HIROSHIMA MATH. J.
23 (1993), 63–77

# Applications of fractional calculus to ordinary and partial differential equations of the second order

Shih-Tong TU, Katsuyuki NISHIMOTO, Shing-Jong JAW
and
Shy-Der LIN

(Received April 27, 1991)

**Abstract.** In this paper, applications of the fractional calculus to the form

$$(z-a)(z-b)\varphi_2 + (C+Dz)\varphi_1 + E\varphi = f \qquad (z \neq a, z \neq b)$$

and the partial differential equation

$$\frac{\partial^2 \mu}{\partial z^2}(z-a)(z-b) + (C+Dz)\frac{\partial \mu}{\partial z} + \delta \cdot \mu(z, t) = A\frac{\partial^2 \mu}{\partial t^2} + B\frac{\partial \mu}{\partial t} \qquad (z \neq a, z \neq b)$$

are discussed.

## § 0. Introduction

Fractional calculus is a very useful and simple means in obtaining particular solutions to certain non-homogeneous linear differential equations. The solutions of linear ordinary differential equations of the Fuchs type [1]–[7], Gauss type [8, 9] and Laguerre's type [10] obtained by K. Nishimoto, S. L. Kalla, H. M. Srivastava, S. Owa, and S. T. Tu, are but a few important discoveries stemming from these researches. Now, we begin with the statement of the following definition of the fractional calculus (fractional integrals and fractional derivatives) given by Nishimoto 1976.

DEFINITION. If $f(z)$ is a regular function and it has no branch point inside $C$ and on $C(C = \{C_-, C_+\}$, $C_-$ is an integral curve along the cut joining two points $z$ and $-\infty + i\ \mathrm{Im}(z)$, and $C_+$ is an integral curve along the cut joining two points $z$ and $\infty + i\ \mathrm{Im}(z))$, $D = \{D_-, D_+\}$, $D_-$ is a domain surrounded by $C_-$, $D_+$ is a domain surrounded by $C_+$,

$$f_\nu = {}_cf_\nu(z) = \frac{\Gamma(\nu+1)}{2\pi i}\int_c \frac{f(\zeta)}{(\zeta - z)^{\nu+1}}\, d\zeta \qquad \begin{pmatrix} \Gamma\text{: Gamma function} \\ \nu \neq -1, -2, \ldots \end{pmatrix}$$

This work was supported, in part, by the National science council of R.O.C. under Grant NSC80–0208–M033–01.

and

$$f_{-n} = \lim_{\nu \to -n} f_\nu \qquad (n = 1, 2, \ldots)$$

where $\zeta \neq z$, $-\pi \leq \arg(\zeta - z) \leq \pi$ for $C_-$ and $0 \leq \arg(\zeta - z) \leq 2\pi$ for $C_+$, then $f_\nu$ $(\nu > 0)$ is the fractional derivative of order $\nu$ and $f_\nu$ $(\nu < 0)$ is the fractional integral of order $|\nu|$, if $|f_\nu| < \infty$. (Consider the principal value for many valued function $f$.)

We call the function $f = f(z)$ such that $|f_\nu| < \infty$ in $D$ as fractional differintegrable functions by arbitrary order $\nu$ and denote the set of them with notation $\mathscr{F}$. We use

$$|f_\nu| < \infty \Longleftrightarrow f \in \mathscr{F} \qquad (\text{in } D).$$

In order to discuss the solutions of ordinary and partial differential equations, we need the following lemmas [1].

Lemma 1 (Linearity). *Let $U(z)$ and $V(z)$ be analytic and one valued functions. If $U_\nu$ and $V_\nu$ exist, then*

(i) $(aU)_\nu = aU_\nu$,
(ii) $(aU + bV)_\nu = aU_\nu + bV_\nu$

*where $a$ and $b$ are constants and $z \in \boldsymbol{C}$, $\nu \in \boldsymbol{R}$.*

Lemma 2 (Index Law). *Let $f(z)$ be an analytic and one valued function. If $(f_\mu)_\nu$ and $(f_\nu)_\mu$ exist, then*

$$(f_\mu)_\nu = (f_\nu)_\mu,$$

*where $z \in \boldsymbol{C}$, $\mu$, $\nu \in \boldsymbol{R}$ and* $\left| \dfrac{\Gamma(\mu + \nu + 1)}{\Gamma(\mu + 1)\Gamma(\nu + 1)} \right| < \infty$.

