(f_j(y))\,dy < a_j$. Consider the function $f = \sup_j 2^j f_j$. Then $U_\alpha f(x) \ge 2^j U_\alpha f_j(x) \ge 2^j$ for $x \in E \cap G_j$, so that

$$U_\alpha f(x) = \infty \qquad \text{on} \quad E.$$

On the other hand, $M = \sup_{r>0} \Phi_p(2r)/\Phi_p(r) < \infty$ and hence

$$\begin{aligned}\int \Phi_p(f(y))\omega(\rho(y))\,dy &\le \textstyle\sum_j \int_{G_j} \Phi_p(2^j f_j(y))\omega(\rho(y))\,dy \\ &\le \textstyle\sum_j M^j \omega(j^{-1}) \int_{G_j} \Phi_p(f_j(y))\,dy \\ &\le \textstyle\sum_j M^j \omega(j^{-1}) a_j.\end{aligned}$$

Now choose $\{a_j\}$ so that the last sum is convergent.

LEMMA 2.3. *Let f be a nonnegative function satisfying condition* (1.2), *and χ be a positive function on* $(0, 1]$ *for which there is a positive constant M such that $\chi(r) \le M\chi(s)$ whenever $0 < r \le s \le 2r \le 1$. Then there exists a set $E \subset R^n$ such that*

(i) $\lim_{x \to 0, x \in R^n - E} [\chi(|x|)]^{-1} U_3(x) = 0$;

(ii) $\sum_{j=1}^{\infty} [K^*]^{-j} \omega(2^{-j}) C_{\alpha,\Phi_p}(E_j; B_j) < \infty$,

where

$$E_j = \{x \in E; 2^{-j} \le |x| < 2^{-j+1}\},$$
$$B_j = \{x \in R^n; 2^{-j-1} < |x| < 2^{-j+2}\},$$

$$K^* = \sup_{0<r,s<1/2} \frac{\Phi_p(s/\chi(r))}{\Phi_p(s/\chi(2r))}.$$

PROOF. For a sequence $\{a_j\}$ of positive numbers, we set

$$E_j = \{x \in R^n;\ 2^{-j} \leq |x| < 2^{-j+1},\ U_3(x) \geq a_j^{-1}\chi(|x|)\}, \qquad j = 1, 2, \ldots,$$

and

$$E = \bigcup_{j=1}^{\infty} E_j.$$

Since $U_3(x) \leq \int_{B_j} |x-y|^{\alpha-n} f(y)\,dy$ if $x \in E_j$, we have by the definition of C_{α,Φ_p},

$$C_{\alpha,\Phi_p}(E_j; B_j) \leq \int_{B_j} \Phi_p(M_1 a_j [\chi(2^{-j})]^{-1} f(y))\,dy$$
$$\leq K^{*j} \int_{B_j} \Phi_p(M_1 a_j [\chi(1)]^{-1} f(y))\,dy.$$

By condition (1.2) we can find a sequence $\{b_j\}$ of positive numbers such that $\lim_{j\to\infty} b_j = \infty$ but

$$\sum_{j=1}^{\infty} \int_{B_j} b_j \Phi_p(f(y))\omega(|y|)\,dy < \infty.$$

By (φ3) there exists $\varepsilon_0 > 1$ such that $\varphi(st)/\varphi(t) \leq M_2 s^{\varepsilon_0}$ whenever $s > 1$ and $t > 0$. Now let $a_j^{p+\varepsilon_0} = b_j$. Then, since $\sum_{j=1}^{\infty} \int_{B_j} \Phi_p(a_j f(y))\omega(|y|)\,dy < \infty$, it follows that

$$\sum_{j=1}^{\infty} [K^*]^{-j} \omega(2^{-j}) C_{\alpha,\Phi_p}(E_j; B_j) < \infty.$$

Since (i) follows readily, Lemma 2.3 is established.

REMARK 2.1. If $\Phi_p(r) = r^p$, $\omega(r) = r^\beta$ and $\chi(r) = r^{-(n-\alpha p+\beta)/p}$, then (ii) implies

$$\sum_{j=1}^{\infty} 2^{-j(n-\alpha p)} C_{\alpha,p}(E_j; B_j) < \infty,$$

where $C_{\alpha,p} = C_{\alpha,\Phi_p}$ is the usual (α, p)-capacity.

3. Fine limits

Our first aim is to establish the following result.

THEOREM 3.1. *If f is a nonnegative measurable function on R^n satisfying conditions* (1.1) *and* (1.2), *then there exists a set $E \subset R^n$ such that*

$$\lim_{x\to 0, x\in R^n - E} U_\alpha f(x) = U_\alpha f(0)$$

and

$$\sum_{j=1}^{\infty} \omega(2^{-j}) C_{\alpha,\Phi_p}(E_j; B_j) < \infty,$$

where E_j and B_j are as in Lemma 2.3.

PROOF. If $U_\alpha f(0) = \infty$, then, by the lower semicontinuity of $U_\alpha f$, we see that $\lim_{x\to 0} U_\alpha f(x) = \infty = U_\alpha f(0)$.

If $U_\alpha f(0) < \infty$, then Lebesgue's dominated convergence theorem implies

$$\lim_{x\to 0} [U_1(x) + U_2(x)] = U_\alpha f(0),$$

since $|x-y|^{\alpha-n} \le 3^{n-\alpha}|y|^{\alpha-n}$ for $y \in R^n - B(x, |x|/2)$. Thus Lemma 2.3 with $\chi \equiv 1$ yields the required assertion.

In case $U_\alpha f(0) = \infty$, we discuss the order of infinity at the origin.

THEOREM 3.2. *Let f be a nonnegative measurable function on R^n satisfying conditions* (1.1) *and* (1.2). *Set $\kappa = \kappa_1 + \kappa_2$. If $\lim_{r\to 0} \kappa(r) = \infty$, then there exists a set $E \subset R^n$ such that*

$$\lim_{x\to 0, x\in R^n - E} [\kappa(|x|)]^{-1} U_\alpha f(x) = 0$$

and

$$\sum_{j=1}^{\infty} K^{-j} \omega(2^{-j}) C_{\alpha,\Phi_p}(E_j; B_j) < \infty,$$

where E_j and B_j are as before, and

$$K = \sup_{0<r,s<1/2} [\Phi_p(s/\kappa(r))]/[\Phi_p(s/\kappa(2r))].$$

PROOF. By Corollary 2.1, we have

$$\limsup_{x\to 0} [\kappa(|x|)]^{-1} U_1(x) \le M\left(\int_{B(0,a)} \Phi_p(f(y))\omega(|y|)\,dy\right)^{1/p}$$

for any $a > 0$, which implies that the left hand side is equal to zero. Further, from Corollary 2.2 it follows that

$$\lim_{x\to 0} [\kappa(|x|)]^{-1} U_2(x) = 0.$$

Thus, applying Lemma 2.3 with $\chi = \kappa$, we can complete the proof of Theorem 3.2.

EXAMPLE 3.1. In case $\eta(r) = r^\beta$, where $\alpha p - n \le \beta \le (p-1)n$, we see that

$$\kappa(r) \sim r^{-(n-\alpha p+\beta)/p} \times \begin{cases} 1 & \text{if } \alpha p - n < \beta < n(p-1) \\ \{\log(1/r)\}^{1-1/p} & \text{if } \beta = \alpha p - n \text{ or } \beta = n(p-1) \end{cases}$$

as $r \to 0$. In addition, if $\omega(r) = r^{\beta}$ (and hence $\varphi(r) \equiv 1$), then E in Theorem 3.2 satisfies

$$\sum_{j=1}^{\infty} 2^{-j(n-\alpha p)} C_{\alpha,p}(E_j; B_j) < \infty.$$

Therefore, by use of the inversion: $x \to x/|x|^2$, Theorem 3.2 gives a generalization of Theorem 4.5 in [5].

If $p > 1$ and

$$\int_0^1 [t^{n-\alpha p}\varphi(t^{-1})]^{-p'/p} t^{-1}\,dt < \infty, \tag{3.1}$$

then we consider the function

$$K(r) = \kappa(r) + [\omega(r)]^{-1/p}\varphi^*(r),$$

where

$$\varphi^*(r) = \left(\int_0^r [t^{n-\alpha p}\varphi(t^{-1})]^{-p'/p} t^{-1}\,dt\right)^{1/p'}.$$

Here note that

$$\varphi^*(r) \geq M[r^{n-\alpha p}\varphi(r^{-1})]^{-1/p} \tag{3.2}$$

and

$$K(r) \geq M[r^{n-\alpha p}\eta(r)]^{-1/p} \tag{3.3}$$

for $r > 0$.

THEOREM 3.3. *Let $p > 1$ and assume that* (3.1) *holds. If f is as in Theorem* 3.2 *and* $\lim_{r\to 0} K(r) = \infty$, *then*

$$\lim_{x\to 0} [K(|x|)]^{-1} U_{\alpha}f(x) = 0.$$

If $K(r)$ is bounded, then $U_{\alpha}f(0)$ is finite and $U_{\alpha}f(x)$ tends to $U_{\alpha}f(0)$ as $x \to 0$.

COROLLARY 3.1 (cf. Theorem 1 in [18]). *Let $p = n/\alpha > 1$ and $\varphi^*(1) < \infty$. If f is a nonnegative measurable function on R^n satisfying* (1.1) *and $\int \Phi_p(f(y))\,dy < \infty$, then $U_{\alpha}f$ is continuous on R^n in the usual sense.*

PROOF OF THEOREM 3.3. Let $0 < \delta < \alpha$. Since

$$U_3(x) = \int_{B(0,|x|/2)} |y|^{\alpha-n} f(x+y)\,dy,$$

we have by Lemma 2.1

$$U_3(x) \le M_1\left(\int_0^{|x|/2}[r^{n-\alpha p}\varphi(r^{-1})]^{-p'/p}r^{-1}\,dr\right)^{1/p'}$$
$$\times\left(\int_{B(0,|x|/2)}\Phi_p(f(x+y))\,dy\right)^{1/p} + M_1|x|^{\alpha-\delta}$$
$$\le M_2\varphi^*(|x|)[\omega(|x|)]^{-1/p}\left(\int_{B(x,|x|/2)}\Phi_p(f(y))\omega(|y|)\,dy\right)^{1/p} + M_1|x|^{\alpha-\delta}.$$

If $K(r)\to\infty$ as $r\to 0$, then it follows that

$$\lim_{x\to 0}[K(|x|)]^{-1}U_3(x) = 0.$$

As in the proof of Theorem 3.2, we have

$$\lim_{x\to 0}[K(|x|)]^{-1}\{U_1(x)+U_2(x)\} = 0,$$

and hence

$$\lim_{x\to 0}[K(|x|)]^{-1}U_\alpha f(x) = 0.$$

If $K(r)$ is bounded, then $U_3(x)\to 0$ as $x\to 0$. Also, Corollary 2.1 implies

$$\limsup_{x\to 0} U_1(x) < \infty,$$

and Corollary 2.2 implies that $U_2(x)$ tends to zero as $x\to 0$. It follows that $U_\alpha f(0) < \infty$ and

$$\lim_{x\to 0} U_\alpha f(x) = \lim_{x\to 0}\{U_1(x)+U_2(x)\} = U_\alpha f(0)$$

as in the proof of Theorem 3.1. Thus we complete the proof of Theorem 3.3.

Here we discuss the best-possibility of Theorem 3.3 as to the order of infinity.

PROPOSITION 3.1. *Let $\alpha p = n$, and suppose $\varphi^*(1) < \infty$,*

$$\lim_{r\to 0}[\omega(r)]^{-1/p}\varphi^*(r) = \infty \quad \textit{and} \quad \lim_{r\to 0} r^{n/p'}[\omega(r)]^{-1/p}\varphi^*(r) = 0.$$

Then, for any positive nondecreasing function on $a(r)$ on $(0,\infty)$ such that $\lim_{r\to 0}a(r)=\infty$, there exists a nonnegative measurable function f on R^n satisfying (1.1) *and* (1.2) *such that*

$$\limsup_{x\to 0} a(|x|)[\omega(|x|)]^{1/p}[\varphi^*(|x|)]^{-1}U_\alpha f(x) = \infty.$$

PROOF. Let $\{j_i\}$ be a sequence of positive integers such that $j_i + 2 < j_{i+1}$ and $\sum_i a_i^{-1/p} < \infty$, where $a_i = a(r_i)$ and $r_i = 2^{-j_i}$. Setting $x^{(i)} = (r_i, 0,\dots,0)\in R^n$, we define

$$f(y) = a_i^{-1/p}[\varphi^*(r_i)]^{-p'/p}[\omega(r_i)]^{-1/p}|x^{(i)}-y|^{-\alpha}[\varphi(|x^{(i)}-y|^{-1})]^{-p'/p}$$

if $y \in B(x^{(i)}, r_i/2)$ for $i = 1, 2, \ldots$, and $f(y) = 0$ on $R^n - \bigcup_{i=1}^{\infty} B(x^{(i)}, r_i/2)$. Then we have

$$\int f(y)\,dy = \sum_i a_i^{-1/p}[\varphi^*(r_i)]^{-p'/p}[\omega(r_i)]^{-1/p}$$
$$\times \int_{B(x^{(i)}, r_i/2)} |x^{(i)} - y|^{-\alpha}[\varphi(|x^{(i)} - y|^{-1})]^{-p'/p}\,dy$$
$$\leq M_1 \sum_i a_i^{-1/p}[\varphi^*(r_i)]^{-p'/p}[\omega(r_i)]^{-1/p} r_i^{n/p'} \varphi^*(r_i)^{p'}$$
$$= M_1 \sum_i a_i^{-1/p}[r_i^{n/p'}\{\omega(r_i)\}^{-1/p}\varphi^*(r_i)] < \infty,$$

so that f satisfies (1.1) by our assumption. Note that $\{a_i^{-1/p}\}$ and $\{r_i^{n/p'}\omega(r_i)^{-1/p}\varphi^*(r_i)\}$ are bounded. Hence, using (3.2), we obtain

$$f(y) \leq M_2[\varphi^*(r_i)]^{-p'/p}[r_i^{n/p'}\varphi^*(r_i)]^{-1}|x^{(i)} - y|^{-\alpha}[\varphi(|x^{(i)} - y|^{-1})]^{-p'/p}$$
$$\leq M_3|x^{(i)} - y|^{-n/p' - \alpha}$$

on $B(x^{(i)}, r_i/2)$. Hence, in view of ($\varphi 3$) and ($\varphi 4$),

$$\varphi(f(y)) \leq M_4 \varphi(|x^{(i)} - y|^{-1})$$

there. Consequently, by condition ($\omega 1$) we establish

$$\int \Phi_p(f(y))\omega(|y|)\,dy \leq M_5 \sum_i a_i^{-1}[\varphi^*(r_i)]^{-p'}$$
$$\times \int_{B(x^{(i)}, r_i)} |x^{(i)} - y|^{-\alpha p}[\varphi(|x^{(i)} - y|^{-1})]^{-p'/p}\,dy \leq M_6 \sum_i a_i^{-1} < \infty,$$

which implies that f satisfies (1.2). Since

$$U_\alpha f(x^{(i)}) \geq a_i^{-1/p}[\varphi^*(r_i)]^{-p'/p}[\omega(r_i)]^{-1/p}$$
$$\times \int_{B(x^{(i)}, r_i/2)} |x^{(i)} - y|^{-n}[\varphi(|x^{(i)} - y|^{-1})]^{-p'/p}\,dy$$
$$\geq M_7 a_i^{-1/p}[\omega(r_i)]^{-1/p}\varphi^*(r_i),$$

we find

$$a(|x^{(i)}|)[\omega(|x^{(i)}|)]^{1/p}[\varphi^*(|x^{(i)}|)]^{-1} U_\alpha f(x^{(i)}) \geq M_7[a(|x^{(i)}|)]^{1/p'} \longrightarrow \infty$$

as $i \to \infty$. Thus f has all the required properties.

REMARK 3.1. In Proposition 3.1, if $\varphi^*(1) = \infty$, then we can find a nonnegative measurable function f on R^n, which satisfies (1.1) and (1.2), and a set A, which is of the form $\bigcup_i [B(0, 2r_i) - B(0, r_i)]$ with some sequence $\{r_i\}$ of positive numbers tending to zero, such that

$$\lim_{x\to 0, x\in A} a(|x|)[\omega(|x|)]^{1/p}[\varphi^*(|x|)]^{-1}U_\alpha f(x) = \infty.$$

4. Radial limits

Before discussing the existence of radial limits of Riesz potentials, we prepare two lemmas concerning the capacity C_{α,Φ_p}.

A mapping $T\colon G \to G'$ is said to be bi-Lipschitzian if there exists $A > 1$ such that

$$A^{-1}|x-y| \le |Tx - Ty| \le A|x-y| \qquad \text{for all } x, y \in G.$$

The following result can be proved easily by the definition of C_{α,Φ_p} (cf. [11, Lemma 3]).

LEMMA 4.1. *Let T be a bi-Lipschitzian mapping from G onto TG. Then*

$$C_{\alpha,\Phi_p}(TE;\ TG) \le M C_{\alpha,\Phi_p}(E;\ G) \qquad \textit{for any } E \subset G,$$

where M is a positive constant which may depend on A (the Lipschitz constant of T).

For a set $E \subset R^n$, we denote by $\tilde{E}$ the set of all $\xi \in \partial B(0, 1)$ such that $r\xi \in E$ for some $r > 0$. By using Lemma 4.1 and applying the methods in the proof of Lemma 5 in [11], we can prove the following lemma.

LEMMA 4.2. *There exists a positive constant M such that*

$$C_{\alpha,\Phi_p}(\tilde{E};\ B(0,4)) \le M C_{\alpha,\Phi_p}(E;\ B(0,4))$$

whenever $E \subset B(0,2) - B(0,1)$.

We consider the quantity

$$\tilde{K} = 2^{-\alpha p} \sup_{t>0} \frac{\varphi(2^{-\alpha}t)}{\varphi(t)} \qquad (\le 2^{-\alpha p} < 1).$$

LEMMA 4.3. *If $\sum_{j=1}^\infty 2^{nj}\tilde{K}^j C_{\alpha,\Phi_p}(E_j;\ B_j) < \infty$, then*

$$C_{\alpha,\Phi_p}(E^*;\ B(0,\ 2)) = 0,$$

where $E^* = \bigcap_{k=1}^\infty (\bigcup_{j=k}^\infty \tilde{E}_j)$.

PROOF. Let f be a nonnegative measurable function on R^n such that $f = 0$ outside B_j and $U_\alpha f(x) \ge 1$ on E_j. If $x \in E_j$, then

$$1 \le \int_{B_j} |x-y|^{\alpha-n} f(y)\,dy = 2^{-\alpha j}\int_{B_0} |2^j x - z|^{\alpha-n} f(2^{-j}z)\,dz.$$

Hence, by the definition of capacity C_{α,Φ_p}, we obtain

$$C_{\alpha,\Phi_p}(2^j E_j; B_0) \le \int_{B_0} \Phi_p(2^{-\alpha j} f(2^{-j}z))\,dz = 2^{jn} \int_{B_j} \Phi_p(2^{-\alpha j} f(y))\,dy$$
$$\le 2^{jn} \tilde{K}^j \int_{B_j} \Phi_p(f(y))\,dy,$$

which implies

$$C_{\alpha,\Phi_p}(2^j E_j; B_0) \le 2^{jn} \tilde{K}^j C_{\alpha,\Phi_p}(E_j; B_j).$$

Therefore it follows from Lemma 4.2 that

$$C_{\alpha,\Phi_p}(\tilde{E}_j; B(0, 4)) \le M_1 C_{\alpha,\Phi_p}(2^j E_j; B(0, 4)) \le M_2 2^{jn} \tilde{K}^j C_{\alpha,\Phi_p}(E_j; B_j)$$

with positive constants M_1 and M_2 independent of j. Thus, $C_{\alpha,\Phi_p}(E^*; B(0, 4)) = 0$, which together with Lemma 2.2 (iii) gives the required result.

Now we show radial limit theorems as generalizations of the results in [11].

By Lemma 4.3 and Theorem 3.1, we have

THEOREM 4.1. *Let f be as in Theorem 3.1, and suppose*

$$\sup_j [2^{nj} \tilde{K}^j]/\omega(2^{-j}) < \infty.$$

Then there exists a set $\tilde{E} \subset \partial B(0, 1)$ such that $C_{\alpha,\Phi_p}(\tilde{E}; B(0, 2)) = 0$ and

$$\lim_{r \to 0} U_\alpha f(r\xi) = U_\alpha f(0) \qquad \textit{for every } \ \xi \in \partial B(0, 1) - \tilde{E}.$$

By Lemma 4.3 and Theorem 3.2, we can prove

THEOREM 4.2. *Let f, κ and K be as in Theorem 3.2, and suppose*

$$\sup_j \frac{2^{nj} \tilde{K}^j}{K^{-j} \omega(2^{-j})} < \infty.$$

If $\lim_{r \to 0} \kappa(r) = \infty$, then there exists a set $\tilde{E} \subset \partial B(0, 1)$ such that $C_{\alpha,\Phi_p}(\tilde{E}; B(0, 2)) = 0$ and

$$\lim_{r \to 0} [\kappa(r)]^{-1} U_\alpha f(r\xi) = 0 \qquad \textit{for every } \ \xi \in \partial B(0, 1) - \tilde{E}.$$

Theorems 4.1 and 4.2 give generalizations of Theorems 1 and 2 in [11].

5. q-th means of potentials

For $q > 0$ and a nonnegative Borel function u on R^n, define

$$S_q(u, r) = \left(\frac{1}{c_n r^{n-1}} \int_{\partial B(0,r)} u(x)^q \, dS(x) \right)^{1/q},$$

where c_n denotes the area of the unit sphere $\partial B(0, 1)$.

Set $R_\alpha(x, y) = |x - y|^{\alpha - n}$, $0 < \alpha < n$.

LEMMA 5.1. *Let $\beta = \delta q (n - \alpha)$ for $\delta > 0$. Then*

$$S_q(R_\alpha(\cdot, y)^\delta, r) \le M[I(|y|, r)]^{1/q},$$

where

$$I(t, r) = \begin{cases} t^{-\beta} & \text{in case } t \ge 2r, \\ r^{-\beta} & \text{in case } r/2 < t < 2r \text{ and } n - 1 - \beta > 0, \\ r^{-\beta}(|t - r|/r)^{n-1-\beta} & \text{in case } r/2 < t < 2r \text{ and } n - 1 - \beta < 0, \\ r^{-\beta} \log(2r/|t - r|) & \text{in case } r/2 < t < 2r \text{ and } n - 1 - \beta = 0, \\ r^{-\beta} & \text{in case } t \le r/2, \end{cases}$$

and M is a positive constant independent of r, t and y.

PROOF. Let $t = |y|$. First we note

$$S_q(R_\alpha(\cdot, y)^\delta, r) \le M_1 \left(\int_0^1 \theta^{n-2} \{(t - r)^2 + tr\theta^2\}^{-\beta/2} d\theta \right)^{1/q}.$$

If $t \ge 2r$, then $S_q(R_\alpha(\cdot, y)^\delta, r) \le M_2 t^{-\beta/q}$. If $t \le r/2$, then $S_q(R_\alpha(\cdot, y)^\delta, r) \le M_3 r^{-\beta/q}$. If $r/2 < t < 2r$, then

$$S_q(R_\alpha(\cdot, y)^\delta, r) \le M_4 \left(r^{-\beta} \int_0^1 \theta^{n-2} \{[(t - r)/r]^2 + \theta^2\}^{-\beta/2} d\theta \right)^{1/q}.$$

Hence we obtain the required inequalities.

For $0 < \beta < n$, we define an outer capacity by setting

$$C_\beta(E) = C_\beta^{(n)}(E) = \inf \mu(R^n), \qquad E \subset R^n,$$

where the infimum is taken over all nonnegative measures μ on R^n such that

$$\int |x - y|^{\beta - n} d\mu(y) \ge 1 \qquad \text{for every } x \in E.$$

For simplicity, let R_+ denote the open interval $(0, \infty)$.

LEMMA 5.2. *Let $0 < \beta < 1$ and μ be a nonnegative measure on R_+ such that $\mu(R_+) < \infty$. Then there exists a set $E \subset R_+$ such that*

$$\lim_{x\to 0, x\in R_+ - E} x^\beta \int_{R_+} |x-y|^{-\beta} d\mu(y) = 0$$

and

$$\sum_j 2^{j\beta} C_{1-\beta}(E_j) < \infty,$$

where $C_{1-\beta} = C_{1-\beta}^{(1)}$ *and* $E_j = \{x \in E; 2^{-j} \le x < 2^{-j+1}\}$.

PROOF. For $x > 0$, we write $\int |x-y|^{-\beta} d\mu(y) = u_1(x) + u_2(x)$, where

$$u_1(x) = \int_{\{y; |x-y| < x/2\}} |x-y|^{-\beta} d\mu(y)$$

and

$$u_2(x) = \int_{\{y\in R_+; |x-y| \ge x/2\}} |x-y|^{-\beta} d\mu(y).$$

If $|x-y| \ge x/2$, then $x^\beta |x-y|^{-\beta} \le 2^\beta$. Hence we can apply Lebesgue's dominated convergence theorem to obtain

$$\lim_{x\to 0} x^\beta u_2(x) = 0.$$

For each positive integer j, we define

$$E_j = \{x; 2^{-j} \le x < 2^{-j+1}, 2^{-j\beta} u_1(x) > a_j^{-1}\},$$

where $\{a_j\}$ is a sequence of positive integers so chosen that

$$\lim_{j\to\infty} a_j = \infty$$

and

$$\sum_j a_j \mu(D_j) < \infty \qquad \text{with } D_j = (2^{-j-1}, 2^{-j+2}).$$

Then it follows from the dual definition of $C_{1-\beta}$ that

$$C_{1-\beta}(E_j) \le a_j 2^{-j\beta} \mu(D_j).$$

If we set $E = \bigcup_j E_j$, then we see easily that E has the required properties.

Let $I_j = [2^{-j}, 2^{-j+1})$. Then we have

$$\int_{I_j} |x-y|^{-\beta} dx \le 2\int_0^{2^{-j}/2} |x|^{-\beta} dx = 2(1-\beta)^{-1}(2^{-j-1})^{1-\beta} \equiv A_\beta 2^{j(\beta-1)}.$$

If $\int |x-y|^{-\beta} d\mu(y) \ge 1$ on I_j, then

$$\int_{I_j} dx \leq \int_{I_j}\left(\int |x-y|^{-\beta}\,d\mu(y)\right)dx$$

$$= \int\left(\int_{I_j} |x-y|^{-\beta}\,dx\right)d\mu(y) \leq A_\beta 2^{j(\beta-1)}\mu(R_+),$$

which implies $2^{\beta j}C_{1-\beta}(I_j) \geq A_\beta^{-1} > 0$. Thus $I_j - E_j \neq \emptyset$ for large j, so that Lemma 5.2 gives the following result.

COROLLARY 5.1. *If μ and β are as in Lemma 5.2, then*

$$\liminf_{x\to 0} x^\beta \int_{R_+} |x-y|^{-\beta}\,d\mu(y) = 0.$$

Now we study the behavior at 0 of spherical means of Riesz potentials.

THEOREM 5.1. *Let $\alpha p > 1$, $q > 0$ and $(n-\alpha p)/p(n-1) < 1/q$. If $\lim_{r\to 0}\kappa(r) = \infty$, and if f is a nonnegative measurable function on R^n satisfying conditions* (1.1) *and* (1.2), *then*

$$\lim_{r\to 0}[\kappa(r)]^{-1}S_q(U_\alpha f, r) = 0.$$

REMARK 5.1. In case $p = 1$, Theorem 5.1 implies a result by Gardiner [4].

PROOF OF THEOREM 5.1. For $x \in R^n$, set $E(x) = B(x, |x|/2)$. First we consider the case $q \geq p > 1$. Take δ such that

$$0 < \delta < 1 \quad\text{and}\quad \frac{n-\alpha p}{p(n-\alpha)} < \delta < \frac{n-1}{q(n-\alpha)}.$$

Since $(\alpha - n)(1-\delta) + n/p' > 0$, by the computations as in the proof of Lemma 2.1 and using Hölder's inequality, we have

$$U_3(x) \leq \left(\int_{E(x)} [R_\alpha(x,y)]^{(1-\delta)p'}[\varphi(|x-y|^{-\varepsilon})]^{-p'/p}\,dy\right)^{1/p'}$$

$$\times\left(\int_{E(x)} [R_\alpha(x,y)]^{\delta p}\Phi_p(f(y))\,dy\right)^{1/p} + \int_{E(x)} |x-y|^{\alpha-n-\varepsilon}\,dy$$

$$\leq M_1|x|^{(\alpha-n)(1-\delta)+n/p'}[\varphi(|x|^{-\varepsilon})]^{-1/p}$$

$$\times\left(\int_{E(x)} [R_\alpha(x,y)]^{\delta p}\Phi_p(f(y))\,dy\right)^{1/p} + M_1|x|^{\alpha-\varepsilon},$$

where $0 < \varepsilon < \alpha$. Using Minkowski's inequality and (φ4), we obtain

$$S_q(U_3, r)^p \leq M_2[r^{(\alpha-n)(1-\delta)+n/p'}]^p[\varphi(r^{-1})\omega(r)]^{-1}$$

$$\times \int_{B(0,2r)} (S_q(R_\alpha(\cdot, y)^\delta, r))^p \Phi_p(f(y))\omega(|y|)\,dy + M_2 r^{(\alpha-\varepsilon)p}.$$

Here we note

(5.1) $$\kappa_1(r) \geq \left(\int_r^{2r} [t^{n-\alpha p}\eta(t)]^{-p'/p} t^{-1}\,dt \right)^{1/p'} \geq M_3[r^{n-\alpha p}\eta(r)]^{-1/p}.$$

Since $\delta q < (n-1)/(n-\alpha)$, by Lemma 5.1, we find

$$S_q(R_\alpha(\cdot, y)^\delta, r) \leq M_4 r^{\delta(\alpha-n)}$$

for $y \in B(0, 2r)$, so that

(5.2) $$S_q(U_3, r)^p \leq M_5[\kappa(r)]^p \int_{B(0,2r)} \Phi_p(f(y))\omega(|y|)\,dy + M_2 r^{(\alpha-\varepsilon)p}.$$

This is true in case $p = 1$, too. Since $S_q(u, r)$ is nondecreasing with respect to q, (5.2) also holds for q smaller than p. Thus the required result holds for U_3 instead of $U_\alpha f$. The same fact is also valid for U_1 and U_2, in view of Corollaries 2.1 and 2.2, and hence Theorem 5.1 is established.

THEOREM 5.2. *Let $q > 0$ and $1/p - \alpha/(n-1) < 1/q$. If f is a nonnegative measurable function on R^n as in Theorem 5.1, then*

$$\liminf_{r\to 0} \kappa(r)^{-1} S_q(U_\alpha f, r) = 0.$$

PROOF. First we consider the case $q \geq p > 1$. Take δ such that

$$\frac{n-1}{q(n-\alpha)} < \delta < 1 \quad \text{and} \quad \frac{n-\alpha p}{p(n-\alpha)} < \delta < \frac{n-1}{q(n-\alpha)} + \frac{1}{p(n-\alpha)}.$$

Then, as in the previous proof, we have

$$S_q(U_3, r)^p \leq M_1[r^{(\alpha-n)(1-\delta)+n/p'}]^p[\varphi(r^{-1})\omega(r)]^{-1}$$
$$\times \int_{B(0,2r)} (S_q(R_\alpha(\cdot, y)^\delta, r))^p \Phi_p(f(y))\omega(|y|)\,dy + M_1 r^{(\alpha-\varepsilon)p}.$$

Set $\beta = -p[n-1-\delta q(n-\alpha)]/q$. Then $0 < \beta < 1$. By Lemma 5.1, we obtain

$$S_q(U_3, r)^p \leq M_2[\kappa(r)]^p \int_{B(0,2r)} \left(\frac{||y|-r|}{r} \right)^{-\beta} \Phi_p(f(y))\omega(|y|)\,dy + M_1 r^{(\alpha-\varepsilon)p}.$$

If $p = 1$, $q \geq 1$ and $(n-1)/(n-\alpha) < q < (n-1)/(n-\alpha-1)$, then the above inequality also holds with $\beta = n-\alpha-(n-1)/q$. Now, applying Corollary 5.1, we see that the required result holds for U_3 instead of $U_\alpha f$, if $q \geq p$. Thus, using the monotonicity of $S_q(u, r)$ with respect to q, Corollaries 2.1 and 2.2, we end the proof.

6. Global fine limits

Let D denote the half space $\{x = (x', x_n) \in R^{n-1} \times R^1; x_n > 0\}$. In this section we study the global fine limit at the boundary ∂D of the Riesz potential $U_\alpha f$, where f is a nonnegative measurable function on R^n satisfying condition (1.1) and

$$(6.1) \qquad \int_G \Phi_p(f(y))\omega(|y_n|)\,dy < \infty \qquad \text{for any bounded open set } G \subset R^n;$$

recall that ω is a positive and monotone function on the interval $(0, \infty)$ satisfying the (Δ_2) condition (see $(\omega 1)$). As an application, we shall study the fine boundary limits of Beppo-Levi-Deny functions u on D satisfying (1.4), and give a generalization of [17, Theorem 1] (see Section 10).

In what follows, let $p > 1$.

Our aim in this section is to establish

THEOREM 6.1. *Assume that*

$(\omega 2)$ $\qquad r^{\beta - 1/p}\omega(r)^{-1/p}$ *is nondecreasing on* $(0, \infty)$ *for some* $\beta < 1$.

Let f be a nonnegative measurable function on R^n satisfying (1.1) *and* (6.1). *If*

$$\lim_{r \to 0} \kappa_1(r) = \infty,$$

then there exists a set $E \subset D$ such that

$$\lim_{x_n \to 0, x \in D' - E} [\kappa_1(x_n)]^{-1} U_\alpha f(x) = 0$$

for any bounded open set $D' \subset D$ and

$$\sum_{j=1}^{\infty} K^{-j}\omega(2^{-j}) C_{\alpha, \Phi_p}(E_j \cap B(0, N); D_j \cap B(0, 2N)) < \infty$$

for any $N > 0$, where $K = K^*$ in Lemma 2.3 with $\chi = \kappa_1$, $E_j = \{x = (x', x_n) \in E; 2^{-j} \le x_n < 2^{-j+1}\}$ and $D_j = \{x = (x', x_n); 2^{-j-1} < x_n < 2^{-j+2}\}$.

REMARK 6.1. In case $\omega(r) = r^\beta$, $(\omega 2)$ holds if and only if $\beta < p - 1$. In fact, if $\beta < p - 1$, then take $\beta_1 \in [(1 + \beta)/p, 1)$ and note that $r^{\beta_1 - 1/p}\omega(r)^{-1/p}$ is nondecreasing on $(0, \infty)$.

Before giving a proof of Theorem 6.1, we prepare the following result similar to Lemma 2.1.

LEMMA 6.1. *Let $\gamma_1, \gamma_2 \ge 0$, $\delta > 0$ and assume that $r^{\beta - 1/p}\omega(r)^{-1/p}$ is* *nondecreasing on* $(0, \infty)$ *for some* $\beta < 1 + \gamma_2$. *Let* f *be a nonnegative measurable function on* R^n. *If* $x = (x', x_n) \in D$ *and* $0 \leq s \leq x_n/2 < r/4$, *then*

$$\int_{D \cap B(x,r) - B(x,s)} |x - y|^{\alpha - n} |\bar{x} - y|^{-\gamma_1} y_n^{\gamma_2} f(y)\, dy$$
$$\leq M F(r) \left\{ \left(\int_{x_n}^{r} [t^{n - \alpha p + (\gamma_1 - \gamma_2) p} \varphi(t^{-1}) \omega(t)]^{-p'/p} t^{-1}\, dt \right)^{1/p'} + x_n^{-\gamma_1 + \gamma_2} [\omega(x_n)]^{-1/p} \left(\int_{s}^{x_n} [t^{n - \alpha p} \varphi(t^{-1})]^{-p'/p} t^{-1}\, dt \right)^{1/p'} \right\}$$
$$+ M \int_{D \cap B(x,r) - B(x,s)} |x - y|^{\alpha - n - \delta} |\bar{x} - y|^{-\gamma_1} y_n^{\gamma_2}\, dy,$$

where $\bar{x} = (x', -x_n)$ *and* $F(r) = \left(\int_{D \cap B(x,r)} \Phi_p(f(y)) \omega(y_n)\, dy \right)^{1/p}$.

PROOF. As in the proof of Lemma 2.1, we have by Hölder's inequality

$$\int_{D \cap B(x,r) - B(x,s)} |x - y|^{\alpha - n} |\bar{x} - y|^{-\gamma_1} y_n^{\gamma_2} f(y)\, dy$$
$$\leq F(r) J + \int_{D \cap B(x,r) - B(x,s)} |x - y|^{\alpha - n - \delta} |\bar{x} - y|^{-\gamma_1} y_n^{\gamma_2}\, dy,$$

where

$$J = \left(\int_{D \cap B(x,r) - B(x,s)} [|x - y|^{\alpha - n} |\bar{x} - y|^{-\gamma_1} \{\varphi(|x - y|^{-\delta}) \omega(y_n)\}^{-1/p} y_n^{\gamma_2}]^{p'}\, dy \right)^{1/p'}.$$

In order to evaluate J, we set

$$J_j = \left(\int_{E_j} [|x - y|^{\alpha - n} |\bar{x} - y|^{-\gamma_1} \{\varphi(|x - y|^{-\delta}) \omega(y_n)\}^{-1/p} y_n^{\gamma_2}]^{p'}\, dy \right)^{1/p'},$$

where

$$E_1 = \{y \in B(x, r) - B(x, s);\ y_n > x_n/2\},$$
$$E_2 = \{y \in D \cap B(x, r) - B(x, s);\ y_n < x_n/2\}.$$

Since $y_n \leq x_n + |x - y|$, we see from condition (ω2) that

$$y_n^{\beta - 1/p} [\omega(y_n)]^{-1/p} \leq (x_n + |x - y|)^{\beta - 1/p} [\omega(x_n + |x - y|)]^{-1/p}$$

for $y \in D$. Set $t = |x - y|$ and $|x_n - y_n| = t \cos\theta$, and note

$$3 y_n \geq |x_n - y_n| + x_n \geq (t + x_n) \cos\theta \qquad \text{for any} \quad y \in E_1.$$

Since $p'(\gamma_2 - \beta + 1/p) > -1$, we see that

$$\int_0^{\pi/2} (\cos\theta)^{p'(\gamma_2-\beta+1/p)}\,d\theta < \infty.$$

If $\gamma_2 - \beta + 1/p < 0$, then, applying polar coordinates about x, we have

$$J_1 \le M_1\left(\int_s^r [t^{\alpha-n}\{\varphi(t^{-1})\omega(x_n+t)\}^{-1/p}(x_n+t)^{-\gamma_1+\beta-1/p}]^{p'}\right.$$
$$\left.\times (x_n+t)^{p'(\gamma_2-\beta+1/p)}t^{n-1}\,dt\right)^{1/p'}$$
$$\le M_2\left(\int_{x_n}^r [t^{\alpha-n/p-\gamma_1+\gamma_2}\{\varphi(t^{-1})\omega(t)\}^{-1/p}]^{p'}t^{-1}\,dt\right)^{1/p'}$$
$$+ M_2 x_n^{-\gamma_1+\gamma_2}[\omega(x_n)]^{-1/p}\left(\int_s^{x_n} [t^{n-\alpha p}\varphi(t^{-1})]^{-p'/p}t^{-1}\,dt\right)^{1/p'}.$$

Similarly, if $\gamma_2 - \beta + 1/p \ge 0$, then, noting $y_n \le x_n + |x-y|$, we derive the same estimate of J_1 as above. Next, since $y_n \le |z-y|$ if $y \in E_2$, where $z = (x', 0)$, by the condition on ω again, we have

$$[\omega(y_n)]^{-1/p} \le y_n^{-\beta+1/p}|z-y|^{\beta-1/p}[\omega(|z-y|)]^{-1/p}$$

for $y \in E_2$. Consequently, by using polar coordinates about z, we obtain

$$J_2 \le M_3 x_n^{\alpha-n-\gamma_1+\beta-1/p}\{\varphi(x_n^{-1})\omega(x_n)\}^{-1/p}\left(\int_{D\cap B(z,x_n/2)} y_n^{p'(1/p-\beta+\gamma_2)}\,dy\right)^{1/p'}$$
$$+ M_3\left(\int_{x_n/2}^r [t^{\alpha-n/p-\gamma_1+\gamma_2}\{\varphi(t^{-1})\omega(t)\}^{-1/p}]^{p'}t^{-1}\,dt\right)^{1/p'}$$
$$\le M_4 x_n^{\alpha-n/p-\gamma_1+\gamma_2}[\varphi(x_n^{-1})\omega(x_n)]^{-1/p}$$
$$+ M_4\left(\int_{x_n/2}^r [t^{\alpha-n/p-\gamma_1+\gamma_2}\{\varphi(t^{-1})\omega(t)\}^{-1/p}]^{p'}t^{-1}\,dt\right)^{1/p'}$$
$$\le M_5\left(\int_{x_n}^r [t^{\alpha-n/p-\gamma_1+\gamma_2}\{\varphi(t^{-1})\omega(t)\}^{-1/p}]^{p'}t^{-1}\,dt\right)^{1/p'};$$

the last inequality follows from the (Δ_2) conditions on φ and ω (see (5.1)). Now our lemma is proved.

REMARK 6.2. If $\alpha - \delta - \gamma_1 + \gamma_2 > 0$, then

$$\int_{D\cap B(x,r)-B(x,s)} |x-y|^{\alpha-n-\delta}|\bar{x}-y|^{-\gamma_1}y_n^{\gamma_2}\,dy \le Mr^{\alpha-\delta-\gamma_1+\gamma_2}.$$

REMARK 6.3. The above proof shows that if ω is as in Lemma 6.1, then

$$\left(\int_{B(x,r)-B(x,s)} [|x-y|^{\alpha-n}\{\varphi(|x-y|^{-1})\omega(|y_n|)\}^{-1/p}|y_n|^{\gamma_2}]^{p'} dy\right)^{1/p'}$$
$$\leq M\left(\int_{x_n}^{r} [t^{\alpha-n/p+\gamma_2}\{\varphi(t^{-1})\omega(t)\}^{-1/p}]^{p'} t^{-1} dt\right)^{1/p'}$$
$$+ Mx_n^{\gamma_2}[\omega(x_n)]^{-1/p}\left(\int_{s}^{x_n} [t^{n-\alpha p}\varphi(t^{-1})]^{-p'/p} t^{-1} dt\right)^{1/p'}.$$

In view of Remark 6.3, we obtain

LEMMA 6.2. *Let $0 < \delta < \alpha$ and assume that ω satisfies (ω2). Let f be a nonnegative measurable function on R^n. If $x = (x', x_n) \in D$ and $0 \leq s \leq 2^{-1}x_n < 4^{-1}r$, then*

$$\int_{B(x,r)-B(x,s)} |x-y|^{\alpha-n} f(y)\, dy$$
$$\leq M\left(\int_{B(x,r)} \Phi_p(f(y))\omega(|y_n|)\, dy\right)^{1/p} \left\{\left(\int_{x_n}^{r} [t^{n-\alpha p}\varphi(t^{-1})\omega(t)]^{-p'/p} t^{-1} dt\right)^{1/p'}\right.$$
$$\left. + [\omega(x_n)]^{-1/p}\left(\int_{s}^{x_n} [t^{n-\alpha p}\varphi(t^{-1})]^{-p'/p} t^{-1} dt\right)^{1/p'}\right\} + Mr^{\alpha-\delta}.$$

PROOF OF THEOREM 6.1. For $x = (x', x_n) \in D$, we write $U_\alpha f(x) = u_1(x) + u_2(x)$, where

$$u_1(x) = \int_{R^n - B(x, x_n/2)} |x-y|^{\alpha-n} f(y)\, dy,$$
$$u_2(x) = \int_{B(x, x_n/2)} |x-y|^{\alpha-n} f(y)\, dy.$$

For $a > 1$ and a bounded open set D' in D, let $D'(a) = \{x = (x', x_n) \in D'; 0 < x_n < a\}$. For $x \in D'(a)$, write

$$u_1(x) = \int_{R^n - B(x, 2a)} |x-y|^{\alpha-n} f(y)\, dy + \int_{B(x,2a)-B(x,x_n/2)} |x-y|^{\alpha-n} f(y)\, dy$$
$$= u_{11}(x) + u_{12}(x).$$

By condition (1.1), we see that u_{11} is bounded on $D'(a)$, so that

$$\lim_{x_n \to 0, x \in D'} [\kappa_1(x_n)]^{-1} u_{11}(x) = 0.$$

For u_{12}, we obtain by Lemma 6.2,

$$u_{12}(x) \le M_1 \kappa_1(x_n)\left(\int_{D''} \Phi_p(f(y))\omega(|y_n|)\,dy\right)^{1/p} + M_1,$$

for any $x \in D'$, where $D'' = \bigcup_{x \in D'} B(x, 2a)$. Hence it follows that $[\kappa_1(x_n)]^{-1} u_{12}(x)$ tends to zero as $x_n \to 0$, $x \in D'$. To complete the proof, take a sequence $\{a_j\}$ of positive numbers such that

$$\sum_{j=1}^{\infty} \int_{B_j} \Phi_p(f(y))\omega(y_n)\,dy < \infty,$$

where $B_j = \{x = (x', x_n) \in D \cap B(0, 2j);\ 0 < x_n < a_j\}$. Further take a sequence $\{b_{j,\ell}\}$ of positive numbers such that

$$\lim_{\ell \to \infty} b_{j,\ell} = \infty$$

and

$$\sum_{j=1}^{\infty}\left(\sum_{\{\ell;\, 2^{-\ell} \le a_j/2\}} \int_{\Delta_{j,\ell}} \Phi_p(b_{j,\ell} f(y))\omega(y_n)\,dy\right) < \infty,$$

where $\Delta_{j,\ell} = B_j \cap D_\ell$ when $2^{-\ell} \le a_j/2$; cf. the proof of Lemma 2.3. As in the proof of Lemma 2.3, we consider the sets

$$E_{j,\ell} = \{x \in D \cap B(0, j);\ 2^{-\ell} \le x_n < 2^{-\ell+1},\ u_2(x) \ge b_{j,\ell}^{-1} \kappa_1(x_n)\}$$

for j and ℓ such that $2^{-\ell} \le a_j/2$; we set $E_{j,\ell} = \emptyset$ for other (j, ℓ). If $x \in E_{j,\ell} \cap B(0, a)$, then, since $B(x, x_n/2) \subset \Delta_{j,\ell} \cap B(0, 2a)$, we find

$$\begin{aligned} C_{\alpha,\Phi_p}(E_{j,\ell} \cap B(0, a);\ D_\ell \cap B(0, 2a)) &\le M_1 \int_{\Delta_{j,\ell}} \Phi_p(b_{j,\ell}\kappa_1(2^{-\ell})^{-1} f(y))\,dy \\ &\le M_4 K^\ell [\omega(2^{-\ell})]^{-1} \int_{\Delta_{j,\ell}} \Phi_p(b_{j,\ell} f(y))\omega(y_n)\,dy. \end{aligned}$$

Define $E = \bigcup_{j,\ell} E_{j,\ell}$. We see that $E_\ell \cap B(0, a) \subset \bigcup_{\{j;\, 2^{-\ell} \le a_j/2\}} E_{j,\ell} \cap B(0, a)$, so that E has all the required properties. Hence the proof of Theorem 6.1 is completed.

REMARK 6.4. If κ_1 is bounded, then we can take $K = 1$ in Theorem 6.1. Hence, in view of the proof of Theorem 6.1, $U_\alpha f(x)$ tends to $U_\alpha f(\xi)$ as $x \to \xi$, $x \in D - E$, for any $\xi \in \partial D$, where

$$\sum_{j=1}^{\infty} \omega(2^{-j}) C_{\alpha,\Phi_p}(E_j \cap B(0, N);\ D_j \cap B(0, 2N)) < \infty$$

for any $N > 0$.

7. T_ψ-limits

Let ψ be a positive nondecreasing continuous function on the interval $(0, \infty)$ satisfying the (Δ_2) condition and the following:

(ψ1) $\quad r^{-1}\psi(r)$ is nondecreasing on the interval $(0, \infty)$.