Lemma 3 (Nishimoto 1979). *Let $U(z)$ and $V(z)$ be analytic and one valued functions. If $U_\nu$ and $V_\nu$ exist, then*

$$(UV)_\nu = \sum_{n=0}^{\infty} \frac{\Gamma(\nu + 1)}{\Gamma(\nu - n + 1)\Gamma(n + 1)} U_{\nu - n} V_n,$$

*where $\nu \in \boldsymbol{R}$ and* $\left| \dfrac{\Gamma(\nu + 1)}{\Gamma(\nu - n + 1)\Gamma(n + 1)} \right| < \infty$.

Lemma 4 (Nishimoto 1979). *If $|\Gamma(\nu - a)/\Gamma(-a)| < \infty$, then*

$$(z^a)_\nu = e^{-i\pi\nu} \frac{\Gamma(\nu - a)}{\Gamma(-a)} z^{a - \nu},$$

*where $\nu$ is a real number and $z \in \boldsymbol{C}$.*

LEMMA 5. $(e^{az})_\nu = a^\nu e^{az}$, $a \neq 0$, $z \in \boldsymbol{C}$, $\nu \in \boldsymbol{R}$.

## § 1. Main Theorem

With the help of above Lemmas, we have the following main results of this paper.

THEOREM 1. *If* $f \in \mathscr{F}$ *and* $f_{-\alpha-1} \neq 0$, *then the nonhomogeneous second order differentil equations*

$$(1.1) \qquad (z-a)(z-b)\varphi_2 + (C+Dz)\varphi_1 + E\varphi = f \qquad (z \neq a,\ z \neq b)$$

*has a particular solution of the form*

$$(1.2) \qquad \varphi = ((f_{-\alpha-1}[(z-a)(z-b)]^{-(\alpha+2)}(z-a)^{(C+aD)/(a-b)}(z-b)^{-(C+bD)/(a-b)})_{-1} \cdot [(z-a)(z-b)]^{1+\alpha}(z-a)^{-(C+aD)/(a-b)}(z-b)^{(C+bD)/(a-b)})_\alpha$$

*for* $a \neq b$

*and*

$$(1.3) \qquad \varphi = ((f_{-\alpha-1}(z-a)^{-(2\alpha+4-D)}e^{-(C+aD)/(z-a)})_{-1} \cdot (z-a)^{2\alpha+2-D}e^{(C+aD)/(z-a)})_\alpha$$

*for* $a = b$.

*Here* $a, b, C, D$ *and* $E$ *are constants,* $\varphi = \varphi(z)$, $f = f(z)$ *is known and* $\alpha = \dfrac{(D-3) \pm \sqrt{(D-1)^2 - 4E}}{2}$ *with* $(D-1)^2 \geq 4E$.

PROOF. For $a \neq b$, we choose suitable $\alpha$ such that

$$(1.4) \qquad (\alpha+1)(D-\alpha-2) = E$$

or

$$(1.5) \qquad \alpha = \frac{(D-3) \pm \sqrt{(D-1)^2 - 4E}}{2}$$

with $(D-1)^2 \geq 4E$, and let $\varphi = W_\alpha$, we have

$$(1.6) \qquad \varphi_1 = W_{\alpha+1}, \quad \varphi_2 = W_{\alpha+2}.$$

Substituting (1.6) into (1.1), (1.1) becomes

$$(1.7) \qquad (z-a)(z-b)W_{\alpha+2} + (C+Dz)W_{\alpha+1} + (\alpha+1)(D-\alpha-2)W_\alpha = f.$$

It follows from Lemma 3 that

(1.8)
$$\begin{aligned}&(W_1(z-a)(z-b))_{\alpha+1}\\&\quad=\sum_{k=0}^{2}\frac{\Gamma(\alpha+2)}{\Gamma(\alpha-k+2)\Gamma(k+1)}W_{\alpha+2-k}((z-a)(z-b))_k\\&\quad=W_{\alpha+2}(z-a)(z-b)+(\alpha+1)(2z-a-b)W_{\alpha+1}+\alpha(\alpha+1)W_\alpha\end{aligned}$$