For $a > 0$ and $\xi \in \partial D$, we set

$$T_\psi(\xi, a) = \{x = (x', x_n) \in R^{n-1} \times R^1;\ \psi(|x - \xi|) < ax_n\}.$$

We say that a function u has a T_ψ-limit ℓ at $\xi \in \partial D$ if

$$\lim_{x \to \xi, x \in T_\psi(\xi, a)} u(x) = \ell$$

for any $a > 0$; if $\psi(r) = r^\gamma$, then we say "T_γ-limit" instead of T_ψ-limit. We here discuss the existence of T_ψ-limits of Riesz potentials $U_\alpha f$ for functions f satisfying condition (6.1), when φ satisfies a condition similar to (1.3).

We consider the quantity

$$C_{\alpha,\Phi_p,\omega}(E; G) = \inf \int_G \Phi_p(g(y))\omega(|y_n|)\,dy$$

for a set E and an open set G, where the infimum is taken over all nonnegative measurable functions g on G such that $\int_G |x - y|^{\alpha-n} g(y)\,dy \geq 1$ for every $x \in E$. For simplicity, we write

$$C_{\alpha,\Phi_p,\omega}(E) = 0$$

if $C_{\alpha,\Phi_p,\omega}(E \cap G; G) = 0$ for any bounded open set $G \subset R^n$. In case $\omega(r) = r^\beta$, we write $C_{\alpha,\Phi_p,\beta}$ for $C_{\alpha,\Phi_p,\omega}$; with this notation, remark $C_{\alpha,\Phi_p,0} = C_{\alpha,\Phi_p}$.

Let h be a positive nondecreasing function on $(0, \infty)$ satisfying the (Δ_2) condition. We denote by H_h the Hausdorff measure with the measure function h. Set

$$E_f = \left\{\xi \in \partial D;\ \int |\xi - y|^{\alpha-n} f(y)\,dy = \infty\right\}$$

and

$$F_{f,h} = \left\{\xi \in \partial D;\ \limsup_{r \to 0}\,[h(r)]^{-1} \int_{B(\xi,r)} \Phi_p(f(y))\omega(|y_n|)\,dy > 0\right\}$$

for a nonnegative measurable function f on R^n.

By the definition of $C_{\alpha,\Phi_p,\omega}$, we have

LEMMA 7.1. *If f is a nonnegative measurable function on R^n satisfying* (1.1) *and* (6.1), *then*

$$C_{\alpha,\Phi_p,\omega}(E_f) = 0.$$

Applying a covering lemma ([25, Lemma 1.6, Chapter 1]), we prove

LEMMA 7.2. *Let h be a positive nondecreasing function on* $(0, \infty)$ *satisfying the* (Δ_2) *condition. Let g be a nonnegative function in* $L^1(R^n)$ *and set*

$$F = \left\{\xi \in \partial D;\ \limsup_{r\to 0} [h(r)]^{-1} \int_{B(\xi,r)} g(y)\,dy > 0\right\}.$$

Then $H_h(F) = 0$.

PROOF. For $\varepsilon > 0$, consider the set

$$F(\varepsilon) = \left\{\xi \in \partial D;\ \limsup_{r\to 0} [h(r)]^{-1} \int_{B(\xi,r)} g(y)\,dy > \varepsilon\right\}.$$

Let $\delta > 0$. By definition, for each $\xi \in F(\varepsilon)$, there exists a number $r(\xi)$ such that $0 < r(\xi) < \delta$ and

$$\int_{B(\xi,r(\xi))} g(y)\,dy \geq \varepsilon h(r(\xi)).$$

By using the covering lemma mentioned above, we can find a disjoint family $\{B(\xi_j, r_j)\}$ of balls such that $\xi_j \in F(\varepsilon)$, $r_j = r(\xi_j)$ and $\{B(\xi_j, 5r_j)\}$ covers $F(\varepsilon)$. Then note

$$\begin{aligned}\sum_j h(5r_j) &\leq M_1 \sum_j h(r_j)\\ &\leq M_1 \varepsilon^{-1} \sum_j \int_{B(\xi_j,r_j)} g(y)\,dy\\ &\leq M_1 \varepsilon^{-1} \int_{D(\delta)} g(y)\,dy,\end{aligned}$$

where $D(\delta) = \bigcup_{\xi\in\partial D} B(\xi, \varepsilon)$. Letting $\delta \to 0$, we find

$$H_h(F(\varepsilon)) = 0,$$

which implies $H_h(F) = 0$.

COROLLARY 7.1. *If f is a nonnegative measurable function on R^n satisfying* (1.1) *and* (6.1), *then*

$$H_h(F_{f,h}) = 0$$

for any measure function h.

REMARK 7.1. *If* $h(0) > 0$, *then* $F_{f,h}$ *is empty.*

LEMMA 7.3. *Let* ω *be a monotone function on* $(0, \infty)$ *satisfying* $(\omega 1)$, $(\omega 2)$ *and*

$(\omega 3)$ $\qquad r^{\beta}\omega(r)$ *is nondecreasing on* $(0, \infty)$ *for some* $\beta < 1$.

Then, for any $a > 0$, *there exists* $M > 1$ *such that*

$$M^{-1}[\kappa_{1,a}(r)]^{-p} \le C_{\alpha,\Phi_p,\omega}(B(0, r); B(0, a)) \le M[\kappa_{1,a}(r)]^{-p}$$

whenever $0 < r < a/2$, *where*

$$\kappa_{1,a}(r) = \left(\int_r^a [t^{n-\alpha p}\eta(t)]^{-p'/p}\frac{dt}{t}\right)^{1/p'}$$

with $\eta(r) = \varphi(r^{-1})\omega(r)$.

PROOF. If suffices to prove the required inequality for $a = 1$, by considering a change of variables: $x \to ax$; in this case, $\kappa_{1,a} = \kappa_1$. Consider the function

$$f_r(y) = \begin{cases} |y|^{-\alpha}[|y|^{n-\alpha p}\eta(|y|)]^{-p'/p} & \text{if } y \in B(0, 1) - B(0, r), \\ 0 & \text{otherwise.} \end{cases}$$

If $x \in B(0, r)$, then $|x - y| \le 2|y|$ for $y \in B(0, 1) - B(0, r)$, so that

$$\begin{aligned} \int |x - y|^{\alpha-n} f_r(y)\, dy &\ge 2^{\alpha-n}\int_{B(0,1)-B(0,r)} |y|^{-n}[|y|^{n-\alpha p}\eta(|y|)]^{-p'/p}dy \\ &\ge M_1[\kappa_1(r)]^{p'}. \end{aligned}$$

Hence it follows that

$$C_{\alpha,\Phi_p,\omega}(B(0, r); B(0, 1)) \le \int \Phi_p\left(\frac{f_r(y)}{M_1[\kappa_1(r)]^{p'}}\right)\omega(|y_n|)\, dy.$$

By $(\omega 2)$, there exists $\beta_1 < 1$ such that $\omega(|y|)^{-1/p} \le M_2|y|^{-\beta_1+1/p}$ for $y \in B(0, 1)$, so that

$$\frac{f_r(y)}{[\kappa_1(r)]^{p'}} \le M_3\frac{|y|^{-\beta}}{[\kappa_1(2^{-1})]^{p'}}$$

whenever $y \in B(0, 1)$, for $\beta = \alpha + (n - \alpha p)p'/p + (\beta_1 - 1/p)p'$. Thus we find

$$\Phi_p\left(\frac{f_r(y)}{M_1[\kappa_1(r)]^{p'}}\right) \le M_4\left(\frac{f_r(y)}{[\kappa_1(r)]^{p'}}\right)^p \varphi(|y|^{-\beta})$$

$$\leq M_5[\kappa_1(r)]^{-pp'}|y|^{-\alpha p}[|y|^{n-\alpha p}\eta(|y|)]^{-p'}\varphi(|y|^{-1}).$$

On the other hand, by ($\omega 3$), $r^{\beta_2}\omega(r)$ is nondecreasing on $(0, \infty)$ for some $\beta_2 < 1$. Consequently we establish

$$\begin{aligned}
&C_{\alpha,\Phi_p,\omega}(B(0, r); B(0, 1)) \\
&\leq M_5[\kappa_1(r)]^{-pp'}\int_{B(0,1)-B(0,r)}[|y|^{n-\alpha p}\eta(|y|)]^{-p'}|y|^{-\alpha p}\varphi(|y|^{-1})\omega(|y_n|)\,dy \\
&\leq M_5[\kappa_1(r)]^{-pp'}\int_{B(0,1)-B(0,r)}[|y|^{n-\alpha p}\eta(|y|)]^{-p'}|y|^{-\alpha p}\eta(|y|)|y|^{\beta_2}|y_n|^{-\beta_2}\,dy \\
&\leq M_6[\kappa_1(r)]^{-p}.
\end{aligned}$$

Conversely, take a nonnegative measurable function g on R^n such that $g = 0$ outside $B(0, 1)$ and $U_\alpha g \geq 1$ on $B(0, r)$. Then we have

$$\begin{aligned}
\int_{B(0,r)} dx &\leq \int_{B(0,r)}\left(\int |x - y|^{\alpha-n}g(y)\,dy\right)dx \\
&= \int\left(\int_{B(0,r)}|x - y|^{\alpha-n}\,dx\right)g(y)\,dy \\
&\leq M_7 r^n\int(r + |y|)^{\alpha-n}g(y)\,dy.
\end{aligned}$$

Let $\varepsilon > 0$ and $0 < \delta < \alpha$. As in the proofs of Lemmas 2.1 and 6.1, Hölder's inequality gives

$$\begin{aligned}
&\int(r + |y|)^{\alpha-n}g(y)\,dy \\
&= \int_{\{y;g(y)>\varepsilon|y|^{-\delta}\}}(r + |y|)^{\alpha-n}g(y)\,dy + \int_{\{y;0<g(y)\leq\varepsilon|y|^{-\delta}\}}(r + |y|)^{\alpha-n}g(y)\,dy \\
&\leq \left(\int_{B(0,1)}[(r + |y|)^{\alpha-n}\{\varphi(\varepsilon|y|^{-\delta})\omega(|y_n|)\}^{-1/p}]^{p'}\,dy\right)^{1/p'} \\
&\quad\times\left(\int\Phi_p(g(y))\omega(|y_n|)\,dy\right)^{1/p} + \varepsilon\int_{B(0,1)}(r + |y|)^{\alpha-n}|y|^{-\delta}\,dy.
\end{aligned}$$

By ($\varphi 3$) and ($\varphi 4$),

$$[\varphi(\varepsilon t^{-\delta})]^{-1/p} \leq M(\varepsilon)[\varphi(t^{-\delta})]^{-1/p} \leq M(\varepsilon)M_8[\varphi(t^{-1})]^{-1/p}$$

for any $t > 0$. By condition ($\omega 2$),

$$\omega(|y_n|)^{-1/p} \leq |y_n|^{1/p-\beta_1}r^{\beta_1-1/p}\omega(r)^{-1/p}$$

for $y \in B(0, r)$, where $\beta_1 < 1$. Hence,

$$\left(\int_{B(0,r)} [(r+|y|)^{\alpha-n}\{\varphi(\varepsilon|y|^{-\delta})\omega(|y_n|)\}^{-1/p}]^{p'} dy\right)^{1/p'}$$

$$\leq M(\varepsilon)M_8 r^{\alpha-n+\beta_1-1/p}[\eta(r)]^{-1/p}\left(\int_{B(0,r)} |y_n|^{p'(1/p-\beta_1)} dy\right)^{1/p'}$$

$$\leq M(\varepsilon)M_9[r^{n-\alpha p}\eta(r)]^{-1/p} \leq M(\varepsilon)M_{10}\kappa_1(r)$$

by (5.1). Similarly,

$$\left(\int_{B(0,1)-B(0,r)} [(r+|y|)^{\alpha-n}\{\varphi(\varepsilon|y|^{-\delta})\omega(|y_n|)\}^{-1/p}]^{p'} dy\right)^{1/p'}$$

$$\leq M(\varepsilon)M_8 \int_{B(0,1)-B(0,r)} t^{p'(\alpha-n)}[\eta(t)t^{p\beta_1-1}]^{-p'/p}|y_n|^{p'(1/p-\beta_1)} dy\Big)^{1/p'}, \qquad t = |y|,$$

$$\leq M(\varepsilon)M_{11}\left(\int_r^1 [t^{n-\alpha p}\eta(t)]^{-p'/p} t^{-1} dt\right)^{1/p'}.$$

Thus we derive

$$\int (r+|y|)^{\alpha-n} g(y)\, dy \leq M(\varepsilon)M_{12}\kappa_1(r)\left(\int \Phi_p(g\,(y))\omega(|y_n|)\, dy\right)^{1/p} + M_{12}\varepsilon,$$

so that

$$1 \leq M(\varepsilon)M_{13}\kappa_1(r)[C_{\alpha,\Phi_p,\omega}(B(0, r);\, B(0, 1))]^{1/p} + M_{13}\varepsilon.$$

If $M_{13}\varepsilon = 1/2$, then we establish

$$M_{14}[\kappa_1(r)]^{-p} \leq C_{\alpha,\Phi_p,\omega}(B(0, r);\, B(0, 1)).$$

By using a covering lemma (cf. [25, Lemma 1.6, Chapter 1]), we have

COROLLARY 7.2. *Let ω be as in Lemma 7.3. If G and G' are bounded open sets in R^n such that $\bar{G}' \subset G$, then there exists $M > 0$, depending on the distance between $\partial G'$ and ∂G, such that*

$$C_{\alpha,\Phi_p,\omega}(E;\, G) \leq M\ \ H_h(E)$$

for any set $E \subset \partial D \cap G'$, where $h(r) = [\kappa_1(r)]^{-p}$.

In view of Theorem 12.2 given later, we have

COROLLARY 7.3. *Let $-1 < \beta < p-1$, and assume $C_{\alpha,\Phi_p,\beta}(E) = 0$. If $E \subset \partial D$, then E has Hausdorff dimension at most $n - \alpha p + \beta$; if $E \subset D$, then E has Hausdorff dimension at most $n - \alpha p$.*

COROLLARY 7.4. *Let ω be as in Lemma 7.3. Then, for $x_0 \in \partial D$,*

$$C_{\alpha,\Phi_p,\omega}(\{x_0\}) = 0 \qquad \text{if and only if} \quad \kappa_1(0) = \infty.$$

For $x_0 \in D$,

$$C_{\alpha,\Phi_p}(\{x_0\}) = 0 \qquad \text{if and only if} \quad \int_0^1 [t^{n-\alpha p}\varphi(t^{-1})]^{-1/(p-1)} t^{-1}\,dt = \infty.$$

THEOREM 7.1. *Assume that (ω2) holds and $\varphi^*(1) < \infty$, that is,*

$$\int_0^1 [r^{n-\alpha p}\varphi(r^{-1})]^{-p'/p} r^{-1}\,dr < \infty. \tag{7.1}$$

Let ψ be as above, and set

$$\begin{aligned} \tau_1(r) &= [\kappa_1(r)]^{-p}, \\ \tau_2(r) &= \inf\nolimits_{r \le t \le 1} \omega(t)\,[\varphi^*(t)]^{-p}, \\ \tau(r) &= \min\{\tau_1(r), \tau_2(r)\}, \\ h(r) &= \tau(\psi(r)) \end{aligned}$$

for $0 < r < 1$. Let f be a nonnegative measurable function on R^n satisfying (1.1) and (6.1). Then there exist $E_1, E_2 \subset \partial D$ such that

$$C_{\alpha,\Phi_p,\omega}(E_1) = 0, \qquad H_h(E_2) = 0$$

and $U_\alpha f(x)$ has a finite T_ψ-limit $U_\alpha f(\xi)$ at $\xi \in \partial D - (E_1 \cup E_2)$. If in addition $\tau(0) > 0$, then $U_\alpha f(x)$ has a limit $U_\alpha f(\xi)$ at any $\xi \in \partial D$; in this case, $E_1 \cup E_2 = \emptyset$.

PROOF. For $x \in D$, we write $U_\alpha f(x) = u_1(x) + u_2(x)$, where

$$u_1(x) = \int_{R^n - B(\xi, 2|x-\xi|)} |x-y|^{\alpha-n} f(y)\,dy$$

and

$$u_2(x) = \int_{B(\xi, 2|x-\xi|)} |x-y|^{\alpha-n} f(y)\,dy.$$

Since $y \in R^n - B(\xi, 2|\xi - x|)$ implies $|\xi - y| \le 2|x-y|$, we can apply Lebesgue's dominated convergence theorem to obtain

$$u_1(x) \longrightarrow U_\alpha f(\xi) \qquad \text{as } x \longrightarrow \xi.$$

If $\xi \in \partial D - E_f$, then $U_f(\xi) < \infty$. By Lemma 7.1, $C_{\alpha,\Phi_p,\omega}(E_f) = 0$. On the other hand, in view of Lemma 6.2 with $r = 3|x-\xi|$, $s = 0$ and f replaced by the restriction of f to the ball $B(\xi, 2|x-\xi|)$, we can establish

$$u_2(x) \le M_1\left([\tau(x_n)]^{-1}\int_{B(\xi,2|x-\xi|)} \Phi_p(f(y))\omega(|y_n|)\,dy\right)^{1/p} + M_1|x-\xi|^{\alpha-\delta},$$

where $0<\delta<\alpha$. If $\xi\in\partial D-F_{f,h}$, then, noting that $[\tau(x_n)]^{-1}\le M(a)[h(|x-\xi|)]^{-1}$ for $x\in T_\psi(\xi, a)$, we see that $u_2(x)$ tends to zero as $x\to\xi$ along $T_\psi(\xi, a)$. In case $\tau(0)>0$, $\tau(x_n)^{-1}$ is bounded for $0<x_n<1$, so that $u_2(x)$ tends to zero as $x\to\xi$, $x\in D$. Since $H_h(F_{f,h})=0$ by Corollary 7.1, the proof of Theorem 7.1 is completed.

By using Theorem 7.1 and Corollary 7.2, we have

THEOREM 7.2. *Assume that* (ω2) *and* (7.1) *hold. Let f be a nonnegative measurable function on R^n satisfying* (1.1) *and* (6.1). *If $\tau_1(r)\le M\tau_2(r)$ for $0<r<1$, then there exists a set $E\subset\partial D$ such that $C_{\alpha,\Phi_p,\omega}(E)=0$ and $U_\alpha f(x)$ has a nontangential limit at any $\xi\in\partial D-E$; that is, $U_\alpha f(x)$ has a finite T_1-limit at any $\xi\in\partial D-E$.*

COROLLARY 7.5. *Let $0<\alpha p-n\le\beta<p-1$. Let f be a nonnegative measurable function on R^n satisfying* (1.1) *and*

$$(7.2)\qquad \int_G \Phi_p(|f(y)|)|y_n|^\beta\,dy<\infty \qquad \text{for any bounded open set } G\subset R^n.$$

Then there exists a set $E\subset\partial D$ such that $C_{\alpha,\Phi_p,\beta}(E)=0$ and $U_\alpha f(x)$ has a nontangential limit at any $\xi\in\partial D-E$.

In fact, in case $\alpha p>n$, $\varphi^*(1)<\infty$ and, moreover, we find

$$\tau_2(r)\sim r^{n-\alpha p+\beta}\varphi(r^{-1}) \quad \text{as} \quad r\longrightarrow 0,$$

so that $\tau_1(r)\le M_1\tau_2(r)$ for $0<r<1$. Now Corollary 7.5 is a direct consequence of Theorem 7.2.

THEOREM 7.3. *Assume that* (7.1) *is satisfied, and let $-1<\beta<p-1$. Let f be a nonnegative measurable function on R^n satisfying* (1.1) *and* (7.2).

(i) *If $n-\alpha p+\beta>0$, then for $\gamma>1$, there exists a set $E_\gamma\subset\partial D$ such that $H_h(E_\gamma)=0$, where $h(r)=\tau_2(r^\gamma)$ with*

$$\tau_2(r)=\inf_{r\le t\le 1} t^\beta\left(\int_0^t [s^{n-\alpha p}\varphi(s^{-1})]^{-1/(p-1)}ds/s\right)^{-p+1},$$

and $U_\alpha f$ has a finite T_γ-limit at any $\xi\in\partial D-E_\gamma$.

(ii) *If $\beta=\alpha p-n>0$, then there exists a set $E\subset\partial D$ such that $C_{\alpha,\Phi_p,\beta}(E)=0$ and $U_\alpha f$ has a finite T_γ-limit at any $\xi\in\partial D-E$ for any $\gamma\ge 1$.*

(iii) *If $\beta=\alpha p-n=0$ or $n-\alpha p+\beta<0$, then $U_\alpha f$ has a finite limit at* *any* $\xi \in \partial D$.

PROOF. First note by (7.1) that $\alpha p \geq n$. Hence, if $n - \alpha p + \beta > 0$, then $\beta > 0$ and

$$\tau_1(r) \geq M_1 r^{n-\alpha p+\beta} \varphi(r^{-1}) \geq M_2 \tau_2(r)$$

for $0 < r < 1$, according to the notation in Theorem 7.1. Now we apply Theorem 7.1, together with Corollary 7.3, in order to prove (i).

If $\beta = \alpha p - n > 0$, then

$$\tau_2(r) \geq M_3 r^{n-\alpha p+\beta} \varphi(r^{-1}),$$

so that $\tau_2(0) > 0$. Further, in this case, $\tau_1(r^\gamma) \sim [\kappa_1(r)]^{-p}$ for any $\gamma > 1$. Hence, if we set $h_\gamma(r) = \tau(r^\gamma)$ with τ in Theorem 7.1, then $h_\gamma(r) \sim [\kappa_1(r)]^{-p}$ for any $\gamma > 1$. It follows from Corollary 7.2 that $C_{\alpha,\Phi_p,\beta}(F_{f,h_\gamma}) = 0$. Now (ii) is a consequence of Theorem 7.1.

If $\beta \leq 0$, then

$$\kappa_1(0) \leq \varphi^*(1) < \infty,$$

on account of (7.1). Further, in this case, $\tau_2(0) > 0$. If $0 < \beta < \alpha p - n$, then $\kappa_1(0) < \infty$, so that $\tau_1(0) > 0$, and further $\tau_2(0) > 0$, as seen above. In the case of (iii), it follows that $\tau(0) > 0$. Thus (iii) also follows from Theorem 7.1.

REMARK 7.2. Theorem 7.2, together with Theorem 7.3, (ii), is best possible as to the size of the exceptional sets; that is, if $E \subset \partial D$ and $C_{\alpha,\Phi_p,\omega}(E) = 0$, then we can find a nonnegative measurable function f on R^n such that $U_\alpha f \not\equiv \infty$, $U_\alpha f = \infty$ on E and

$$\int \Phi_p(f(y))\omega(|y_n|)\,dy < \infty$$

(cf. the proof of Lemma 2.2, (iv)). Clearly, $U_\alpha f$ does not have a finite T_ψ-limit at any $\xi \in E$, by the lower semicontinuity of $U_\alpha f$.

REMARK 7.3. In Theorem 7.2, if (7.1) does not hold, then we can not generally expect the existence of limits of u along $T_\psi(\xi, a)$.

In fact, by Corollary 7.4, $C_{\alpha,\Phi_p}(F) = 0$ for any countable set $F \subset D$. Hence we can find a nonnegative measurable function f on D such that $U_\alpha f \not\equiv \infty$, $U_\alpha f = \infty$ on F and

$$\int_D \Phi_p(f(y))\omega(y_n)\,dy < \infty \tag{7.3}$$

(see Lemma 2.2, (iv)). If in addition F is everywhere dense in D, then we see easily that $U_\alpha f$ does not have a finite T_ψ-limit at any boundary point of D.

8. Curvilinear limits

Let ψ be a positive nondecreasing continuous function on $[0, \infty)$ satisfying conditions (Δ_2) and $(\psi 1)$, as before. Take continuous functions ψ_j, $j = 2, 3, \ldots, n-1$, on $[0, \infty)$ such that $\psi_j(0) = 0$ and

$$|\psi_j(t) - \psi_j(s)| \leq M|t - s| \qquad \text{for any } s, t \geq 0.$$

For convenience, let $\psi_1(r) = r$, $\psi_n(r) = \psi(r)$ and $\Psi(r) = (\psi_1(r), \ldots, \psi_n(r))$. For $\xi \in \partial D$, we define

$$\xi(r) = \xi + \Psi(r) \quad \text{and} \quad L_\Psi(\xi) = \{\xi(r); 0 < r < 1\}.$$

THEOREM 8.1. *Let ω be a positive nondecreasing function on $(0, \infty)$ satisfying both $(\omega 1)$ and $(\omega 2)$. Assume further that there exists a positive nondecreasing function ω^* on $(0, \infty)$ satisfying the following conditions*:

(i) $\omega^*(2r) \leq M\omega^*(r)$ *on* $(0, \infty)$;

(ii) $\int_0^r \omega^*(s)^{1/p} s^{-1}\, ds \leq M\omega(r)^{1/p}$ *for any* $r > 0$,

where M is a positive constant. Let τ_1 be as in Theorem 7.1,

$$\tau_2^*(r) = \inf_{r \leq t \leq 1} t^{n-\alpha p}\omega^*(t)\varphi(t^{-1})$$

and

$$h^*(r) = \min\{\tau_1(\psi(r)),\ \tau_2^*(\psi(r))\}$$

for $0 < r < 1$. Let f be a nonnegative measurable function on R^n satisfying conditions (1.1) and (6.1). Then there exist two sets E_1 and E_2 such that $C_{\alpha,\Phi_p,\omega}(E_1) = 0$, $H_{h^*}(E_2) = 0$ and

$$\lim_{r \to 0} U_\alpha f(\xi(r)) = U_\alpha f(\xi) \qquad \text{for any } \xi \in \partial D - (E_1 \cup E_2).$$

PROOF. Letting $a = 10^{-1}$ and $\xi \in \partial D$, we write $U_\alpha f(x) = u_1(x) + u_2(x) + u_3(x)$, where

$$u_1(x) = \int_{R^n - B(\xi, 2|x-\xi|)} |x - y|^{\alpha-n} f(y)\, dy,$$

$$u_2(x) = \int_{B(\xi, 2|x-\xi|) - B(x, ax_n)} |x - y|^{\alpha-n} f(y)\, dy,$$

$$u_3(x) = \int_{B(x, ax_n)} |x - y|^{\alpha-n} f(y)\, dy.$$

If $\xi \in \partial D - E_f$, then, as in the proof of Theorem 7.1, $u_1(x)$ has the finite limit $U_\alpha f(\xi)$ at ξ. Further Lemma 6.2 yields

$$|u_2(x)| \le M_1 \kappa_1(x_n)\left(\int_{B(\xi, 2|x-\xi|)} \Phi_p(f(y))\omega(|y_n|)\,dy\right)^{1/p} + M_1|x-\xi|^{\alpha-\delta}$$

for $x \in D \cap B(\xi, 1)$, where $0 < \delta < \alpha$. If we set

$$E' = \left\{\xi \in \partial D;\ \limsup_{r\to 0}\,[\tau_1(\psi(r))]^{-1}\int_{B(\xi,r)} \Phi_p(f(y))\omega(|y_n|)\,dy > 0\right\},$$

then Lemma 7.2 implies $H_{h^*}(E') = 0$. Moreover, if $b > 0$ and $x \in T_\psi(\xi, b)$, then we have

$$\kappa_1(x_n) \le M(b)\,[\tau_1(\psi(|x-\xi|))]^{-1/p}$$

for some positive constant $M(b)$. Hence we see that u_2 has T_ψ-limit zero when $\xi \in \partial D - E'$. If $x = \xi(r) \in L_\Psi(\xi)$, $r > 0$, then

$$|x-\xi| = |\Psi(r)| = (\textstyle\sum_{j=1}^n |\psi_j(r)|^2)^{1/2} \le M_2 r,$$

so that

$$\psi(|x-\xi|) \le \psi(M_2 r) \le M_3\psi(r) = M_3 x_n.$$

Consequently, $L_\Psi(\xi) \subset T_\psi(\xi, M_3)$, and it follows that $u_2(x)$ tends to zero as $x \to \xi$ along the curve $L_\Psi(\xi)$ when $\xi \in \partial D - E'$. Thus it suffices to prove that $u_3(x)$ tends to zero as $x \to \xi$ along the curve $L_\Psi(\xi)$, for any $\xi \in \partial D$ except those in a set E'' such that $H_{h^*}(E'') = 0$. For this purpose, we may assume that $f = 0$ outside $D \cap B(0, N)$ for some $N > 1$, so that f satisfies (7.3). Set

$$X_j = \{x \in D;\ 2^{-j} \le x_n < 2^{-j+1},\ u_3(x) > a_j^{-1}\},$$

where $\{a_j\}$ is a sequence of positive numbers such that

$$\lim_{j\to\infty} a_j = \infty, \qquad \lim_{j\to\infty} j^{-1}a_j = 0 \tag{8.1}$$

and

$$\textstyle\sum_j a_j^p \displaystyle\int_{D_j} \Phi_p(f(y))\omega(y_n)\,dy < \infty$$

with $D_j = \{x \in D;\ 2^{-j-1} < x_n < 2^{-j+2}\}$. For a set $X \subset D$, we denote by $\tilde{X}$ the set of all $\xi \in \partial D$ such that $\xi(r) \in X$ for some r with $0 < r < 1$. We consider the set

$$E'' = \textstyle\bigcap_k(\bigcup_{j>k} \tilde{X}_j).$$

Then it is easy to see that $u_3(\xi(r))$ tends to zero as $r \to 0$ whenever $\xi \in \partial D - E''$. What remains is to prove that $H_{h^*}(E'') = 0$. If $x \in X_j$, then

$$a_j^{-1} < \int_{B(x,ax_n)} |x-y|^{\alpha-n} f(y)\,dy$$
$$= (n-\alpha)\int_0^{ax_n} F_1(x,r) r^{\alpha-n-1}\,dr + (ax_n)^{\alpha-n} F_1(x, ax_n),$$

where $F_1(x,r) = \int_{B(x,r)} f(y)\,dy$. By Lemma 2.1, we have

$$F_1(x,r) \le M_4[r^{n-\varepsilon} + r^{n/p'}\{\varphi(r^{-1})\}^{-1/p}\{F_{\Phi_p}(x,r)\}^{1/p}], \tag{8.2}$$

where $0 < \varepsilon < \min\{1, \alpha\}$ and $F_{\Phi_p}(x,r) = \int_{B(x,r)} \Phi_p(f(y))\,dy$. Let $x \in X_j$ and assume that

$$F_{\Phi_p}(x,r) < \tilde{M} a_j^{-p} \omega(x_n)^{-1} \tau_2^*(r) \tag{8.3}$$

for any r with $0 < r \le ax_n$. Then it follows from (8.2) that

$$1 \le M_4(n-\alpha)\Big(a_j \int_0^{ax_n} r^{\alpha-\varepsilon-1}\,dr$$
$$+ \tilde{M}^{1/p}\{\omega(x_n)\}^{-1/p} \int_0^{ax_n} \{\tau_2^*(r) r^{\alpha p-n}\varphi(r^{-1})^{-1}\}^{1/p} r^{-1}\,dr\Big)$$
$$+ M_4(a_j(ax_n)^{\alpha-\varepsilon} + \tilde{M}^{1/p}(ax_n)^{\alpha-n/p}\{\varphi((ax_n)^{-1})\omega(ax_n)\}^{-1/p}\{\tau_2^*(ax_n)\}^{1/p}).$$

Since $\tau_2^*(r) \le \omega^*(r)[r^{n-\alpha p}\varphi(r^{-1})]$, in view of conditions (i) and (ii) for ω^*, we see that

$$1 \le M_5\Big(a_j(ax_n)^{\alpha-\varepsilon} + \tilde{M}^{1/p}\{\omega(x_n)\}^{-1/p}\int_0^{ax_n}\{\omega^*(r)\}^{1/p} r^{-1}\,dr\Big)$$
$$+ M_4\tilde{M}^{1/p}\{\omega(ax_n)\}^{-1/p}\{\omega^*(ax_n)\}^{1/p} \le M_6 a_j 2^{-j(\alpha-\varepsilon)} + M_6\tilde{M}^{1/p},$$

where M_6 does not depend on j nor $\tilde{M}$. In view of (8.1), there is j_0 such that $M_6 a_j 2^{-j(\alpha-\varepsilon)} < 2^{-1}$ for any $j \ge j_0$. Thus, if $x \in X_j$, $j \ge j_0$, and (8.3) holds for all $r \in (0, ax_n]$, then $\tilde{M}$ must satisfy

$$M_6\tilde{M}^{1/p} \ge 2^{-1}.$$

Now, if we take $\tilde{M}$ so small that $M_6\tilde{M}^{1/p} < 2^{-1}$, then, for any $x \in X_j$, $j \ge j_0$, we can find $r(x)$, $0 < r(x) \le ax_n$, such that

$$F_{\Phi_p}(x, r(x)) \ge \tilde{M} a_j^{-p}\{\omega(x_n)\}^{-1}\tau_2^*(r(x)).$$

Since $\{B(x, r(x));\, x \in X_j\}$ covers X_j, there exists a mutually disjoint (finite or) countable family $\{B(x_{j,k}, r_{j,k})\}$, $r_{j,k} = r(x_{j,k})$, such that $x_{j,k} \in X_j$ for all k and $\{B(x_{j,k}, 5r_{j,k})\}$ covers X_j. Then

(8.4) $$\sum_k \tau_2^*(r_{j,k}) \le M_7 a_j^p \int_{D_j} \Phi_p(f(y))\omega(y_n)\,dy.$$

Now we are ready to show

$$H_{h^*}(E'') = 0.$$

Let $\xi_{j,k}$ be the point on ∂D such that $x_{j,k} = \xi_{j,k}(s_{j,k})$ for some $s_{j,k} > 0$. Since $\psi(r)$ is strictly increasing on account of condition $(\psi 1)$, for any $r > 0$ we can find only one r^* satisfying $\psi(r^*) = r$. If $\xi \in \partial D$, $\zeta \in \partial D$, $x = \xi(t)$, $y = \zeta(s)$ and $y \in B(x, r)$ with $r < ax_n < 1$, then condition $(\psi 1)$ gives

$$\psi(|s-t|) \le |\psi(s) - \psi(t)| = |x_n - y_n| \le |x - y| < r = \psi(r^*),$$

so that $|s - t| < r^*$. Also, if $0 < r < 1$, then $r^* = \psi^{-1}(r) < \psi^{-1}(1)$, which together with $(\psi 1)$ yields

$$\frac{\psi(r^*)}{r^*} \le \frac{1}{\psi^{-1}(1)} \quad \text{or} \quad r < \frac{r^*}{\psi^{-1}(1)}.$$

Hence

$$|\xi - \zeta| \le |x - y| + \sum_{i=1}^{n-1} |\psi_i(s) - \psi_i(t)| \le r + |s-t| + (n-2)M|s-t| \le M_8 r^*.$$

This implies $\bigcup_{j \ge \ell}(\bigcup_k \{B(\xi_{j,k}, M_8(5r_{j,k})^*)\}) \supset E''$ for any $\ell \ge j_0$, so that

$$\begin{aligned} H_{h^*}^{(\delta_\ell^*)}(E'') &\le \textstyle\sum_{j \ge \ell}(\sum_k h^*(M_8(5r_{j,k})^*)) \\ &\le M_9 \textstyle\sum_{j \ge \ell}(\sum_k \tau_2^*(r_{j,k})) \\ &\le M_{10} \textstyle\sum_{j \ge \ell} a_j^p \displaystyle\int_{D_j} \Phi_p(f(y))\omega(y_n)\,dy \end{aligned}$$

by (8.4), where $\delta_\ell^* = \sup_{j \ge \ell}\{\sup_k M_8(5r_{j,k})^*\}$. Here note

$$\psi(\delta_\ell^*) = \sup_{j \ge \ell}\{\sup_k \psi(M_8(5r_{j,k})^*)\} \le M_{11} 2^{-\ell+1},$$

so that $\lim_{\ell \to \infty} \delta_\ell^* = 0$. Thus it follows that $H_{h^*}(E'') = 0$, and the proof of Theorem 8.1 is completed.

COROLLARY 8.1. *Let $\alpha p - n \le \beta < p - 1$. Let $\psi(r) = r^\gamma$ for $\gamma \ge 1$; in this case, $\Psi(r) = (r, \psi_2(r), \dots, \psi_{n-1}(r), r^\gamma)$. Further, let f be as in Theorem 7.3.*

- (i) *If $\beta > 0$, $n - \alpha p + \beta > 0$ and $\gamma > 1$, then there exists a set $E \subset \partial D$ such that $H_h(E) = 0$ with $h(r) = \inf_{r \le t \le 1} t^{\gamma(n-\alpha p+\beta)}\varphi(t^{-1})$ and $U_\alpha f$ has a finite limit along the curve $L_\Psi(\xi)$, for any $\xi \in \partial D - E$.*
- (ii) *If $\beta > 0$, $n - \alpha p + \beta > 0$ and $\gamma = 1$, then there exists a set $E \subset \partial D$ such that $C_{\alpha,\Phi_p,\beta}(E) = 0$ and $U_\alpha f$ has a finite limit along the curve $L_\Psi(\xi)$, for any $\xi \in \partial D - E$.*

(iii) *If* $\beta \leq 0$ *and* $\gamma > 1$, *then there exists a set* $E \subset \partial D$ *such that* $H_{\gamma(n-\alpha p+\delta)}(E) = 0$ *for any* $\delta > 0$, *that is, E has Hausdorff dimension at most* $\gamma(n-\alpha p)$, *and* $U_\alpha f$ *has a finite limit along the curve* $L_\Psi(\xi)$, *for any* $\xi \in \partial D - E$.

PROOF. If $\beta > 0$, then we can take

$$\omega^*(r) = \omega(r) = r^\beta \quad \text{and} \quad \tau_2^*(r) = \inf_{r \leq t \leq 1} t^{n-\alpha p+\beta}\varphi(t^{-1})$$

in Theorem 8.1 If in addition $n - \alpha p + \beta > 0$, then $h^* = h$. In case $\gamma > 1$, $C_{\alpha,\Phi_p,\beta}(F) = 0$ implies $H_{h^*}(F) = 0$ by Corollary 7.3. Thus (i) follows from Theorem 8.1. In case $\gamma = 1$, $\tau_1(r) \leq M\tau_2^*(r)$ by (5.1) and $h^*(r) \sim [\kappa_1(r)]^{-p}$. In view of Corollary 7.2, $H_h(F) = 0$ implies $C_{\alpha,\Phi_p,\beta}(F) = 0$. Hence (ii) also follows from Theorem 8.1.

If $\beta \leq 0$, then, for $\delta > 0$, consider

$$\omega_\delta(r) = \omega_\delta^*(r) = r^\delta.$$

Since $n - \alpha p + \delta > n - \alpha p + \beta \geq 0$, we can apply (i) with $\beta = \delta$ to establish (iii).

Here we give radial limit results as generalizations of [12, Theorems 3 and 4].

THEOREM 8.2. *Let* $-1 < \beta < p - 1$ *and* f *be as in Theorem* 7.3. *Then there exists a set* $E \subset \partial D$ *satisfying*

(i) $C_{\alpha,\Phi_p,\beta}(E) = 0$;
(ii) *to each* $\xi \in \partial D - E$, *there corresponds a set* E_ξ *such that* $C_{\alpha,\Phi_p}(E_\xi) = 0$ *and*

$$\lim_{r\to 0} U_\alpha f(\xi + r\zeta) = U_\alpha f(\xi) \qquad \text{for any} \quad \zeta \in D \cap \partial B(0, 1) - E_\xi.$$

This fact can be proved by [14, Theorem 2′] and the contractive property of the capacity C_{α,Φ_p}, which is derived in the same manner as that of $C_{\alpha,p}$ (see [11, Lemma 5]). More precisely, to complete the proof, apply the proofs of [12, Theorem 4] and [14, Theorem 4].

THEOREM 8.3. *Let* ω *be a nonnegative nondecreasing function on* $[0, \infty)$ *satisfying* (ω1) *and* (ω2). *Let* $\zeta \in D$ *be fixed. If f is a nonnegative measurable function on* R^n *satisfying* (1.1) *and* (6.1), *then there exists a set* $E \subset \partial D$ *such that* $C_{\alpha,\Phi_p,\omega}(E) = 0$ *and*

$$\lim_{t\downarrow 0} U_\alpha f(\xi + t\zeta) = U_\alpha f(\xi) \qquad \text{at any} \quad \xi \in \partial D - E.$$

PROOF. Define

$$u_1(x) = \int_{R^n - B(x, x_n/2)} |x - y|^{\alpha - n} f(y)\, dy,$$

$$u_2(x) = \int_{B(x, x_n/2)} |x - y|^{\alpha - n} f(y)\, dy$$

for $x \in D$. If $x = \xi + t\zeta$, $\xi \in \partial D$, $t > 0$ and $y \in R^n - B(x, x_n/2)$, then

$$|y - \xi| \leq |y - x| + t|\zeta| \leq [1 + 2(|\zeta|/\zeta_n)]|x - y|,$$

so that

$$\lim_{t \downarrow 0} u_1(\xi + t\zeta) = U_\alpha f(\xi)$$

for every $\xi \in \partial D$. In fact, if $U_\alpha f(\xi) = \infty$, then it follows readily from Fatou's lemma; if $U_\alpha f(\xi) < \infty$, then apply Lebesgue's dominated convergence theorem. As in the proof of Theorem 6.1, we can find a set $E \subset D$ such that

$$\lim_{x_n \to 0, x \in D \cap A(a) - E} u_2(x) = 0$$

and

$$\sum_{j=1}^{\infty} \omega(2^{-j}) C_{\alpha, \Phi_p}(E_j \cap A(a); D_j \cap A(2a)) < \infty$$

for any $a > 0$, where $A(a) = \{x = (x_1, \ldots, x_n); |x_j| < a \text{ for any } j\}$. Define

$$E_j^* = \{(x', 0); (x', t) \in E_j \text{ for some } t > 0\},$$

$$\tilde{E}_j = \{(x', 0); (x', 0) + t\zeta \in E_j \text{ for some } t > 0\}.$$

Letting $D_j' = \{(x', x_n); |x_n| < 2^{-j+2}\}$, we have by the contractive property of C_{α, Φ_p} (cf. [10, Lemma 1]),

$$C_{\alpha, \Phi_p}(E_j^* \cap A(a); D_j' \cap A(2a)) \leq C_{\alpha, \Phi_p}(E_j \cap A(a); D_j' \cap A(2a)),$$

so that

$$\begin{aligned} C_{\alpha, \Phi_p, \omega}(E_j^* \cap A(a); A(2a)) &\leq C_{\alpha, \Phi_p, \omega}(E_j^* \cap A(a); D_j' \cap A(2a)) \\ &\leq \omega(2^{-j+2}) C_{\alpha, \Phi_p}(E_j^* \cap A(a); D_j' \cap A(2a)) \\ &\leq M_1 \omega(2^{-j}) C_{\alpha, \Phi_p}(E_j \cap A(a); D_j \cap A(2a)). \end{aligned}$$

On the other hand,

$$C_{\alpha, \Phi_p, \omega}(\tilde{E}_j \cap A(a); A(2a)) \leq M_2 C_{\alpha, \Phi_p, \omega}(E_j^* \cap A(a); A(2a))$$

(cf. [11, Lemma 3]). Hence if we set $\tilde{E} = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} \tilde{E}_j$, then $C_{\alpha, \Phi_p, \omega}(\tilde{E}) = 0$ and

$$\lim_{t \to 0} u_2(\xi + t\zeta) = 0 \qquad \text{whenever} \quad \xi \in \partial D - \tilde{E}.$$

Thus Theorem 8.3 is obtained.

REMARK 8.1. In case $\varphi \equiv 1$, these results are considered in Wu [27] and Mizuta [14].

Finally we study the best-possibility of our theorems, as far as the exceptional sets are concerned.

THEOREM 8.4. *Let* $n = 2$. *Let* ω *and* ψ *be positive nondecreasing continuous functions on* $(0, \infty)$ *satisfying the* (Δ_2) *condition, together with the following*:

(i) ψ *satisfies* $(\psi 1)$.
(ii) ω *satisfies both* $(\omega 2)$ *and* $(\omega 3)$.

Suppose there exists $c > 2\psi(1)$ *such that* $2\kappa_1(cr) < \kappa_1(\psi(r))$ *for* $0 < r < 1$, *and set* $h(r) = [\kappa_1(\psi(r))]^{-p}$. *If* $E \subset \partial D$ *and* $H_h(E) = 0$, *then there exists a nonnegative measurable function* f *on* D *such that* $U_\alpha f \not\equiv \infty$, $\int_D \Phi_p(f(y))\omega(y_2)\,dy < \infty$ *and*

$$\limsup_{r\to 0} U_\alpha f(\xi + (r, \psi(r))) = \infty \qquad \text{for any} \quad \xi \in E.$$

PROOF. For each positive integer i, we can find a family $\{B_{i,j}\}$ of discs such that $B_{i,j} = B(x_{i,j}, r_{i,j})$, $\sum_j h(r_{i,j}) < 2^{-i}$ and $E \subset \bigcup_j B_{i,j}$. Here we may assume further that $x_{i,j} \in \partial D$ and $r_{i,j} < 1$. Let $z_{i,j,\ell} = x_{i,j} + (\ell r_{i,j}, 0)$ and $C_{i,j,\ell} = B(z_{i,j,\ell}, cr_{i,j}) - B(z_{i,j,\ell}, 2^{-1}\psi(r_{i,j}))$ for $\ell = 0, 1$. For simplicity, set $\tilde{\eta}(r) = r^{2-\alpha p}\varphi(r^{-1})\omega(r)$, and define

$$f_{i,j,\ell}(y) = i[h(r_{i,j})]^{p'/p}|y - z_{i,j,\ell}|^{-\alpha}[\tilde{\eta}(|y - z_{i,j,\ell}|)]^{-p'/p}$$

for $y \in D \cap C_{i,j,\ell}$; and define $f_{i,j,\ell}(y) = 0$ otherwise. Consider the function $f = \sup_{i,j,\ell} f_{i,j,\ell}$. Since, by $(\omega 2)$, $r^{\beta_1 - 1/p}\omega(r)^{-1/p}$ is nondecreasing on $(0, \infty)$ for some $\beta_1 < 1$, we note

$$f_{i,j,\ell}(y) \le M_1|y - z_{i,j,\ell}|^{-\beta}$$

for $\beta > \alpha + (2 - \alpha p)p'/p + (\beta_1 - 1/p)p'$. Hence we have

$$\begin{aligned}
&\int_D \Phi_p(f_{i,j,\ell}(y))\omega(y_2)\,dy \\
&\le M_2 i^p[h(r_{i,j})]^{p'} \int_{D\cap C_{i,j,\ell}} |y - z_{i,j,\ell}|^{-\alpha p}[\tilde{\eta}(|y - z_{i,j,\ell}|)]^{-p'}\varphi(|y - z_{i,j,\ell}|^{-1})\omega(y_2)\,dy \\
&\le M_3 i^p[h(r_{i,j})]^{p'} \int_{C_{i,j,\ell}} |y - z_{i,j,\ell}|^{-2}[\tilde{\eta}(|y - z_{i,j,\ell}|)]^{-p'+1}\,dy
\end{aligned}$$

$$\leq M_4 i^p [h(r_{i,j})]^{p'} \int_{2^{-1}\psi(r_{i,j})}^{cr_{i,j}} [\tilde{\eta}(r)]^{-p'/p} r^{-1}\, dr$$

$$\leq M_5 i^p h(r_{i,j}),$$

so that

$$\int_D \Phi_p(f(y))\omega(y_2)\, dy \leq \sum_{i,j,\ell} \int_D \Phi_p(f_{i,j,\ell}(y))\omega(y_2)\, dy \leq 2M_5 \sum_i i^p 2^{-i} < \infty.$$

Next we see that for $x \in D \cap B(z_{i,j,\ell}, \psi(r_{i,j}))$,

$$\int |x-y|^{\alpha-2} f(y)\, dy \geq M_6 i [h(r_{i,j})]^{p'/p} \int_{C_{i,j,\ell}} |y - z_{i,j,\ell}|^{-2} [\tilde{\eta}(|y - z_{i,j,\ell}|)]^{-p'/p} dy$$

$$= M_7 i [h(r_{i,j})]^{p'/p} \int_{2^{-1}\psi(r_{i,j})}^{cr_{i,j}} [\tilde{\eta}(r)]^{-p'/p} r^{-1}\, dr \geq M_8 i.$$

Let $\xi \in E$. For each i, there is j such that $\xi \in B_{i,j}$. Further observe that the curve $L_\Psi(\xi)$ intersects at least one of two half balls $D \cap B(z_{i,j,\ell}, \psi(r_{i,j}))$, $\ell = 0, 1$. Consequently,

$$\limsup_{r \to 0} U_\alpha f(\xi + (r, \psi(r))) \geq \limsup_{i \to \infty} M_8 i = \infty.$$

REMARK 8.2. Let $\omega(r) = r^\beta$. If $-1 < \beta < p-1$, then ω satisfies both (ω2) and (ω3). If in addition $2 - \alpha p + \beta > 0$, then one can take c so large that

$$2\kappa_1(cr) < \kappa_1(\psi(r)) \qquad \text{for any} \quad 0 < r < 1;$$

in this case, $h(r) \sim r^{2-\alpha p+\beta}\varphi(r^{-1})$ as $r \to 0$, in Theorem 8.4. Moreover, if α is a positive integer m, then, as will be shown later (see Lemma 12.1),

$$\int_G \Phi_p(|\nabla_m U_m f(x)|)|x_n|^\beta\, dx < \infty \qquad \text{for any bounded open set} \quad G \subset R^n.$$

9. Beppo-Levi-Deny functions

For an open set $G \subset R^n$, we denote by $BL_m(L^p(G))$ the space of all functions $u \in L^p_{loc}(G)$ such that $D^\lambda u \in L^p(G)$ for any λ with $|\lambda| = m$, where $D^\lambda = (\partial/\partial x)^\lambda = (\partial/\partial x_1)^{\lambda_1} \cdots (\partial/\partial x_n)^{\lambda_n}$; if the restriction of u to any relatively compact open subset G' of G belongs to $BL_m(L^p(G'))$, then we write $u \in BL_m(L^p_{loc}(G))$ (see [3]).