(1.9)
$$\begin{aligned}(2\alpha+2-D)(Wz)_{\alpha+1}&=(2\alpha+2-D)(W_{\alpha+1}z+(\alpha+1)W_\alpha)\\&=(2\alpha+2-D)zW_{\alpha+1}+(2\alpha+2-D)(\alpha+1)W_\alpha\end{aligned}$$

and

(1.10)
$$([C+(a+b)(\alpha+1)]W)_{\alpha+1}=[C+(a+b)(\alpha+1)]W_{\alpha+1}.$$

Therefore, with the aid of (1.8), (1.9) and (1.10), (1.7) gives

(1.11)
$$\begin{aligned}&(W_1(z-a)(z-b))_{\alpha+1}-(2\alpha+2-D)(Wz)_{\alpha+1}+[C+(a+b)(\alpha+1)]W_{\alpha+1}\\&=(z-a)(z-b)W_{\alpha+2}+(C+Dz)W_{\alpha+1}+[\alpha(\alpha+1)-(2\alpha+2-D)(\alpha+1)]W_\alpha\\&=f.\end{aligned}$$

That is, $(W_1(z-a)(z-b)-[(2\alpha+2-D)z-(C+(a+b)(\alpha+1))]W)_{\alpha+1}=f$. This is equivalent to

(1.12)
$$W_1-\frac{(2\alpha+2-D)z-(C+(a+b)(\alpha+1))}{(z-a)(z-b)}W=f_{-\alpha-1}\frac{1}{(z-a)(z-b)}.$$

Let

(1.13)
$$\begin{aligned}P(z)&=-\int\frac{(2\alpha+2-D)z-(C+(a+b)(\alpha+1))}{(z-a)(z-b)}dz\\&=-\left\{\left[(1+\alpha)-\frac{C+aD}{a-b}\right]\log(z-a)+\left[(1+\alpha)+\frac{C+bD}{a-b}\right]\log(z-b)\right\}.\end{aligned}$$

Then (1.12) is equivalent to

(1.14)
$$(W\cdot e^{P(z)})_1=f_{-\alpha-1}\frac{1}{(z-a)(z-b)}e^{P(z)}.$$

Thus we have

(1.15)
$$W=\left(f_{-\alpha-1}\frac{1}{(z-a)(z-b)}e^{P(z)}\right)_{-1}e^{-P(z)}.$$

Therefore, a particular solution $\varphi$ of (1.1) is given by

(1.16)

$$\varphi = W_\alpha = \left(\left(f_{-\alpha-1}\frac{1}{(z-a)(z-b)}e^{P(z)}\right)_{-1} e^{-P(z)}\right)_\alpha$$

$$= ((f_{-\alpha-1}[(z-a)(z-b)]^{-(\alpha+2)}(z-a)^{(C+aD)/(a-b)}(z-b)^{-(C+bD)/(a-b)})_{-1}$$

$$[(z-a)(z-b)]^{\alpha+1}(z-a)^{-(C+aD)/(a-b)}(z-b)^{(C+bD)/(a-b)})_\alpha,$$

where $\alpha$ is given in (1.5). Conversely, if (1.16) holds true. From (1.13) and (1.15) we have

$$(e^{-P(z)})_1 = e^{-P(z)}(-P(z))_1$$

$$= e^{-P(z)}\frac{(2\alpha+2-D)z-(C+(a+b)(\alpha+1))}{(z-a)(z-b)},$$

and

$$W_1 = \left(\left(f_{-\alpha-1}\frac{1}{(z-a)(z-b)}e^{P(z)}\right)_{-1} e^{-P(z)}\right)_1$$

$$= f_{-\alpha-1}\frac{1}{(z-a)(z-b)} + \left(f_{-\alpha-1}\frac{1}{(z-a)(z-b)}e^{P(z)}\right)_{-1} e^{-P(z)}$$

$$\cdot\frac{(2\alpha+2-D)z-(C+(a+b)(\alpha+1))}{(z-a)(z-b)}.$$

Then, substituting (1.15) and (1.16) into the left hand side (L.H.S.) of (1.1), we obtain

$$\text{L.H.S. of (1.1)} = (W_1(z-a)(z-b))_{\alpha+1} - (2\alpha+2-D)(Wz)_{\alpha+1}$$

$$+ [C+(a+b)(\alpha+1)]W_{\alpha+1}$$

$$= (W_1(z-a)(z-b) - [\{(2\alpha+2-D)z$$

$$- [C+(a+b)(\alpha+1)]\} W])_{\alpha+1}$$

$$= \left(f_{-\alpha-1} + \left(f_{-\alpha-1}\frac{1}{(z-a)(z-b)}e^{P(z)}\right)_{-1}\right.$$