In order to study the boundary behavior of Beppo-Levi-Deny functions on D, we have to prepare an integral representation for functions in $BL_m(L^p(R^n))$. The following sobolev integral representation for infinitely differentiable functions with compact support is fundamental (cf. Reshetnyak [22]).

LEMMA 9.1. *If* $\psi \in C_0^\infty(R^n)$, *then*

$$\psi(x) = \sum_{|\lambda|=m} a_\lambda \int \frac{(x-y)^\lambda}{|x-y|^n} D^\lambda \psi(y)\, dy,$$

where $\{a_\lambda\}$ *are constants*; $a_\lambda = m/(c_n \lambda!)$.

Our first aim in this section is to show the following result.

THEOREM 9.1. *If* u *is a function in* $BL_m(L^p_{loc}(R^n))$ *such that*

$$\int (1+|y|)^{m-n} |D^\lambda u(y)|\, dy < \infty \tag{9.1}$$

for any λ *with length* m, *then there exists a polynomial* P *of degree at most* $m-1$ *such that*

$$u(x) = \sum_{|\lambda|=m} a_\lambda \int \frac{(x-y)^\lambda}{|x-y|^n} D^\lambda u(y)\, dy + P(x)$$

holds for almost every x *in* R^n.

REMARK 9.1. In [8, Theorem 3.1], this representation is given under the assumption of the existence of $\{\varphi_j\} \subset C_0^\infty(R^n)$ such that

$$\lim_{j\to\infty} \sum_{|\lambda|=m} \| D^\lambda(\varphi_j - u)\|_p = 0.$$

PROOF OF THEOREM 9.1. Let $\psi \in C_0^\infty(R^n)$ and $|\mu| = m$. By condition (9.1), we can apply Fubini's lemma and Lemma 9.1 to obtain

$$\begin{aligned}
&\int \left(\sum_{|\lambda|=m} a_\lambda \int \frac{(x-y)^\lambda}{|x-y|^n} D^\lambda u(y)\, dy\right) D^\mu \psi(x)\, dx \\
&= \sum_{|\lambda|=m} a_\lambda \int \left(\int \frac{(x-y)^\lambda}{|x-y|^n} D^\mu \psi(x)\, dx\right) D^\lambda u(y)\, dy \\
&= \sum_{|\lambda|=m} a_\lambda \int \left(\int \frac{z^\lambda}{|z|^n} D^\mu \psi(z+y)\, dz\right) D^\lambda u(y)\, dy \\
&= \sum_{|\lambda|=m} a_\lambda \int \frac{z^\lambda}{|z|^n} \left(\int D^\mu \psi(z+y) D^\lambda u(y)\, dy\right) dz \\
&= \sum_{|\lambda|=m} a_\lambda \int \frac{z^\lambda}{|z|^n} \left(\int D^\lambda \psi(z+y) D^\mu u(y)\, dy\right) dz \\
&= \int \left(\sum_{|\lambda|=m} a_\lambda \int \frac{z^\lambda}{|z|^n} D^\lambda \psi(z+y)\, dz\right) D^\mu u(y)\, dy
\end{aligned}$$

$$= (-1)^m \int \psi(y) D^\mu u(y)\, dy$$

$$= \int u(y) D^\mu \psi(y)\, dy.$$

Hence it follows that $u(x) - \sum_{|\lambda|=m} a_\lambda \int \frac{(x-y)^\lambda}{|x-y|^n} D^\lambda u(y)\, dy$ is equal a.e. to a polynomial of degree at most $m-1$.

COROLLARY 9.1. *If u is a function in $BL_n(L^1_{loc}(R^n))$, then there exists a continuous function on R^n which is equal to u a.e. on R^n.*

PROOF. For any $\psi \in C_0^\infty(G)$, ψu can be seen as a function in $BL_n(L^1(R^n))$, and hence by Theorem 9.1 there exists a polynomial P such that

$$(\psi u)(x) = \sum_{|\lambda|=n} a_\lambda \int \frac{(x-y)^\lambda}{|x-y|^n} D^\lambda(\psi u)(y)\, dy + P(x) \tag{9.2}$$

for almost every $x \in R^n$. It is easy to see that the right hand side of (9.2) is continuous on R^n. Hence the required assertion follows.

Here we relax condition (9.1). To do this, we introduce the kernel functions:

$$k_\lambda(x) = x^\lambda |x|^{-n}$$

and

$$k_{\lambda,\ell}(x, y) = \begin{cases} k_\lambda(x-y), & \text{if } |y| < 1, \\ k_\lambda(x-y) - \sum_{|\mu| \le \ell} \frac{x^\mu}{u!} (D^\mu k_\lambda)(-y), & \text{if } |y| \ge 1 \end{cases}$$

(see [16], [19]). We now show an extension of Theorem 9.1, in the same manner as [16, Theorem 1] and [19, Theorems 1 and 1′].

THEOREM 9.2. *Let $u \in BL_m(L^p(R^n))$. Then there exists a polynomial P of degree at most $m-1$ such that*

$$u(x) = \sum_{|\lambda|=m} a_\lambda \int k_{\lambda,\ell}(x, y) D^\lambda u(y)\, dy + P(x)$$

holds for almost every x in R^n, where ℓ is the integer such that $\ell \le m - n/p < \ell + 1$.

REMARK 9.2. In view of [16] and [19], the function u is also represented as

$$u(x) = \sum_{|\lambda|=m} b_\lambda \int k^*_{\lambda,\ell}(x, y) D^\lambda u(y)\, dy + P(x),$$

where $k^*_{\lambda,\ell}$ is defined as above with k_λ replaced by $k^*_\lambda = D^\lambda R_{2m}$, R_{2m} denoting the Riesz kernel of order $2m$, $\{b_\lambda\}$ are constants, ℓ is the integer given in Theorem 9.2 and P is a polynomial. More precisely, $\{b_\lambda\}$ is chosen so that

$$\Delta^m = c\sum_{|\lambda|=m} b_\lambda D^{2\lambda}$$

with some constant c. In the latter representation of u, the logarithmic term may appear, and hence Corollary 9.1 may not follow from this representation.

PROOF OF THEOREM 9.2. Set

$$U(x) = \sum_{|\lambda|=m} a_\lambda \int k_{\lambda,\ell}(x, y) D^\lambda u(y)\, dy.$$

By the mean value theorem, we see that

$$|k_{\lambda,\ell}(x, y)| \leq M_1 |x|^{\ell+1} |y|^{m-n-\ell-1}$$

whenever $|y| \geq 1$ and $|y| \geq 2|x|$ (cf. Lemma 2 in [19]). Hence, if $x \in B(0, a)$, $a > 1$, then Hölder's inequality gives

$$\int_{R^n - B(0,2a)} |k_{\lambda,\ell}(x, y)| |D^\lambda u(y)|\, dy$$

$$\leq M_1 a^{\ell+1} \int_{R^n - B(0,2a)} |y|^{m-n-\ell-1} |D^\lambda u(y)|\, dy < \infty$$

for any λ with length m. Since

$$\int_{B(0,2a)} k_{\lambda,\ell}(x, y) D^\lambda u(y)\, dy = \int_{B(0,2a)} k_\lambda(x - y) D^\lambda u(y)\, dy + \text{a polynomial},$$

U is defined almost everywhere and $U \in L^1_{loc}(R^n)$. Note that $\int x^\sigma D^\lambda \psi(x)\, dx = 0$ whenever $|\sigma| < |\lambda|$ and $\psi \in C_0^\infty(R^n)$. Hence, as in the proof of Theorem 9.1, we have for $\psi \in C_0^\infty(R^n)$, $|\mu| = m$ and $|\nu| = m$,

$$\int U(x) D^{\mu+\nu}\psi(x)\, dx = \sum_{|\lambda|=m} a_\lambda \int \left(\int k_{\lambda,\ell}(x, y) D^{\mu+\nu}\psi(x)\, dx \right) D^\lambda u(y)\, dy$$

$$= \sum_{|\lambda|=m} a_\lambda \int \left(\int k_\lambda(x - y) D^{\mu+\nu}\psi(x)\, dx \right) D^\lambda u(y)\, dy.$$

For a positive integer j, set $k_\lambda^{(j)} = x^\lambda \{|x|^2 + (1/j)^2\}^{-n/2}$. Then, in view of Lemma 3.3 in [8], we see that

$$\int k_\lambda^{(j)}(x - y) D^{\mu+\nu}\psi(x)\, dx \longrightarrow \int k_\lambda(x - y) D^{\mu+\nu}\psi(x)\, dx$$

as $j \to \infty$ in $L^q(R^n)$ for any number $q > 1$. Hence we apply Fubini's lemma again to establish

$$\int\left(\int k_\lambda(x-y)D^{\mu+\nu}\psi(x)\,dx\right)D^\lambda u(y)\,dy$$

$$= \lim_{j\to\infty}\int\left(\int k_\lambda^{(j)}(x-y)D^{\mu+\nu}\psi(x)\,dx\right)D^\lambda u(y)\,dy$$

$$= (-1)^m \lim_{j\to\infty}\int\left(\int D^\mu k_\lambda^{(j)}(x-y)D^\nu\psi(x)\,dx\right)D^\lambda u(y)\,dy$$

$$= (-1)^m \lim_{j\to\infty}\int D^\mu k_\lambda^{(j)}(z)\left(\int D^\nu\psi(z+y)D^\lambda u(y)\,dy\right)dz$$

$$= (-1)^m \lim_{j\to\infty}\int D^\mu k_\lambda^{(j)}(z)\left(\int D^\lambda\psi(z+y)D^\nu u(y)\,dy\right)dz$$

$$= \int\left(\int k_\lambda(x-y)D^{\mu+\lambda}\psi(x)\,dx\right)D^\nu u(y)\,dy.$$

Therefore, as in the proof of Theorem 9.1, we find

$$\int U(x)D^{\mu+\nu}\psi(x)\,dx = \int u(x)D^{\mu+\nu}\psi(x)\,dx.$$

Thus $P(x) \equiv u(x) - U(x)$ is equal a.e. to a polynomial of degree at most $2m-1$. By the above considerations, we see also that if $|\mu| = m$, then

$$\left|\int U(x)D^\mu\psi(x)\,dx\right| \le \left|\sum_{|\lambda|=m} a_\lambda \int\left(\int k_\lambda(x-y)\,D^\mu\psi(x)\,dx\right)D^\lambda u(y)\,dy\right|$$
$$\le M\|\psi\|_{p'}\left(\sum_{|\lambda|=m}\|D^\lambda u\|_p\right),$$

on account of Lemma 3.3 in [8]. This implies $D^\mu P \in L^p(R^n)$ for $|\mu| = m$, so that the degree of the polynomial P is at most $m-1$.

THEOREM 9.3. *Let $u \in BL_m(L^p(G))$ satisfy*

$$\sum_{|\lambda|=m}\int_G \Phi_p(|D^\lambda u(x)|)\,dx < \infty.$$

If $\varphi^*(1) < \infty$, that is, $\int_0^1 [r^{n-mp}\varphi(r^{-1})]^{1/(1-p)}r^{-1}\,dr < \infty$, then there exists a continuous function u^* on G such that $u = u^*$ a.e. on G.

PROOF. For any $\psi \in C_0^\infty(G)$, ψu can be seen as a function in $BL_m(L^p(R^n))$ by [24, Chap. 9, Théorème XV (Kryloff)], and hence by Theorem 9.1 there exists a polynomial P such that

$$(9.3)\qquad (\psi u)(x) = \sum_{|\lambda|=m} a_\lambda \int \frac{(x-y)^\lambda}{|x-y|^n} D^\lambda(\psi u)(y)\,dy + P(x)$$

for almost every $x \in R^n$. In view of the proof of Theorem 3.3, note that if G' is a bounded open set in R^n and $\int_{G'} \Phi_p(|f(y)|)\,dy < \infty$, then the function

$$\int_{G'} \frac{(x-y)^\lambda}{|x-y|^n} f(y)\,dy$$

is continuous on G' when $|\lambda| = m$; in case $mp > n$, the continuity is well known as a part of Sobolev's imbedding theorem. Hence, if in addition $\psi = 1$ on a neighborhood of a point $x_0 \in G$, say, $\psi = 1$ on $B(x_0, r_0)$, then

$$\int_{R^n - B(x_0,r_0)} \frac{(x-y)^\lambda}{|x-y|^n} D^\lambda(\psi u)(y)\,dy$$

is continuous on $B(x_0, r_0)$ and

$$\int_{B(x_0,r_0)} \frac{(x-y)^\lambda}{|x-y|^n} D^\lambda(\psi u)(y)\,dy = \int_{B(x_0,r_0)} \frac{(x-y)^\lambda}{|x-y|^n} D^\lambda u(y)\,dy$$

is continuous at x_0, by the above consideration. Thus we can find a continuous function u^* on G which is equal to u a.e. on G.

REMARK 9.3. In case $mp > n$, $\varphi^*(1) < \infty$. Hence Theorem 9.3 gives an extension of Sobolev's imbedding theorem, concerning the continuity of Beppo-Levi-Deny functions.

10. Boundary limits of Beppo-Levi-Deny functions

In this section we study the boundary limits of Beppo-Levi-Deny functions u on the half space $D = \{x = (x', x_n) \in R^{n-1} \times R^1;\, x_n > 0\}$ satisfying (1.4).

We say that a function u on an open set $G \subset R^n$ is (m, Φ_p)-quasicontinuous on G if for any $\varepsilon > 0$ and any bounded open set $G' \subset G$, there exists an open set $G'' \subset G'$ such that $C_{m,\Phi_p}(G''; G') < \varepsilon$ and the restriction of u to $G' - G''$ is continuous. As in Lemma 2.3 in [8], if u is a function in $BL_m(L^p_{loc}(D))$ satisfying (1.4), then we can find a function u^* such that $u^* = u$ a.e. on D and u^* is (m, Φ_p)-quasicontinuous on D. In case $mp > n$, u^* may be taken as a continuous function on D (cf. Remark 9.3).

THEOREM 10.1. *Let u be a function in $BL_m(L^p_{loc}(D))$ satisfying* (1.4). *If*

(10.1) $$\int_0^1 [\varphi(t^{-1})\omega(t)]^{-p'/p}\, dt < \infty,$$

then there exists a function $u^* \in BL_m(L^1_{loc}(R^n))$ *such that* $u^* = u$ *a.e. on* D *and* u^* *is* (m, Φ_p)*-quasicontinuous on* D.

PROOF. Let $a > 1$. As in the proof of Lemma 2.1, using Hölder's inequality, we have

$$\begin{aligned}\int_{D\cap B(0,a)} |D^\lambda u(x)|\, dx &\le \left(\int_{D\cap B(0,a)} \Phi_p(|D^\lambda u(x)|)\omega(x_n)\, dx\right)^{1/p} \\ &\times \left(\int_{D\cap B(0,a)} [\varphi(x_n^{-\delta})\omega(x_n)]^{-p'/p}\, dx\right)^{1/p'} + \int_{D\cap B(0,a)} x_n^{-\delta}\, dx \\ &\le M_1\left(\int_{D\cap B(0,a)} \Phi_p(|D^\lambda u(x)|)\omega(x_n)\, dx\right)^{1/p}\left(\int_0^a [\varphi(t^{-1})\omega(t)]^{-p'/p}\, dt\right)^{1/p'} \\ &+ M_1 a^{n-\delta}\end{aligned}$$

for any λ with length m, where $0 < \delta < 1$. This implies that the restriction of u to the set $D \cap B(0, a)$ belongs to $BL_m(L^1(D \cap B(0, a)))$. Hence, in view of the extension theorem in Stein's book [25, Chap. 6], we can find a function $\tilde{u}$ in $BL_m(L^1_{loc}(R^n))$ such that $\tilde{u} = u$ a.e. on D. For this $\tilde{u}$ we have only to take an (m, Φ_p)-quasicontinuous representation on D.

REMARK 10.1. Condition $(\omega 2)$ implies (10.1).

As applications of the results in Sections 6–8 concerning Riesz potentials, we can study the existence of boundary limits of Beppo-Levi-Deny functions, generalizing the results in the case $m = 1$; see Wallin [26] and Mizuta [9], [12], [17].

For this purpose, let

$$\kappa_1(r) = \left(\int_r^1 [t^{n-mp}\eta(t)]^{-p'/p} t^{-1}\, dt\right)^{1/p'},$$

$$\varphi^*(r) = \left(\int_0^r [t^{n-mp}\varphi(t^{-1})]^{-p'/p} t^{-1}\, dt\right)^{1/p'},$$

$$\tau_1(r) = [\kappa_1(r)]^{-p},$$

$$\tau_2(r) = \inf_{r \le t \le 1} \omega(t)\, [\varphi^*(t)]^{-p}$$

for $0 < r \le 2^{-1}$.

THEOREM 10.2. *Let* ω *be as in Theorem* 6.1, *and let* u *be an* (m, Φ_p)*-quasicontinuous function on* D *satisfying condition* (1.4). *If* $\kappa_1(0) = \infty$, *then there exists a set* $E \subset D$ *such that*

$$\lim_{x_n \to 0, x \in G - E} [\kappa_1(x_n)]^{-1} u(x) = 0$$

for any bounded open set $G \subset D$ *and*

$$\sum_{j=1}^{\infty} K^{-j}\omega(2^{-j}) C_{m,\Phi_p}(E_j \cap B(0, a); D_j \cap B(0, 2a)) < \infty \tag{10.2}$$

for any $a > 0$, *where* K, E_j *and* D_j *are defined as in Theorem* 6.1.

PROOF. It follows from condition (ω2) that (10.1) holds. Hence, by Theorem 10.1, there exists a function $u^* \in BL_m(L^1_{loc}(R^n))$ which is equal to u on D. For $a > 1$, take $\zeta \in C_0^\infty(R^n)$ such that $\zeta = 1$ on $B(0, 2a)$. Then it follows from Theorem 9.1 that

$$(\zeta u^*)(x) = \sum_{|\lambda| = m} a_\lambda \int \frac{(x - y)^\lambda}{|x - y|^n} D^\lambda(\zeta u^*)(y)\, dy + P(x) \tag{10.3}$$

holds for almost every $x \in R^n$, where P is a polynomial. Since u is (m, Φ_p)-quasicontinuous on D, (10.2) holds for every $x \in D$ except those in a set E' with $C_{m,\Phi_p}(E') = 0$. But, since E' satisfies (10.2) clearly, we may assume that (10.3) holds for every $x \in D$. Set $f_a(y) = \sum_{|\lambda| = m} |(\partial/\partial y)^\lambda(\zeta u^*)(y)|$. Then it satisfies

$$\int_{B(0,2a)} \Phi_p(f_a(y))\omega(|y_n|)\, dy < \infty.$$

In view of Theorem 6.1, we can find $E_a \subset D \cap B(0, a)$ such that

$$\sum_{j=1}^{\infty} K^{-j}\omega(2^{-j}) C_{m,\Phi_p}(E_{a,j}; D_j \cap B(0, 2a)) < \infty,$$

where $E_{a,j} = \{x \in E_a;\ 2^{-j} \le x_n < 2^{-j+1}\}$, and

$$\lim_{x_n \to 0, x \in D \cap B(0,a) - E_a} [\kappa_1(x_n)]^{-1} u(x) = 0.$$

Now, as in the proof of Theorem 6.1, we can find a sequence $\{j_a\}$ of positive integers such that $E = \bigcup_{a=1}^{\infty} (\bigcup_{j \ge j_a} E_{a,j})$ has all the required properties.

Similarly, by Theorems 7.2 and 7.3, we obtain the following results.

THEOREM 10.3. *Assume that* (ω2) *holds and* $\varphi^*(1) < \infty$. *Let* u *be a continuous function on* D *satisfying condition* (1.4). *If* $\tau_1(r) \le M\tau_2(r)$ *for* $0 < r < 1$, *then there exists a set* $E \subset \partial D$ *such that* $C_{m,\Phi_p,\omega}(E) = 0$ *and* u *has a nontangential limit at any* $\xi \in \partial D - E$.

COROLLARY 10.1. *Assume that* $0 < mp - n \le \beta < p - 1$ *and* u *is a continuous function on* D *satisfying*

$$\sum_{|\lambda| = m} \int_G \Phi_p(|D^\lambda u(x)|) x_n^\beta\, dx < \infty \tag{10.4}$$

for any bounded open set $G \subset D$. *Then there exists a set* $E \subset \partial D$ *such that* $C_{m,\Phi_p,\beta}(E) = 0$ *and* u *has a nontangential limit at any* $\xi \in \partial D - E$.

THEOREM 10.4. *Let* $-1 < \beta < p - 1$, $\varphi^*(1) < \infty$ *and* u *be as in Corollary* 10.1.

(i) *If* $n - mp + \beta > 0$, *then for* $\gamma > 1$, *there exists a set* $E_\gamma \subset \partial D$ *such that* $H_h(E_\gamma) = 0$ *with* $h(r) = \tau_2(r^\gamma)$ *and* u *has a finite* T_γ*-limit at any* $\xi \in \partial D - E_\gamma$.

(ii) *If* $\beta = mp - n > 0$, *then there exists a set* $E \subset \partial D$ *such that* $C_{m,\Phi_p,\beta}(E) = 0$ *and* u *has a finite* T_γ *limit at any* $\xi \in \partial D - E$ *for any* $\gamma \geq 1$.

(iii) *If* $\beta = mp - n = 0$ *or* $n - mp + \beta < 0$, *then* u *has a finite limit at any* $\xi \in \partial D$.

In the above theorem,

$$\tau_2(r) = \inf_{r \leq t \leq 1} t^\beta \left(\int_0^t [s^{n-mp}\varphi(s^{-1})]^{-p'/p} s^{-1}\, ds \right)^{-p/p'}.$$

THEOREM 10.5. *Let* ω *and* ω^* *be as in Theorem* 8.1, *and set*

$$\tau_2^*(r) = \inf_{r \leq t \leq 1} t^{n-mp}\omega^*(t)\varphi(t^{-1}),$$

$$\tau^*(r) = \min\{\tau_1(r), \tau_2^*(r)\},$$

$$h^*(r) = \tau^*(\psi(r))$$

for $0 < r < 1$. *If* u *is an* (m, Φ_p)*-quasicontinuous function satisfying* (1.4), *then there exist* E_1 *and* E_2 *such that* $C_{\alpha,\Phi_p,\omega}(E_1) = 0$, $H_{h^*}(E_2) = 0$ *and* u *has a finite limit along* $L_\Psi(\xi)$, *for any* $\xi \in \partial D - (E_1 \cup E_2)$.

PROOF. For simplicity we assume that u vanishes outside some bounded set. In this case,

$$u(x) = \sum_{|\lambda|=m} a_\lambda \int k_\lambda(x-y) D^\lambda u(y)\, dy + P(x)$$

holds for every $x \in D - E'$, where P is a polynomial and E' is a subset of D with $C_{m,\Phi_p}(E') = 0$. Denote by u^* the function defined by the above summation about λ. Since $C_{m,\Phi_p}(E') = 0$, we can find a nonnegative measurable function f on D such that $U_m f \not\equiv \infty$, $U_m f = \infty$ on E' and (7.3) holds. Then, in view of Theorem 8.1, there exist E_1' and E_2' such that $C_{m,\Phi_p,\omega}(E_1') = 0$, $H_{h^*}(E_2') = 0$ and $U_m f$ has a finite limit at $\xi \in \partial D - (E_1' \cup E_2')$. This implies that if $\xi \in \partial D - (E_1' \cup E_2')$, then $u = u^* + P$ on $L_\Psi(\xi) \cap B(\xi, r_\xi)$ for some $r_\xi > 0$. Now we apply the same discussions as in Theorem 8.1 to the function u^*, and complete the proof.

Noting Corollary 8.1, we have

COROLLARY 10.2. *Let* $mp - n \le \beta < p - 1$, $\gamma \ge 1$ *and* Ψ *be of the form* $(r, \psi_2(r), \dots, \psi_{n-1}(r), r^\gamma)$ *as in Corollary* 8.1. *Further, let* u *be an* (m, Φ_p)-*quasicontinuous function on* D *satisfying* (10.4). *Then*:

(i) *If* $\beta > 0$, $n - mp + \beta > 0$ *and* $\gamma > 1$, *then there exists a set* $E \subset \partial D$ *such that* $H_h(E) = 0$ *with* $h(r) = \inf_{r \le t \le 1} t^{\gamma(n-mp+\beta)} \varphi(t^{-1})$ *and* u *has a finite limit along the curve* $L_\Psi(\xi)$, *for any* $\xi \in \partial D - E$.

(ii) *If* $\beta > 0$, $n - mp + \beta > 0$ *and* $\gamma = 1$, *then there exists a set* $E \subset \partial D$ *such that* $C_{m,\Phi_p,\beta}(E) = 0$ *and* u *has a finite limit along the curve* $L_\Psi(\xi)$, *for any* $\xi \in \partial D - E$.

(iii) *If* $\beta \le 0$ *and* $\gamma > 1$, *then there exists a set* $E \subset \partial D$ *such that* E *has Hausdorff dimension at most* $\gamma(n - mp)$ *and* u *has a finite limit along the curve* $L_\Psi(\xi)$, *for any* $\xi \in \partial D - E$.

By Theorems 8.2 and 8.3 we derive radial limit results for Beppo-Levi-Deny functions on D.

THEOREM 10.6. *Let* $-1 < \beta < p - 1$ *and let* u *be an* (m, Φ_p)-*quasicontinuous function on* D *satisfying* (10.4). *Then there exists a set* $E \subset \partial D$ *such that* $C_{m,\Phi_p,\beta}(E) = 0$ *and if* $\xi \in \partial D - E$, *then* $u(\xi + r\zeta)$ *has a finite limit as* $r \to 0$ *for every* $\zeta \in D \cap \partial B\,(0, 1)$ *except those in a set* E_ξ *with* $C_{m,\Phi_p}(E_\xi) = 0$.

THEOREM 10.7. *Let* ω *be a nonnegative nondecreasing function on* ([0, ∞) *satisfying* (ω1) *and* (ω2). *Let* $\zeta \in D$ *be fixed. If* u *is an* (m, Φ_p)-*quasicontinuous function on* D *satisfying* (1.4), *then there exists a set* $E \subset \partial D$ *such that* $C_{m,\Phi_p,\omega}(E) = 0$ *and* $u(\xi + t\zeta)$ *has a finite limit as* $t \downarrow 0$ *at any* $\xi \in \partial D - E$.

11. Green potentials

In the half space D, we consider the function

$$G_\alpha(x, y) = \begin{cases} |x - y|^{\alpha-n} - |\bar{x} - y|^{\alpha-n} & \text{in case } \alpha < n, \\ \log(|\bar{x} - y|/|x - y|) & \text{in case } \alpha = n, \end{cases}$$

where $\bar{x} = (x_1, \dots, x_{n-1}, -x_n)$ for $x = (x_1, \dots, x_{n-1}, x_n)$, and define

$$G_\alpha f(x) = \int_D G_\alpha(x, y) f(y)\, dy$$

for a nonnegative locally integrable function f on D.

The following lemma can be proved by elementary calculations (cf. [14, Lemma 9]):

LEMMA 11.1. *If $\alpha < n$, then there exist positive constants M_1 and M_2 such that*

$$M_1 \frac{x_n y_n}{|x-y|^{n-\alpha}|\bar{x}-y|^2} \leq G_\alpha(x, y) \leq M_2 \frac{x_n y_n}{|x-y|^{n-\alpha}|\bar{x}-y|^2};$$

if $\alpha = n$, then for any ε, $0 < \varepsilon < 1$, there exist positive constants M_3 and $M(\varepsilon)$ such that

$$M_3 \frac{x_n y_n}{|\bar{x}-y|^2} \leq G_n(x, y) \leq M(\varepsilon) \frac{x_n y_n}{|x-y|^\varepsilon |\bar{x}-y|^{2-\varepsilon}}.$$

COROLLARY 11.1. *For any nonnegative measurable function f on D, $G_\alpha f \not\equiv \infty$ if and only if*

$$\int_D (1+|y|)^{\alpha-n-2} y_n f(y)\, dy < \infty. \tag{11.1}$$

In this section we are concerned only with the case $\alpha < n$.

We can derive the following result from the Corollary 3.1.

THEOREM 11.1. *Let f be a nonnegative measurable function on D satisfying* (11.1) *such that*

$$\int_{D'} \Phi_p(f(y))\, dy < \infty \qquad \text{for any bounded open set } D' \text{ with closure in } D.$$

If (7.3) *is fulfilled, then $G_\alpha f$ is continuous on D.*

THEOREM 11.2. *Let ω be a positive monotone function on the interval $(0, \infty)$ satisfying* (ω1) *and*

(ω4) $\quad r^{\beta-1/p}\omega(r)^{-1/p}$ *is nondecreasing on* $(0, \infty)$ *for some* $\beta < 2$.

Define

$$\kappa_3(r) = r\left(\int_r^1 [t^{n-\alpha p+p}\eta(t)]^{-p'/p} t^{-1}\, dt\right)^{1/p'}$$

for $0 < r \leq 2^{-1}$, where $\eta(t) = \varphi(t^{-1})\omega(t)$ as before. Let f be a nonnegative measurable function on D satisfying (11.1) *and*

$$\int_{D'} \Phi_p(f(y))\omega(y_n)\, dy < \infty \qquad \text{for any bounded open set } D' \subset D. \tag{11.2}$$

Then there exists a set $E \subset D$ such that

$\lim_{x_n \to 0, x \in D'-E} [\kappa_3(x_n)]^{-1} G_\alpha f(x) = 0 \qquad$ *if* $\lim_{r \to 0} \kappa_3(r) = \infty$,

$\lim_{x_n \to 0, x \in D' - E} G_\alpha f(x) = 0$ *if* $\kappa_3(r)$ *is bounded on* $(0, 2^{-1})$

for any bounded open set D' *and*

$$\sum_{j=1}^{\infty} K^{-j} \omega(2^{-j}) C_{\alpha, \Phi_p}(E_j \cap B(0, a); D_j \cap B(0, 2a)) < \infty$$

for any $a > 0$, *where* $K = K^*$ *in Lemma* 2.3 *with* $\chi = \max\{1, \kappa_3\}$.

PROOF. First, from condition (11.1), we can apply Lebesgue's dominated convergence theorem to see that, if $D' \subset D \cap B(0, N)$, $N > 1$, then

$$\lim_{x_n \to 0, x \in D'} \int_{D - B(0, 2N)} G_\alpha(x, y) f(y) \, dy = 0.$$

For $x = (x', x_n) \in D$, $0 < a < 1$ and $N > 1$, we write

$$G_N f(x) = \int_{D \cap B(0, 2N) - B(x, x_n/2)} G_\alpha(x, y) f(y) \, dy,$$

$$G_{1,a,N} f(x) = \int_{\{y \in D \cap B(0, 2N) - B(x, x_n/2); y_n \geq a\}} G_\alpha(x, y) f(y) \, dy,$$

$$G_{2,a,N} f(x) = \int_{\{y \in D \cap B(0, 2N) - B(x, x_n/2); y_n < a\}} G_\alpha(x, y) f(y) \, dy.$$

Then we see easily that $G_{1,a,N} f(x)$ tends to zero as $x_n \to 0$, $x \in D$. Further we have by Lemma 11.1,

$$G_{2,a,N} f(x) \leq M_1 x_n \int_{\{y \in D \cap B(0, 2N) - B(x, x_n/2); y_n < a\}} |x - y|^{\alpha - n} |\bar{x} - y|^{-2} y_n f(y) \, dy.$$

By $(\omega 4)$ we can apply Lemma 6.1 with $\delta > 0$ such that $\alpha - 1 < \delta < \alpha$, and obtain

$$G_{2,a,N} f(x) \leq M_2 \kappa_3(x_n) \left(\int_{\{y \in D \cap B(0, 2N); y_n < a\}} \Phi_p(f(y)) \omega(y_n) \, dy \right)^{1/p} + M_2$$

for $0 < x_n < 2^{-1}$. Thus, if $\lim_{r \to 0} \kappa_3(r) = \infty$, then we find

$$\limsup_{x_n \to 0, x \in D} [\kappa_3(x_n)]^{-1} G_N f(x) \leq M_2 \left(\int_{\{y \in D \cap B(0, 2N); y_n < a\}} \Phi_p(f(y)) \omega(y_n) \, dy \right)^{1/p}$$

Letting $a \to 0$, we establish

$$\lim_{x_n \to 0, x \in D} [\kappa_3(x_n)]^{-1} G_N f(x) = 0.$$

By Lemma 11.1, note

$$\int_{B(x,x_n/2)} G_\alpha(x, y) f(y)\,dy \le \int_{B(x,x_n/2)} |x-y|^{\alpha-n} f(y)\,dy.$$

The right hand side is just equal to $u_2(x)$ in Theorem 6.1. Hence, considering $E_{j,\ell}$ as in the proof of Theorem 6.1, with κ_1 replaced by κ_3, and noting Remark 6.4, we complete the proof.

Next we discuss the existence of tangential limits of Green potentials $G_\alpha f$ for f satisfying conditions (11.1) and (11.2).

THEOREM 11.3. *Assume that* (7.1) *and* (ω4) *hold. Let* ψ *be a positive nondecreasing function on* $(0, \infty)$ *satisfying conditions* (Δ_2) *and* (ψ1), *and define*

$$\tau_3(r) = \inf\nolimits_{r \le t < 1} [\kappa_3(r)]^{-p},$$

$$\tau_0 = \min\{\tau_2(r), \tau_3(r)\},$$

$$h_0(r) = \tau_0(\psi(r))$$

for $0 < r < 1$, *where* τ_2 *is as in Theorem* 7.1. *If* f *is as in Theorem* 11.2, *then there exists a set* $E \subset \partial D$ *such that* $H_{h_0}(E) = 0$ *and*

$$\lim\nolimits_{x \to \xi, x \in T_\psi(\xi, a)} G_\alpha f(x) = 0$$

for any $a > 0$ *and any* $\xi \in \partial D - E$. *If in addition* $\tau_0(0) > 0$, *then*

$$\lim\nolimits_{x \to \xi, x \in D} G_\alpha f(x) = 0$$

for any $\xi \in \partial D$.

Before proving this theorem, we note the following lemma (cf. [13, Lemma 3]).

LEMMA 11.2. *For* $\xi \in \partial D$, *set* $g_\xi(x) = \int_{D - B(\xi, 2|x-\xi|)} G_\alpha(x, y) f(y)\,dy$. *Then*

$$\lim\nolimits_{x \to \xi, x \in D} g_\xi(x) = 0 \quad \textit{if and only if} \quad \lim\nolimits_{r \to 0} r^{\alpha-n-1} \int_{D \cap B(\xi, r)} y_n f(y)\,dy = 0.$$

PROOF OF THEOREM 11.3. For $\xi \in \partial D$, we write $G_\alpha f = v_1 + v_2 + g_\xi$, where

$$v_1(x) = \int_{D \cap B(\xi, 2|x-\xi|) - B(x, a x_n)} G_\alpha(x, y) f(y)\,dy,$$

$$v_2(x) = \int_{B(x, a x_n)} G_\alpha(x, y) f(y)\,dy.$$

Define

$$E = \left\{\xi \in \partial D;\ \limsup_{r\to 0} h_0(r)^{-1} \int_{D\cap B(\xi,r)} \Phi_p(f(y))\omega(y_n)\,dy > 0\right\}.$$

Then, by Lemma 7.2, we see that $H_{h_0}(E) = 0$. By (ω4),

$$\int_{D\cap B(\xi,r)} [\omega(y_n)^{-1/p} y_n]^{p'}\,dy \le [r^{\beta-1/p}\omega(r)^{-1/p}]^{p'} \int_{D\cap B(\xi,r)} y_n^{p'(-\beta+(1/p)+1)}\,dy$$
$$= M_1 r^{n+p'}[\omega(r)]^{-p'/p}.$$

Hence, as in the proof of Lemma 2.1, we have for δ, $0 < \delta < \alpha$,

$$r^{\alpha-n-1}\int_{D\cap B(\xi,r)} y_n f(y)\,dy = r^{\alpha-n-1}\int_{\{y\in D\cap B(\xi,r);\,f(y)>r^{-\delta}\}} y_n f(y)\,dy$$
$$+ r^{\alpha-n-1}\int_{\{y\in D\cap B(\xi,r);\,f(y)\le r^{-\delta}\}} y_n f(y)\,dy$$
$$\le r^{\alpha-n-1}\left(\int_{D\cap B(\xi,r)} \Phi_p(f(y))\omega(y_n)\,dy\right)^{1/p}$$
$$\times \left(\int_{\{y\in D\cap B(\xi,r);\,f(y)>r^{-\delta}\}} [\varphi(f(y))\omega(y_n)]^{-p'/p} y_n^{p'}\,dy\right)^{1/p'}$$
$$+ r^{\alpha-n-1-\delta}\int_{D\cap B(\xi,r)} y_n\,dy$$
$$\le M_2[r^{n-\alpha p}\eta(r)]^{-1/p}\left(\int_{D\cap B(\xi,r)} \Phi_p(f(y))\omega(y_n)\,dy\right)^{1/p} + M_1 r^{\alpha-\delta}.$$

Here note

$$\kappa_3(r) \ge M_3[r^{n-\alpha p}\eta(r)]^{-1/p}$$

and

$$h_0(r) \le \tau_0(\psi(1)r) \le M_4[\kappa_3(r)]^{-p}$$

for $0 < r < 1$. Therefore, if $\xi \in \partial D - E$, then it follows that

$$\lim_{r\to 0} r^{\alpha-n-1}\int_{D\cap B(\xi,r)} y_n f(y)\,dy = 0.$$

Lemma 11.2 implies that $g_\xi(x)$ tends to zero as $x \to \xi$, $x \in D$. By Lemmas 6.1 and 11.1, we find

$$v_1(x) \le M_2\kappa_3(x_n)\left(\int_{D\cap B(\xi,2|x-\xi|)} \Phi_p(f(y))\omega(y_n)\,dy\right)^{1/p} + M_2|x-\xi|^{\alpha-\delta}$$

for any $x \in D \cap B(\xi, 1)$. Thus, since $\kappa_3(x_n) \le M_3[h_0(|x-\xi|)]^{-1/p}$ for $x \in T_\psi(\xi, a)$, if $\xi \in \partial D - E$, then $v_1(x)$ tends to zero as $x \to \xi$, $x \in T_\psi(\xi, a)$. Finally, Lemma 6.1 yields

$$v_2(x) \le M_4[\tau_2(x_n)]^{-1/p}\left(\int_{B(x, x_n/2)} \Phi_p(f(y))\omega(y_n)\,dy\right)^{1/p} + M_4 x_n^{\alpha-\delta}.$$

Hence it follows that $v_2(x)$ tends to zero as $x \to \xi$, $x \in T_\psi(\xi, a)$, if $\xi \in \partial D - E$. In case $\tau_0(0) > 0$, $\lim_{x \to \xi, x \in D} G_\alpha f(x) = 0$ for any $\xi \in \partial D$. Now Theorem 11.3 is proved.

In the same way as Theorem 7.3, we can derive the following result.

COROLLARY 11.2. *Assume that* (7.1) *holds. Let* $-1 < \beta < 2p - 1$ *and let* f *be a nonnegative measurable function on* D *satisfying* (11.1) *and*

$$\int_{D'} \Phi_p(f(y))y_n^\beta\,dy < \infty \qquad \textit{for any bounded open set} \quad D' \subset D.$$

(i) *If* $n - \alpha p + \beta > 0$, *then, for each* $\gamma \ge 1$, *there exists* $E_\gamma \subset \partial D$ *such that* $H_{h_\gamma}(E_\gamma) = 0$, *where* $h_\gamma(r) = \tau_2(r^\gamma)$ *with*

$$\tau_2(r) = \inf\nolimits_{r \le t \le 1} t^\beta\left(\int_0^t [s^{n-\alpha p}\varphi(s^{-1})]^{-p'/p} s^{-1}\,ds\right)^{-p/p'},$$

and $G_\alpha f(x)$ *has* T_γ*-limit zero at any* $\xi \in \partial D - E_\gamma$.

(ii) *If* $n - \alpha p + \beta \le 0$, *then* $G_\alpha f(x)$ *has limit zero at any* $\xi \in \partial D$.

In fact, if $\beta < 2p - 1$, then $\omega(r) = r^\beta$ satisfies condition (ω4). If in addition $n - \alpha p + p + \beta > 0$, then the corresponding τ_2 and τ_3 in Theorem 11.3 satisfy

$$\tau_3(r) \ge M_1 r^{n-\alpha p+\beta}\varphi(r^{-1}) \ge M_2\tau_2(r),$$

so that (i) follows from Theorem 11.3. On the other hand, in case $-p < n - \alpha p + \beta \le 0$, the above facts also imply $\tau_3(0) > 0$; in case $n - \alpha p + p + \beta \le 0$,

$$\kappa_3(r) \le \left(\int_0^1 [\varphi(t^{-1})]^{-p'/p} t^{p'-1}\,dt\right)^{1/p'} < \infty,$$

so that $\tau_3(0) > 0$. Thus, if $n - \alpha p + \beta \le 0$, then

$$\tau_3(0) > 0.$$

Further, in case $0 < \beta \le \alpha p - n$,

$$\tau_2(r) \ge M_2 r^{n-\alpha p+\beta}\varphi(r^{-1}),$$

so that $\tau_2(0) > 0$. In case $\beta \leq 0$, $\tau_2(0) > 0$, too. Thus, if $n - \alpha p + \beta \leq 0$, then $\tau_2(0) > 0$. Now, if $n - \alpha p + \beta \leq 0$, then $\tau_0(0) > 0$ and the proof of Theorem 11.3 yields the required conclusion of (ii).

In case $\beta \geq 2p - 1$, $\omega(r) = r^{\beta}$ does not satisfy condition (ω4). We can, however, give some results concerning nontangential limits.

PROPOSITION 11.1. *Let* $\beta \geq 2p - 1$. *For* τ_2 *in Corollary* 11.2, (i), *define*

$$h(r) = \min\{r^{n-\alpha+1}, \tau_2(r)\} \qquad \text{for} \quad r > 0.$$

If f *is as in Corollary* 11.2, *then there exists a set* $E \subset \partial D$ *such that* $H_h(E) = 0$ *and* $G_\alpha f$ *has nontangential limit zero at any* $\xi \in \partial D - E$.

PROOF. Consider the set

$$A = \left\{\xi \in \partial D;\ \limsup_{r \to 0} r^{\alpha-n-1} \int_{D \cap B(\xi,r)} y_n f(y)\, dy > 0\right\}.$$

Lemma 7.2 together with (11.1) implies $H_h(A) = 0$. It follows from Lemma 11.2 that g_ξ has limit zero at any $\xi \in \partial D - A$. Further, in the proof of Theorem 11.3,

$$v_1(x) \leq M_1 x_n^{\alpha-n-1} \int_{D \cap B(\xi, 2|x-\xi|)} y_n f(y)\, dy,$$

which implies that v_1 has nontangential limit zero at any $\xi \in \partial D - A$. Since v_2 can be evaluated in the same manner as in the proof of Theorem 11.3, the required result now follows.

PROPOSITION 11.2. *Let* f *be a nonnegative measurable function on* D *satisfying* (11.1) *and*

$$\int_G \Phi_p(f(y))\, y_n^{2p-1}\, dy < \infty$$

for any bounded open set $G \subset D$. *Suppose* $\int_0^1 [\varphi(t^{-1})]^{-p'/p} t^{-1}\, dt < \infty$, *and define*

$$h(r) = \inf_{r \leq t \leq 1} t^{n-1-p(\alpha-2)} \left(\int_0^t [\varphi(s^{-1})]^{-p'/p} s^{-1}\, ds\right)^{-p/p'}.$$

If $\alpha p \geq n$, *then there exists a set* $E \subset \partial D$ *such that* $H_h(E) = 0$ *and* $G_\alpha f$ *has nontangential limit zero at any* $\xi \in \partial D - E$.

PROOF. As in the proof of Theorem 11.3, we have

$$r^{\alpha-n-1}\int_{D\cap B(\xi,r)} y_n f(y)\,dy \le r^{\alpha-n-1}\left(\int_{D\cap B(\xi,r)} \Phi_p(f(y))y_n^{2p-1}\,dy\right)^{1/p}$$
$$\times\left(\int_{D\cap B(\xi,r)} y_n^{-1}[\varphi(y_n^{-\delta})]^{-p'/p}\,dy\right)^{1/p'} + r^{\alpha-n-1}\int_{D\cap B(\xi,r)} y_n^{1-\delta}\,dy$$
$$\le M_1 r^{\alpha-2-(n-1)/p}\left(\int_0^r [\varphi(t^{-1})]^{-p'/p}t^{-1}\,dt\right)^{1/p'}\left(\int_{D\cap B(\xi,r)} \Phi_p(f(y))y_n^{2p-1}\,dy\right)^{1/p}$$
$$+ M_1 r^{\alpha-\delta}$$
$$\le M_1\left([h(r)]^{-1}\int_{D\cap B(\xi,r)} \Phi_p(f(y))y_n^{2p-1}\,dy\right)^{1/p} + M_1 r^{\alpha-\delta},$$

where $0 < \delta < \min\{2, \alpha\}$. Hence $H_h(A) = 0$ by Lemma 7.2, for the set A in the proof of Proposition 11.1. On the other hand, if $\omega(r) = r^{2p-1}$, then τ_2 in Theorem 11.3 satisfies

$$\tau_2(r) \ge \inf_{r\le t\le 1} t^{2p-1} \times t^{n-\alpha p}\left(\int_0^t [\varphi(s^{-1})]^{-p'/p}s^{-1}\,ds\right)^{-p/p'} = h(r).$$

Thus, as in the proof of Proposition 11.1, we see that $G_\alpha f$ has nontangential limit zero at any $\xi \in \partial D - E$, where $H_h(E) = 0$.

By the proofs of Theorems 8.1 and 11.3, we can derive the following result.

THEOREM 11.4. *Let τ_2^* be as in Theorem* 8.1 *and τ_3, ψ be as in Theorem* 11.3. *Define*

$$h_0^*(r) = \min\{\tau_2^*(\psi(r)), \tau_3(\psi(r))\}$$

for $0 < r \le 1$; define $h_0^*(r) = h_0^*(1)$ for $r > 1$. If f is as in Theorem 11.2, *then there exists a set $E \subset \partial D$ such that $H_{h_0^*}(E) = 0$ and*

$$\lim_{r\to 0} G_\alpha f(\xi(r)) = 0 \qquad \textit{for any} \quad \xi \in \partial D - E,$$

where $\xi(r) = \xi + \Psi(r)$ with $\Psi(r) = (r, \psi_2(r), \dots, \psi_{n-1}(r), \psi(r))$ is as in Section 8.

COROLLARY 11.3. *Let $-1 < \beta < 2p - 1$ and $\Psi(r) = (r, \psi_2(r), \dots, \psi_{n-2}(r), r^\gamma)$, $\gamma \ge 1$, as in Corollary* 8.1. *Further let f be as in Corollary* 11.2.

(i) *If $\beta > 0$ and $n - \alpha p + \beta > 0$, then there exists a set $E \subset \partial D$ such that $H_h(E) = 0$ and $G_\alpha f$ has limit zero along the curve $L_\Psi(\xi)$, for any $\xi \in \partial D - E$, where $h(r) = \tau_2(r^\gamma)$ with $\tau_2(r) = \inf_{r\le t\le 1} t^\beta\left(\int_0^t [s^{n-\alpha p}\varphi(s^{-1})]^{-p'/p} ds/s\right)^{-p/p'}$.*

(ii) *If $\beta \le 0$ and $n - \alpha p \ge 0$, then there exists a set $E \subset \partial D$ such that E*

has Hausdorff dimension at most $\gamma(n-\alpha p)$ and $G_\alpha f$ has limit zero along the curve $L_\Psi(\xi)$, for any $\xi \in \partial D - E$.

REMARK 11.1. Our results give generalizations of the results in Rippon [23], Wu [27], Aikawa [1] and Mizuta [14].

12. Singular integrals

In view of Theorem 9.2, if $u \in BL_m(L^p(R^n))$, then

$$u(x) = \sum_{|\lambda|=m} a_\lambda \int k_{\lambda,\ell}(x, y) D^\lambda u(y)\, dy + P(x)$$

for almost every $x \in R^n$, where $\ell < m$ and P is a polynomial of degree at most $m-1$. Conversely, it is known (cf. [16, Lemma 3]) that each integral in the above equality belongs to $BL_m(L^p(R^n))$.

Let us begin with the following result, concerning the Φ_p estimate for the derivatives of potentials.

LEMMA 12.1 (cf. [9, Lemma 6], [18]). *Let $-1 < \beta < p-1$ and f be a nonnegative measurable function on R^n such that*

$$\int_{R^n} (1+|y|)^{m-n} f(y)\, dy < \infty \quad \text{and} \quad \int_{R^n} \Phi_p(f(y)|y_n|^{\beta/p})\, dy < \infty.$$

Set

$$u(x) = \int_{R^n} k_\lambda(x-y) f(y)\, dy,$$

where $k_\lambda(x) = x^\lambda/|x|^n$ and $|\lambda| = m$. Then u is a function in $BL_m(L^q_{\mathrm{loc}}(R^n))$ for q such that $1 < q < \min\{p, p/(\beta+1)\}$. Further, u is (m, Φ_p)-quasicontinuous on D and satisfies

$$\int \Phi_p(|\nabla_m u(x)|\, |x_n|^{\beta/p})\, dx \le M \int \Phi_p(f(y)|y_n|^{\beta/p})\, dy$$

with a positive constant M independent of f, where $|\nabla_m u(x)| = (\sum_{|\lambda|=m} |D^\lambda u(x)|^2)^{1/2}$.

PROOF. First of all, if we note $\int_G \Phi_p(f(y))\, dy < \infty$ for any relatively compact open set G in D, then u is (m, p)-quasicontinuous on D in the sense of [8]. If the required inequality of the present lemma is obtained, then we see that u is (m, Φ_p)-quasicontinuous on D. If $1 < q < \min\{p, p/(\beta+1)\}$, then we have by Hölder's inequality

$$\int_G f(y)^q dy \le \left(\int_G f(y)^p |y_n|^\beta dy \right)^{q/p} \left(\int_G |y_n|^{-\beta q/(p-q)} dy \right)^{1-q/p} < \infty$$

for any bounded open set $G \subset R^n$. Consequently it follows from [8, Lemma 3.3] that $u \in BL_m(L^q_{loc}(R^n))$. For $\varepsilon > 0$, set $k^{(\varepsilon)}_\lambda(x) = x^\lambda(|x|^2 + \varepsilon^2)^{-n/2}$, and consider the function

$$u_\varepsilon(x) = \int k^{(\varepsilon)}_\lambda(x-y) f(y)\, dy.$$

In view of [8, Lemma 3.3], we see that $(\partial/\partial x)^\nu u_\varepsilon(x)$ tends to $(\partial/\partial x)^\nu u(x)$ in $L^q_{loc}(R^n)$ as $\varepsilon \to 0$ for any ν with length m. First we show

$$\int |(\partial/\partial x)^\nu u_\varepsilon(x)|^p |x_n|^\beta dx \le M_1 \int f(y)^p |y_n|^\beta dy, \tag{12.1}$$

where $|\nu| = m$ and M_1 is a positive constant independent of ε and f. For this, note

$$(\partial/\partial x)^\nu u_\varepsilon(x) = \int (\partial/\partial x)^\nu k^{(\varepsilon)}_\lambda(x-y) f(y)\, dy.$$

Setting $v_\varepsilon(x) = \int (\partial/\partial x)^\nu k^{(\varepsilon)}_\lambda(x-y) g(y)\, dy$ with $g(y) = f(y)|y_n|^{\beta/p}$, we have

$$\int |\nabla_m v_\varepsilon(x)|^p dx \le M_2 \int g(y)^p dy, \tag{12.2}$$

in view of the proof of [8, Lemma 3.2] (see also Stein [25, Theorem 2, Section 3.2, Chapter 2]). Further, we obtain

$$\begin{aligned} ||x_n|^{\beta/p}(\partial/\partial x)^\nu u_\varepsilon(x) - (\partial/\partial x)^\nu v_\varepsilon(x)| &\le M_3 \int \frac{|1 - [|x_n|/|y_n|]^{\beta/p}|}{|x-y|^n} g(y)\, dy \\ &= M_3 \int \frac{|1 - [|x_n|/|y_n|]^{\beta/p}|}{|x_n - y_n|} G(x', x_n, y_n)\, dy_n, \end{aligned}$$

where $G(x', x_n, y_n) = \int_{R^{n-1}} \frac{|x_n - y_n|}{(|x'-y'|^2 + |x_n - y_n|^2)^{n/2}} g(y', y_n)\, dy'$. As in the proof of Lemma 6 in [9], using Minkowski's inequality (see [25, Appendix A. 1]) and the property of Poisson integral in the half space, we find

$$\begin{aligned} &\| |x_n|^{\beta/p}(\partial/\partial x)^\nu u_\varepsilon(\cdot, x_n) - (\partial/\partial x)^\nu v_\varepsilon(\cdot, x_n)\|_{L^p(R^{n-1})} \\ &\le M_3 \int \frac{|1 - [|x_n|/|y_n|]^{\beta/p}|}{|x_n - y_n|} \|G(\cdot, x_n, y_n)\|_p\, dy_n \end{aligned}$$

$$\leq M_4 \int \frac{|1 - [|x_n|/|y_n|]^{\beta/p}|}{|x_n - y_n|} \| g(\cdot, y_n) \|_p dy_n.$$

Moreover, by [25, Appendix A. 3], the L^p-norm in R^1 of the right hand side is dominated by $M_5 \|g\|_p$ as long as

$$\int_0^\infty |1 - r^{-\beta/p}| |1 - r|^{-1} r^{-1/p} dr < \infty,$$

which is true because $-1 < \beta < p - 1$. Thus (12.1) is obtained with the aid of (12.2). Letting $\varepsilon \to 0$, we establish

$$\int |(\partial/\partial x)^\nu u(x)|^p |x_n|^\beta dx \leq M_6 \int f(y)^p |y_n|^\beta dy, \tag{12.3}$$

which proves the case $\varphi \equiv 1$. Now we apply the usual interpolation methods (cf. [28], [25, Appendix B]) and prove

$$\int \Phi_p(|(\partial/\partial x)^\nu u(x)| |x_n|^{\beta/p}) dx \leq M \int \Phi_p(f(y)|y_n|^{\beta/p}) dy.$$

For this purpose, let $\gamma = \beta/p$ and note from (12.3)

$$\int [|(\partial/\partial x)^\nu u(x)| |x_n|^\gamma]^q dx \leq M_q \int [f(y)|y_n|^\gamma]^q dy \tag{12.4}$$

for any q such that $q > 1$ and $-1 < \gamma q < q - 1$. Since $-1/p < \gamma < 1/p'$, we can take q_1, q_2 such that

$$1 < q_1 < p < q_2 \quad \text{and} \quad -\frac{1}{q_2} < \gamma < \frac{1}{q_1'};$$

recall that p' and q_1' are the exponents conjugate to p and q_1, respectively. For $a > 0$, decompose f as $f_{a,1} + f_{a,2}$, where

$$f_{a,1}(y) = \begin{cases} f(y) & \text{if } g(y) \geq a, \\ 0 & \text{otherwise,} \end{cases} \qquad g(y) = f(y)|y_n|^\gamma,$$

and write $u_{a,1}$ and $u_{a,2}$ for u with $f = f_{a,1}$ and $f_{a,2}$, respectively. Applying (12.4), we have

$$\int [|(\partial/\partial x)^\nu u_{a,i}(x)| |x_n|^\gamma]^{q_i} dx \leq M_7 \int [f_{a,i}(y)|y_n|^\gamma]^{q_i} dy$$

for $i = 1, 2$. Here remark that M_7 does not depend on a. Since $u = u_{a,1} + u_{a,2}$,

$$m_n(\{x; |(\partial/\partial x)^\nu u(x)|\,|x_n|^\gamma > 2a\})$$
$$\le \int\left[\left(\frac{|(\partial/\partial x)^\nu u_{a,1}(x)|\,|x_n|^\gamma}{a}\right)^{q_1} + \left(\frac{|(\partial/\partial x)^\nu u_{a,2}(x)|\,|x_n|^\gamma}{a}\right)^{q_2}\right]dx$$
$$\le M_7 a^{-q_1}\int [f_{a,1}(y)|y_n|^\gamma]^{q_1}\,dy + M_7 a^{-q_2}\int [f_{a,2}(y)|y_n|^\gamma]^{q_2}\,dy,$$

where m_n denotes the n-dimensional Lebesgue measure. Hence,

$$\int \Phi_p(|(\partial/\partial x)^\nu u(x)|\,|x_n|^\gamma)\,dx = \int m_n(\{x; |(\partial/\partial x)^\nu u(x)|\,|x_n|^\gamma > 2a\})\,d\Phi_p(2a)$$
$$\le M_7\int g(y)^{q_1}\left(\int_0^{g(y)} a^{-q_1}\,d\Phi_p(2a)\right)dy$$
$$+ M_7\int g(y)^{q_2}\left(\int_{g(y)}^{\infty} a^{-q_2}\,d\Phi_p(2a)\right)dy.$$

By ($\varphi 1$) and ($\varphi 5$),

$$s^{-q_1-\delta}\Phi_p(2s) \le M_8 t^{-q_1-\delta}\Phi_p(2t) \quad \text{and} \quad s^{-q_2+\delta}\Phi_p(2s) \ge M_8 t^{-q_2+\delta}\Phi_p(2t)$$

whenever $0 < s < t$, where $\delta > 0$ is chosen so that $q_1 + \delta < p < q_2 - \delta$. Hence it follows that

$$\int_0^{g(y)} a^{-q_1}\,d\Phi_p(2a) = \int_0^{g(y)} \Phi_p(2a)\,d(-a^{-q_1}) + [g(y)]^{-q_1}\Phi_p(2g(y))$$
$$\le q_1 M_8 \Phi_p(2g(y))\,[g(y)]^{-q_1-\delta}\int_0^{g(y)} a^{\delta-1}\,da + [g(y)]^{-q_1}\Phi_p(2g(y))$$
$$\le M_9 \Phi_p(g(y))\,[g(y)]^{-q_1}.$$

Similarly,

$$\int_{g(y)}^{\infty} a^{-q_2}\,d\Phi_p(2a) \le M_{10}\Phi_p(g(y))g(y)^{-q_2}.$$

Now we find

$$\int \Phi_p(|(\partial/\partial x)^\nu u(x)|\,|x_n|^\gamma)\,dx \le M_{11}\int \Phi_p(g(y))\,dy = M_{11}\int \Phi_p(f(y)|y_n|^\gamma)\,dy,$$

which yields the required inequality. Thus the proof of Lemma 12.1 is completed.

REMARK 12.1. If we replace k_λ by R_m or $k_\lambda^* = D^\lambda R_{2m}$, then the same conclusions as in Lemma 12.1 still hold.

LEMMA 12.2. *Let* $-1<\beta<p-1$. *For a nonnegative measurable function* f *on* R^n,

$$\int_G \Phi_p(f(y)|y_n|^{\beta/p})\,dy<\infty \qquad \textit{for any bounded open set } G\subset R^n$$

if and only if

$$\int_G \Phi_p(f(y))|y_n|^{\beta}\,dy<\infty \qquad \textit{for any bounded open set } G\subset R^n.$$

PROOF. Let $\varepsilon>0$ and $\beta(1+\varepsilon^{-1})>-1$. Then, for a bounded open set $G\subset R^n$, we have

$$\begin{aligned}
&\int_G \Phi_p(f(y)|y_n|^{\beta/p})\,dy\\
&\le \int_{\{y\in G;\, f(y)^{\varepsilon}\ge |y_n|^{\beta/p}\}} \Phi_p(f(y)|y_n|^{\beta/p})\,dy+\int_{\{y\in G;\, f(y)^{\varepsilon}< |y_n|^{\beta/p}\}} \Phi_p(f(y)|y_n|^{\beta/p})\,dy\\
&\le \int_G [f(y)|y_n|^{\beta/p}]^p\varphi(f(y)^{1+\varepsilon})\,dy+\int_G \Phi(|y_n|^{(1+\varepsilon^{-1})\beta/p})\,dy\\
&\le M(\varepsilon)\left\{\int_G \Phi_p(f(y))|y_n|^{\beta}\,dy+\int_G |y_n|^{(1+\varepsilon^{-1})\beta}\varphi(|y_n|^{\beta})\,dy\right\}.
\end{aligned}$$

Since $\beta(1+\varepsilon^{-1})>-1$, the last integral is convergent. Thus the "if" part follows. The "only if" part can be proved similarly.

THEOREM 12.1. *Let* $-1<\beta<p-1$ *and* f *be a nonnegative measurable function on* R^n *such that*

$$\int_{R^n} \Phi_p(f(y))|y_n|^{\beta}\,dy<\infty. \tag{12.5}$$

If $\ell\le m-n/p-\beta/p<\ell+1$, *then the function*

$$u(x)=\int k_{\lambda,\ell}(x,y)f(y)\,dy$$

satisfies

$$\int_G \Phi_p(|\nabla_m u(x)|)|x_n|^{\beta}\,dx<\infty \qquad \textit{for any bounded open set } G\subset R^n. \tag{12.6}$$

PROOF. Since $f\in L^q(R^n)$, $1<q<\min\{p,p/(1+\beta)\}$, by the proof of Lemma 12.1, we see that $u\in BL_m(L^q_{loc}(R^n))$ by [19, Lemma 5]. For $a>0$, set

$$u_a'(x) = \int_{B(0,2a)} k_{\lambda,\ell}(x, y) f(y)\,dy,$$

$$u_a''(x) = \int_{R^n - B(0,2a)} k_{\lambda,\ell}(x, y) f(y)\,dy.$$

Since u_a'' is infinitely differentiable on $B(0, 2a)$, it satisfies

$$\int_{B(0,a)} \Phi_p(|\nabla_m u_a''(x)|)|x_n|^\beta\,dx < \infty.$$

On the other hand, $u_a'(x)$ is of the form $v_a(x) = \int_{B(0,2a)} k_\lambda(x-y) f(y)\,dy + w_a(x)$, where w_a is a polynomial. Lemma 12.1 implies

$$\int_{R^n} \Phi_p(|\nabla_m v_a(x)|\,|x_n|^{\beta/p})\,dx \le M \int_{B(0,2a)} \Phi_p(f(y)|y_n|^{\beta/p})\,dy < \infty.$$

Hence, if we note Lemma 12.2, then we have

$$\int_{B(0,a)} \Phi_p(|\nabla_m u_a'(x)|)|x_n|^\beta\,dx < \infty.$$

Therefore,

$$\int_{B(0,a)} \Phi_p(|\nabla_m u(x)|)|x_n|^\beta\,dx < \infty.$$

Since a is arbitrary, Theorem 12.1 is obtained.

LEMMA 12.3. *Let ω be a positive monotone function on $(0, \infty)$ satisfying $(\omega 1)$ and $(\omega 2)$. If $C_{\alpha,\Phi_p,\omega}(E) = 0$, then there exists a nonnegative measurable function f on R^n such that*

$$\int (1 + |y|)^{\alpha-n} f(y) < \infty,$$

$$\int \Phi_p(f(y))\omega(|y_n|)\,dy < \infty$$

and

$$U_\alpha f(x) = \infty \qquad \text{for any} \quad x \in E.$$

PROOF. For any $a > 0$, $C_{\alpha,\Phi_p,\omega}(E \cap B(0, a); B(0, a)) = 0$ by our assumption. Hence we can find a nonnegative measurable function f_a such that $f_a = 0$

outside $B(0, a)$, $U_\alpha f_a = \infty$ on $E \cap B(0, a)$ and $\int_{B(0,a)} \Phi_p(f_a(y))\omega(|y_n|)\,dy < \infty$. As in the proof of Lemma 6.1, we establish

$$\int (1 + |y|)^{\alpha-n} f_a(y)\,dy \le M(a) \int_{B(0,a)} \Phi_p(f_a(y))\omega(|y_n|)\,dy$$

for some constant $M(a) > 0$. For a sequence $\{\varepsilon_j\}$ of positive numbers, consider the function $f = \sup_j \varepsilon_j f_j$. Then

$$U_\alpha f(x) \ge \varepsilon_j U_\alpha f_j = \infty \qquad \text{for any } x \in E \cap B(0, j),$$

which shows that

$$U_\alpha f(x) = \infty \qquad \text{for any } x \in E.$$

On the other hand,

$$\int \Phi_p(f(y))\omega(|y_n|)\,dy \le \sum_j \int_{B(0,j)} \Phi_p(\varepsilon_j f_j(y))\omega(|y_n|)\,dy$$

and

$$\int (1 + |y|)^{\alpha-n} f(y)\,dy \le \sum_j \varepsilon_j M(j) \int_{B(0,j)} \Phi_p(f_j(y))\omega(|y_n|)\,dy.$$

Now choose $\{\varepsilon_j\}$ so small that the last two sums are convergent.

LEMMA 12.4. *Let* $-1 < \beta < p - 1$ *and let* f *be a nonnegative measurable function on* R^n *satisfying* (12.5). *If we define*

$$E = \left\{\xi \in \partial D;\ \int_{B(\xi,1)} |\xi - y|^{m-n} f(y)\,dy = \infty\right\},$$

then $C_{m-\beta/p,\Phi_p}(E) = 0$.

PROOF. For $a > 0$, consider the function

$$u_a(x) = \int_{B(0,a)} |x - y|^{m-n} f(y)\,dy.$$

Then Lemma 12.2 yields

$$\int_{B(0,a)} \Phi_p(f(y)|y_n|^{\beta/p})\,dy < \infty.$$

Hence, in view of Lemma 12.1 and Remark 12.1, we see that

$$\int \Phi_p(|\nabla_m u_a(x)|\,|x_n|^{\beta/p})\,dx < \infty.$$

Define

$$E' = \left\{ \xi \in \partial D;\ \int_{B(\xi,1)} |\xi - y|^{m-\beta/p-n}[|\nabla_m u_a(y)|\,|y_n|^{\beta/p}]\,dy = \infty \right\}.$$

Then it follows from the definition of $C_{m-\beta/p,\Phi_p}$ that $C_{m-\beta/p,\Phi_p}(E') = 0$. If we show $E \cap B(0, a) \subset E'$, then we obtain $C_{m-\beta/p,\Phi_p}(E \cap B(0, a)) = 0$, so that $C_{m-\beta/p,\Phi_p}(E)=0$. If $\xi \in \partial D \cap B(0, a) - E'$, then $\int_{B(\xi,1)\cap T_1(\xi,1)} |\xi - y|^{m-n}|\nabla_m u_a(y)|\,dy < \infty$, which together with [12, Lemma 3] implies

$$\int_{B(\xi,1)\cap T_1(\xi,1)} |\xi - y|^{1-n}|\nabla_1 u_a(y)|\,dy < \infty.$$

By using polar coordinates, we deduce that $u(\xi + r\eta)$ is absolutely continuous on $[0, 1]$ for almost every $\eta \in \partial B(0, 1) \cap T_1(0, 1)$, and hence it follows that $u(\xi) < \infty$. Thus, $\xi \notin E$, so that $E \cap B(0, a) \subset E'$. Now the proof is completed.

THEOREM 12.2. *Let* $-1 < \beta < p - 1$. *For* $E \subset \partial D$, $C_{m,\Phi_p,\beta}(E) = 0$ *if and only if* $C_{m-\beta/p,\Phi_p}(E) = 0$.

PROOF. The "only if" part follows from Lemmas 12.3 and 12.4. We show the "if" part. For this purpose, assume $C_{m-\beta/p,\Phi_p}(E) = 0$. Then, by Lemma 12.3, there exists a nonnegative measurable function f on R^n such that

$$\int (1 + |y|)^{\alpha-n} f(y)\,dy < \infty,$$

$$\int \Phi_p(f(y))\,dy < \infty$$

and

$$U_\alpha f(x) = \infty \qquad \text{for any } x \in E,$$

where $\alpha = m - \beta/p$. Consider the Bessel potential

$$F(x') = g_\alpha * f(x', 0) = \int g_\alpha((x', 0) - y) f(y)\,dy$$

and the Poisson integral

$$u(x', x_n) = P_{x_n} * F(x');$$

see Stein's book [25] for the definitions of Bessel kernel g_α and Poisson kernel P_t. First we treat the case when f is bounded and has compact support. Thus $f \in L^q(R^n)$ for any $q > 1$. Then F belongs to the Lipschitz space $\Lambda^{q,q}_{\alpha-1/q}(R^{n-1})$ and

$$\|F\|_{\Lambda^{q,q}_{\alpha-1/q}(R^{n-1})} \leq M(q)\|f\|_q$$

as long as $\alpha > 1/q$, on account of [25, §4.3 of Chapter 6]. In view of [25, (62') and (63) in p. 152],

$$\left(\int_D \{x_n^{k-(\alpha-1/q)}|\nabla_k u(x)|\}^q x_n^{-1}\,dx\right)^{1/q} \leq M(q, k)\|F\|_{\Lambda^{q,q}_{\alpha-1/q}(R^{n-1})}$$

for any integer k greater than $\alpha - 1/p$. If we set $k = m > (1+\beta)/q$, then

$$\int_D [|\nabla_m u(x)|x_n^{\beta/p}]^q\,dx \leq M(q)' \int f(y)^q\,dy.$$

As in the proof of Lemma 12.1, we find

$$\int_D \Phi_p(|\nabla_m u(x)|x_n^{\beta/p})\,dx \leq M \int \Phi_p(f(y))\,dy.$$

Since the constant M does not depend on f, this inequality holds for general f, so that

$$\int_D \Phi_p(|\nabla_m u(x)|x_n^{\beta/p})\,dx < \infty.$$

By the property of Poisson integral,

$$\lim_{x\to\xi, x\in D} u(x) = \infty \qquad \text{for any } \xi \in E.$$

As in the proof of Lemma 12.4, set

$$E' = \left\{\xi \in \partial D;\ \int_{B(\xi,1)} |\xi - y|^{m-n}|\nabla_m u(y)|\,dy = \infty\right\}.$$

Then it follows that $C_{m,\Phi_p,\beta}(E') = 0$ and $u(\xi + r\zeta)$ has a finite limit as $r \to 0$ for almost every $\zeta \in \partial B(0,1)\cap D$ whenever $\xi \in \partial D - E'$. Therefore $E \subset E'$ and hence $C_{m,\Phi_p,\beta}(E) = 0$, as required.

By Theorem 12.2, we can rewrite our theorems by replacing the condition $C_{\alpha,\Phi_p,\beta}(E) = 0$ by the condition $C_{\alpha-\beta/p,\Phi_p}(E) = 0$. Among them, we give the following results.

THEOREM 12.3 (cf. Corollary 10.1). *Let $0 < mp - n \leq \beta < p - 1$. If u is a continuous function on D satisfying* (10.4), *then there exists a set $E \subset \partial D$ such that $C_{m-\beta/p,\Phi_p}(E) = 0$ and u has a nontangential limit at any $\xi \in \partial D - E$.*

THEOREM 12.4 (cf. Theorem 10.4, (ii)). *Let $0 < mp - n < p - 1$. If u is* *a continuous function on D satisfying*

$$\int_G \Phi_p(|\nabla_m u(x)|)|x_n|^{mp-n}\,dx < \infty \qquad \text{for any bounded open set} \quad G \subset D,$$

then there exists a set $E \subset \partial D$ *such that* $C_{n/p,\Phi_p}(E) = 0$ *and* u *has a finite* T_γ*-limit at any* $\xi \in \partial D - E$ *for any* $\gamma \geq 1$.

THEOREM 12.5 (cf. Theorem 10.6). *Let* $-1 < \beta < p - 1$ *and let* u *be an* (m, Φ_p)*-quasicontinuous function on* D *satisfying* (10.4). *Then there exists a set* $E \subset \partial D$ *such that* $C_{m-\beta/p,\Phi_p}(E) = 0$ *and if* $\xi \in \partial D - E$, *then* $u(\xi + r\zeta)$ *has a finite limit as* $r \to 0$ *for every* $\zeta \in \partial D \cap B(0, 1)$ *except those in a set* E_ξ *with* $C_{m,\Phi_p}(E_\xi) = 0$.

THEOREM 12.6 (cf. Theorem 10.7). *Let* $0 \leq \beta < p - 1$ *and* $\zeta \in D$. *If* u *is an* (m, Φ_p)*-quasicontinuous function on* D *satisfying* (10.4), *then there exists a set* $E \subset \partial D$ *such that* $C_{m-\beta/p,\Phi_p}(E) = 0$ *and* $u(\xi + r\zeta)$ *has a finite limit as* $r \to 0$ *at every* $\xi \in \partial D - E$.

We now give an integral representation for Beppo-Levi-Deny functions in the half space D.

THEOREM 12.7. *Let* $-1 < \beta < p - 1$ *and let* u *be a function in* $BL_m(L^p_{loc}(D))$ *such that*

$$\int_D \Phi_p(|\nabla_m u(x)|x_n^{\beta/p})\,dx < \infty. \tag{12.7}$$

If ℓ *is the integer such that* $\ell \leq m - n/p - \beta/p < \ell + 1$, *then*

$$u(x) = \sum_{|\lambda|=m} b_\lambda \int_D k^*_{\lambda,\ell}(x, y) D^\lambda u(y)\,dy + h(x)$$

for almost every $x \in D$, *where* h *is a function which is polyharmonic of order* m *in* D *satisfying* (12.7); *see Remark* 9.2 *for* b_λ *and* $k^*_{\lambda,\ell}$.

This is a Riesz-type decomposition of Beppo-Levi-Deny functions as the sum of potentials and polyharmonic functions.

PROOF OF THEOREM 12.7. For $\chi \in C_0^\infty(D)$, we have by Fubini's theorem and [16, (3)]

$$\int\left(\sum_{|\lambda|=m} b_\lambda \int_D k^*_{\lambda,\ell}(x, y) D^\lambda u(y)\,dy\right)\Delta^m \chi(x)\,dx$$

$$= \sum_{|\lambda|=m} b_\lambda \int_D \left(\int k^*_{\lambda,\ell}(x, y)\Delta^m \chi(x)\,dx\right) D^\lambda u(y)\,dy$$

$$= c^* \sum_{|\lambda|=m} b_\lambda \int_D D^\lambda \chi(y) D^\lambda u(y)\, dy$$

$$= \int_D \chi(y) \Delta^m u(y) dy$$

where $c^* = (-1)^m c$ with c in Remark 9.2. Thus Lemma 12.1 establishes the required assertion.

THEOREM 12.8. *Let* $-1 < \beta < p-1$ *and* ℓ *be the integer such that* $\ell \leq m - n/p - \beta/p < \ell + 1$. *If* u *is a function in* $BL_m(L^p_{loc}(D))$ *satisfying* (12.7), *then there exist a function* $u^* \in BL_m(L^1_{loc}(R^n))$ *satisfying*

$$\int_{R^n} \Phi_p(|\nabla_m u^*(x)|\,|x_n|^{\beta/p})\, dx < \infty \tag{12.8}$$

and a polynomial P *of degree at most* $m-1$ *such that*

$$u(x) = \sum_{|\lambda|=m} b_\lambda \int_{R^n} k^*_{\lambda,\ell}(x, y) D^\lambda u^*(y)\, dy + P(x)$$

for almost every $x \in D$.

To show this theorem, by the extension theorem in Stein's book [25, Chapter 6], we can find a function u^* satisfying (12.8) such that $u^* = u$ a.e. on D. In view of the proof of Theorem 12.8,

$$u^*(x) = \sum_{|\lambda|=m} b_\lambda \int_D k^*_{\lambda,\ell}(x, y) D^\lambda u^*(y)\, dy + h(x)$$

for almost every $x \in D$, where h is a function which is polyharmonic of order m in R^n satisfying (12.8). As in the proof of [8, Lemma 4.1], we see that h is a polynomial of degree at most $m-1$.

In the same way we can prove

THEOREM 12.9. *If* β, ℓ *and* u *are as above, then there exist a function* $u^* \in BL_m(L^1_{loc}(R^n))$ *satisfying* (12.8) *and a polynomial* P *of degree at most* $m-1$ *such that*

$$u(x) = \sum_{|\lambda|=m} a_\lambda \int_{R^n} k_{\lambda,\ell}(x, y) D^\lambda u^*(y)\, dy + P(x)$$

for almost every $x \in D$.

13. Logarithmic potentials

For a nonnegative measurable function f on R^n, we define

$$Lf(x) = \int \log \frac{1}{|x-y|} f(y)\,dy,$$

where we always assume that

$$\int f(y)\log(2+|y|)\,dy < \infty. \tag{13.1}$$

In this case $Lf(x) > -\infty$ for all $x \in R^n$ and $|Lf| \not\equiv \infty$.

In what follows, we investigate the behavior of logarithmic potentials Lf at the origin, where f satisfies (13.1) and

$$\int \Phi_1(f(y))\omega(|y|)\,dy < \infty. \tag{13.2}$$

For $x \in R^n - \{0\}$, we write $Lf(x) = L_1(x) + L_2(x) + L_3(x)$, where

$$L_1(x) = \int_{R^n - B(0,2|x|)} \log(1/|x-y|) f(y)\,dy,$$

$$L_2(x) = \int_{B(0,2|x|) - B(x,|x|/2)} \log(1/|x-y|) f(y)\,dy,$$

$$L_3(x) = \int_{B(x,|x|/2)} \log(1/|x-y|) f(y)\,dy.$$

Then we can easily find

$$L_1(x) \le \int_{R^n - B(0,2|x|)} \log(2/|y|) f(y)\,dy$$

and

$$L_2(x) \le \log(2/|x|) \int_{B(0,2|x|)} f(y)\,dy.$$

For nonnegative functions φ and ω as before, we set

$$\kappa_1'(r) = \sup_{r \le t \le 1} [\log(1/t)][\eta(t)]^{-1} \quad \text{with} \quad \eta(r) = \varphi(r^{-1})\omega(r)$$

for $0 < r \le 1/2$ and $\kappa_1'(r) = \kappa_1'(1/2)$ for $r > 1/2$.

The following results can be proved in the same manner as the lemmas in Section 2.

LEMMA 13.1. *Let* $0 < \delta < n$. *If* $0 < 2|x| < a < 1$, *then*

$$L_1(x) \le \int_{R^n - B(0,a)} \log(2/|y|) f(y)\, dy + M a^{n-\delta} \log(2/a)$$
$$+ M\kappa_1'(|x|)\left(\int_{B(0,a)} \Phi_1(f(y))\omega(|y|)\, dy\right),$$

where M *is a positive constant independent of* x *and* a.

LEMMA 13.2. *If* $0 < \delta < n$, *then there exists a positive constant* M *such that*

$$L_2(x) \le M\kappa_2'(|x|)\left(\int_{B(0,2|x|)} \Phi_1(f(y))\omega(|y|)\, dy\right) + M|x|^{n-\delta}\log(1/|x|)$$

for any $x \in B(0, 1/2) - \{0\}$, *where*

$$\kappa_2'(r) = \left(\log \frac{2}{r}\right) \sup_{0<t\le r} [\eta(t)]^{-1}$$

for $0 < r \le 1/2$ *and* $\kappa_2'(r) = \kappa_2'(1/2)$ *for* $r > 1/2$.

For an open set $G \subset R^n$, we define

$$C_{n,\Phi_1}(E; G) = \inf_g \int_G \Phi_1(g(y))\, dy,$$

where the infimum is taken over all nonnegative measurable functions g on R^n such that g vanishes outside G and

$$L^+ g(x) = \int \max\left\{0, \log \frac{1}{|x-y|}\right\} g(y)\, dy \ge 1 \qquad \text{for every } x \in E.$$

LEMMA 13.3. *Let* f *be a nonnegative measurable function on* R^n *satisfying condition* (13.2), *and* χ *be a positive function on* $(0, \infty)$ *for which there are positive constants* M *and* r_0 *such that* $\chi(r) \le M\chi(s)$ *whenever* $0 < r \le s \le 2r < r_0$. *Then there exists a set* $E \subset R^n$ *such that*

(i) $\lim_{x\to 0, x\in R^n - E} [\chi(|x|)]^{-1} L_3(x) = 0$;

(ii) $\sum_{j=1}^{\infty} [K^*]^{-j} \omega(2^{-j}) C_{n,\Phi_1}(E_j; B_j) < \infty$,

where

$$E_j = \{x \in E;\, 2^{-j} \le |x| < 2^{-j+1}\},$$
$$B_j = \{x \in R^n;\, 2^{-j-1} < |x| < 2^{-j+2}\},$$
$$K^* = \sup_{0<r,s\le r_0/2} \frac{\Phi_1(s/\chi(r))}{\Phi_1(s/\chi(2r))}.$$

Using these lemmas, we obtain the following theorems on the existence of fine limits for logarithmic potentials.

THEOREM 13.1 (cf. Theorem 3.1). *If f is a nonnegative measurable function on R^n satisfying conditions* (13.1) *and* (13.2), *then there exists a set $E \subset R^n$ such that*

$$\lim_{x \to 0, x \in R^n - E} Lf(x) = Lf(0)$$

and

$$\sum_{j=1}^{\infty} \omega(2^{-j}) C_{n,\Phi_1}(E_j; B_j) < \infty.$$

In case $Lf(0) = \infty$, we are concerned with the order of infinity at the origin.

THEOREM 13.2 (cf. Theorem 3.2). *Let f be a nonnegative measurable function on R^n satisfying conditions* (13.1) *and* (13.2), *and set $\kappa' = \kappa_1' + \kappa_2'$. If $\lim_{r \to 0} \kappa'(r) = \infty$, then there exists a set $E \subset R^n$ such that*

$$\lim_{x \to 0, x \in R^n - E} [\kappa'(|x|)]^{-1} Lf(x) = 0$$

and

$$\sum_{j=1}^{\infty} K^{-j} \omega(2^{-j}) C_{n,\Phi_1}(E_j; B_j) < \infty,$$

where E_j and B_j are as before, and

$$K = \sup_{0 < r, s \le 1/2} [\Phi_1(s/\kappa'(r))]/[\Phi_1(s/\kappa'(2r))].$$

THEOREM 13.3 (cf. Theorem 5.1). *Under the same assumptions as in Theorem* 13.2,

$$\lim_{r \to 0} [\kappa'(r)]^{-1} S_q(Lf, r) = 0$$

for $q > 0$.

For this, it suffices to treat only L_3. In case $q \ge 1$, setting $A(r) = B(0, 3r/2) - B(0, r/2)$, $0 < r < 2^{-1}$, we have

$$\begin{aligned} S_q(L_3, r) &\le \int_{A(r)} [S_q(\log|\cdot - y|, r)] f(y)\, dy \\ &\le M_1 \log(1/r) \int_{A(r)} f(y)\, dy \\ &\le M_1 [\log(1/r)] [\varphi(r^{-1})]^{-1} \int_{A(r)} \Phi_1(f(y))\, dy + M_1 [\log(1/r)] r^{-1} \int_{A(r)} dy \end{aligned}$$

$$\leq M_2 \kappa_1'(r) \int_{A(r)} \Phi_1(f(y))\omega(|y|)\,dy + M_2[\log(1/r)]r^{n-1},$$

so that

$$\lim_{r\to 0}[\kappa'(r)]^{-1} S_q(L_3, r) = 0.$$

THEOREM 13.4 (cf. Theorem 3.3). *Let f be as above. Set*

$$K(r) = \kappa'(r) + [\omega(r)]^{-1} \sup_{0<t<r}[\log(1/t)][\varphi(t^{-1})]^{-1},$$

and assume $K(r) < \infty$ for $r > 0$. If $\lim_{r\to 0} K(r) = \infty$, then

$$\lim_{x\to 0}[K(|x|)]^{-1} Lf(x) = 0.$$

If $K(r)$ is bounded, then $Lf(0)$ is finite and $Lf(x)$ tends to $Lf(0)$ as $x \to 0$.

COROLLARY 13.1 (cf. Corollary 3.1). *Let f be a nonnegative measurable function on R^n satisfying* (13.1) *and*

$$\int f(y)\log(2 + f(y))\,dy < \infty, \tag{13.3}$$

then Lf is continuous on R^n.

REMARK 13.1. If f is a nonnegative function in $L^p(R^n)$, $p > 1$, satisfying condition (13.1), then Lf is continuous as a consequence of Corollary 13.1. In this case, in view of Lemma 4.3 in [8], we find $\int_{R^n} |\nabla_n(Lf)(x)|^p\,dx < \infty$.

REMARK 13.2. If f is a nonnegative measurable function on R^n satisfying condition (13.1), then there exists a set E, which is thin at the origin, such that

$$\lim_{x\to 0, x\in R^n - E} Lf(x) = Lf(0)$$

and

$$\lim_{x\to 0, x\in R^n - E}[\log(1/|x|)]^{-1} Lf(x) = 0.$$

These facts follow readily from Theorems 13.1 and 13.2. For other generalizations of these facts, see Mizuta [15].

Next we consider the boundary limits of Green potentials of order n. We recall (see Corollary 11.1) that, for a nonnegative measurable function f on D, $G_n f \not\equiv \infty$ if and only if

$$\int_D (1 + |y|)^{-2} y_n f(y)\,dy < \infty. \tag{13.4}$$

From Corollary 13.1, we have

THEOREM 13.5. *If f is a nonnegative measurable function on D satisfying* (13.4) *such that*

$$\int_{D'} f(y)\log(2+f(y))\,dy<\infty \tag{13.5}$$

for any bounded open set D' with closure in D, then $G_n f$ is continuous on D.

LEMMA 13.4. *Let ω be a positive monotone function on $(0, \infty)$ satisfying* $(\omega 1)$ *and*

$(\omega 5)$ $\quad r^{\beta-1}[\omega(r)]^{-1}$ *is nondecreasing on* $(0, \infty)$ *for some* $\beta<2$.

Set $\kappa_3'(r)=\sup_{r\le t\le 1}[\eta(t)]^{-1}$ *for* $0<r\le 2^{-1}$ *and* $\kappa_3'(r)=\kappa_3'(2^{-1})$ *for* $r>2^{-1}$. *Then*

$$G_n(x, y)[\eta(y_n)]^{-1}\le M\kappa_3'(x_n) \quad \text{whenever} \quad 0<y_n<1 \quad \text{and} \quad 0<x_n<2|x-y|.$$

PROOF. If $y_n\ge x_n>0$ and $|x-y|\ge x_n/2$, then Lemma 11.1 implies

$$G_n(x, y)[\eta(y_n)]^{-1}\le M_1[\eta(y_n)]^{-1}\le M_1\kappa_3'(x_n).$$

If $0<y_n<x_n\le 2|x-y|$, then Lemma 11.1 implies

$$\begin{aligned} G_n(x, y)[\eta(y_n)]^{-1} &\le M_2 x_n^{-1} y_n[\eta(y_n)]^{-1} \\ &= M_2 x_n^{-1}\cdot y_n^{2-\beta}[\varphi(y_n^{-1})]^{-1}\cdot y_n^{\beta-1}[\omega(y_n)]^{-1} \\ &\le M_3[\eta(x_n)]^{-1}\le M_3\kappa_3'(x_n). \end{aligned}$$

Thus the present lemma is proved.

By Lemma 13.4 and the proof of Theorem 11.2, we have

THEOREM 13.6. *Let ω be as in Lemma* 13.4. *If* $\lim_{r\to 0}\kappa_3'(r)=\infty$ *and f is a nonnegative measurable function on D satisfying* (13.4) *and*

$$\int_{D'}\Phi_1(f(y))\omega(y_n)\,dy<\infty \quad \text{for any bounded open set } D'\subset D, \tag{13.6}$$

then there exists a set $E\subset D$ such that

$$\lim_{x_n\to 0,\, x\in D'-E}[\kappa_3'(x_n)]^{-1}G_n f(x)=0$$

for any bounded open set $D'\subset D$ and

$$\textstyle\sum_{j=1}^{\infty}K^{-j}\omega(2^{-j})C_{n,\Phi_1}(E_j\cap B(0, a);\, D_j\cap B(0, 2a))<\infty$$

for any $a>0$, where $K=K^*$ in Lemma 13.3 *with $\chi=\kappa_3'$.*

THEOREM 13.7. *Assume*

$$\varphi(r^{-1}) \geq M \log(2t/r) \quad \text{whenever} \quad 0 < r < t \tag{13.7}$$

for a positive constant M and

$$(\omega 6) \qquad r[\omega(r)]^{-1} \quad \text{is nondecreasing on} \quad (0, \infty).$$

Let ψ be a positive nondecreasing continuous function on $(0, \infty)$ *satisfying conditions* (Δ_2) *and* $(\psi 1)$, *and set*

$$h'(r) = \tau_2'(\psi(r)) \quad \text{with} \quad \tau_2'(r) = \inf_{r \leq t \leq 1} \{\omega(t) \inf_{0<s<t} [\log(2t/s)]^{-1} \varphi(s^{-1})\}$$

for $0 < r < 1$. *If f is a nonnegative measurable function on D satisfying* (13.4) *and* (13.6), *then there exists a set* $E \subset \partial D$ *such that* $H_{h'}(E) = 0$ *and*

$$\lim_{x \to \xi, x \in T_\psi(\xi, a)} G_n f(x) = 0$$

for any $\xi \in \partial D - E$ *and* $a > 0$.

PROOF. For $\xi \in \partial D$, as in the proof of Theorem 11.3, we write $G_n f = v_1 + v_2 + g_\xi$, and consider the set

$$E = \left\{\xi \in \partial D;\ \limsup_{r \to 0} [h'(r)]^{-1} \int_{D \cap B(\xi, r)} \Phi_1(f(y)) \omega(y_n)\, dy > 0\right\}.$$

Then, by (13.6) and Lemma 7.2, we see that $H_{h'}(E) = 0$. Using $(\omega 6)$, we have for δ, $0 < \delta < 2$,

$$\begin{aligned} r^{-1} \int_{D \cap B(\xi, r)} y_n f(y)\, dy &\leq M_1 [\varphi(r^{-1}) \omega(r)]^{-1} \int_{D \cap B(\xi, r)} \Phi_1(f(y)) \omega(y_n)\, dy + M_1 r^{n-\delta} \\ &\leq M_2 [\tau_2'(r)]^{-1} \int_{D \cap B(\xi, r)} \Phi_1(f(y)) \omega(y_n)\, dy + M_1 r^{n-\delta}. \end{aligned}$$

Hence, if $\xi \in \partial D - E$, then

$$\lim_{r \to 0} r^{-1} \int_{D \cap B(\xi, r)} y_n f(y)\, dy = 0.$$

Since Lemma 11.2 is still true in the present case ($\alpha = n$), $g_\xi(x)$ tends to zero as $x \to \xi$, $x \in D$. By Lemmas 11.1 and 13.4, we find

$$\begin{aligned} v_1(x) \leq & \int_{D \cap B(\xi, 2|x - \xi|) - B(x, x_n/2)} G_n(x, y) [\varphi(y_n^{-\delta})]^{-1} \Phi_1(f(y))\, dy \\ & + \int_{D \cap B(\xi, 2|x - \xi|)} G_n(x, y) y_n^{-\delta}\, dy \end{aligned}$$

$$\leq M_3[\tau_2'(x_n)]^{-1}\int_{D\cap B(\xi,2|x-\xi|)}\Phi_1(f(y))\omega(y_n)\,dy + M_3|x-\xi|^{n-\delta}$$

and

$$v_2(x) \leq M_4\int_{B(x,x_n/2)}\log(3x_n/|x-y|)f(y)\,dy$$

$$\leq M_5[\omega(x_n)]^{-1}\{\sup\nolimits_{0<r<x_n/2}[\log(3x_n/r)][\varphi(r^{-1})]^{-1}\}\int_{B(x,x_n/2)}\Phi_1(f(y))\omega(y_n)\,dy$$

$$+ M_5 x_n^{n-\delta}$$

$$\leq M_6[\tau_2'(x_n)]^{-1}\int_{B(x,x_n/2)}\Phi_1(f(y))\omega(y_n)\,dy + M_5 x_n^{n-\delta}.$$

Hence it follows that

$$\lim\nolimits_{x\to\xi, x\in T_\psi(\xi,a)}[v_1(x)+v_2(x)] = 0$$

for any $\xi\in\partial D - E$ and any $a>0$. Now Theorem 13.7 is proved.

The case $p>1$ is quite similar to Theorem 11.3. In fact we can prove

THEOREM 13.8. *Assume that* $p>1$ *and* $(\omega 4)$ *holds. Let* ψ *be a positive nondecreasing continuous function on* $(0,\infty)$ *satisfying* $(\psi 1)$, *and set*

$$\kappa_4'(r) = r\left(\int_r^1 [t^{n-np+p}\eta(t)]^{-p'/p}t^{-1}\,dt\right)^{1/p'},$$

$$\tau_4'(r) = \inf\nolimits_{r\leq t\leq 1}[\kappa_4'(t)]^{-p},$$

$$h''(r) = \tau_4'(\psi(r))$$

for $0<r<2^{-1}$. *If* f *is a nonnegative measurable function on* D *satisfying* (13.4) *and* (11.2), *then there exists a set* $E\subset\partial D$ *such that* $H_{h''}(E)=0$ *and*

$$\lim\nolimits_{x\to\xi, x\in T_\psi(\xi,a)} G_n f(x) = 0$$

for any $\xi\in\partial D - E$ *and* $a>0$. *If in addition* $\tau_4'(0)>0$, *then*

$$\lim\nolimits_{x\to\xi, x\in D} G_n f(x) = 0$$

for any $\xi\in\partial D$.

REMARK 13.3. If $\omega(r)=r^\beta$ and $\beta>n(p-1)$, then

$$\tau_4'(r) \sim r^{n-np+\beta}\varphi(r^{-1}) \qquad \text{as} \quad r\longrightarrow 0.$$

Here we may assume $n-np+\beta\leq n-1$, when we evaluate the size of the exceptional sets in the boundary ∂D. In the bordering case $\beta = np-1$, ω

does not satisfy (ω4). In this case, however, by (13.4),

$$\lim_{r\to 0} r^{-1} \int_{D\cap B(\xi,r)} y_n f(y)\, dy = 0$$

for every $\xi \in \partial D - E$, where $H_1(E) = 0$. Hence the proofs of Theorems 13.7 and 11.3 show that $G_n f$ has nontangential limit zero at almost every boundary point of D.

Corresponding to Theorems 8.1 and 11.4, we also l

THEOREM 13.9. *Let ω and ω^* be positive nondecreasing functions on the interval* $(0, \infty)$ *satisfying* (ω1), (ω6) *and, further,*

$$\int_0^r \omega^*(s) s^{-1}\, ds \le \omega(r) \qquad \text{for any} \quad r >$$

Let ψ be as in Theorem 13.8, *and define*

$$h^*(r) = \tau_2^*(\psi(r)) \quad \text{with} \quad \tau_2^*(r) = \inf_{r\le t\le 1} \{\omega^*(t) \inf_{0<s<t} [\log(2t/s)]^{-1} \varphi(s^{-1})\}$$

for $0 < r < 1$. *If f is a nonnegative measurable function on D satisfying conditions* (13.4) *and* (13.6), *then there exists a set E such that $H_{h^*}(E) = 0$ and*

$$\lim_{r\to 0} G_n f(\xi(r)) = 0 \qquad \text{for any} \quad \xi \in \partial D - E,$$

where $\xi(r) = \xi + \Psi(r)$ with $\Psi(r) = (r, \psi_2(r), \ldots, \psi_{n-1}(r), \psi(r))$.

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