$$\cdot e^{-P(z)}((2\alpha+2-D)z - [C+(a+b)(\alpha+1)])$$

$$- ((2\alpha+2-D)z - [C+(a+b)(\alpha+1)])\left(f_{-\alpha-1}\right.$$

$$\left.\left.\cdot\frac{1}{(z-a)(z-b)}e^{P(z)}\right)_{-1} e^{-p(z)}\right)_{\alpha+1}$$

$$= (f_{-\alpha-1})_{\alpha+1} = (f)_0 = f.$$

Next, we consider the case $a = b$. For $a = b$, (1.1) becomes

$$(1.17) \qquad (z-a)^2\varphi_2 + (C+Dz)\varphi_1 + E\varphi = f \qquad (z \neq a).$$

Define $\alpha$ and $\varphi$ as in (1.4) and (1.6) respectively, and do the same way as (1.7), (1.8), (1.9) and (1.10). Then (1.17) gives

$$(1.18) \qquad (W_1(z-a)^2)_{\alpha+1} - (((2\alpha+2-D)z - [C+2a(\alpha+1)])W)_{\alpha+1} = f$$

that is,

$$(1.19) \qquad W_1 - \frac{(2\alpha+2-D)z - [C+2a(\alpha+1)]}{(z-a)^2} W = f_{-\alpha-1}\frac{1}{(z-a)^2}.$$

Letting

$$Q(z) = -\int \frac{(2\alpha+2-D)z - (C+2a(\alpha+1))}{(z-a)^2}dz$$
$$= -(2\alpha+2-D)\log(z-a) - \frac{C+aD}{z-a},$$

we get

$$(1.20) \qquad e^{Q(z)} = (z-a)^{-(2\alpha+2-D)}e^{-(C+aD)/(z-a)}.$$

From (1.19), we have

$$(1.21) \qquad (W\cdot e^{Q(z)})_1 = f_{-\alpha-1}\frac{1}{(z-a)^2}e^{Q(z)}.$$

Thus (1.21) has a solution of the form

$$(1.22) \qquad W = \left(f_{-\alpha-1}\frac{1}{(z-a)^2}e^{Q(z)}\right)_{-1} e^{-Q(z)}.$$

Therefore, a solution $\varphi$ of (1.17) is given by

(1.23)

$$\varphi = W_\alpha$$
$$= \left(\left(f_{-\alpha-1}\frac{1}{(z-a)^2}e^{Q(z)}\right)_{-1} e^{-Q(z)}\right)_\alpha$$
$$= \left(\left(f_{-\alpha-1}\frac{1}{(z-a)^2}(z-a)^{-(2\alpha+2-D)}e^{-(C+aD)/(z-a)}\right)_{-1}\right.$$
$$\left.\cdot (z-a)^{(2\alpha+2-D)}e^{(C+aD)/(z-a)}\right)_\alpha$$

$$= ((f_{-\alpha-1}(z-a)^{-(2\alpha+4-D)}e^{-(C+aD)/(z-a)})_{-1}(z-a)^{2\alpha+2-D}e^{(C+aD)/(z-a)})_\alpha .$$

Conversely, if (1.23) holds true, from (1.20) and (1.22) we have

$$(e^{-Q(z)})_1 = e^{-Q(z)}(-Q(z))_1$$
$$= e^{-Q(z)}\frac{(2\alpha+2-D)z-(C+2a(\alpha+1))}{(z-a)^2}$$

and

$$W_1 = \left(\left(f_{-\alpha-1}\frac{1}{(z-a)^2}e^{Q(z)}\right)_{-1} e^{-Q(z)}\right)_1$$
$$= f_{-\alpha-1}\frac{1}{(z-a)^2} + \left(f_{-\alpha-1}\frac{1}{(z-a)^2}e^{Q(z)}\right)_{-1}$$
$$\cdot e^{-Q(z)}\frac{(2\alpha+2-D)z-(C+2a(\alpha+1))}{(z-a)^2}.$$

Substituting (1.23) into L.H.S. of (1.17), we obtain

$$\text{L.H.S. of (1.17)} = (z-a)^2 W_{\alpha+2} + (C+Dz)W_{\alpha+1} + EW_\alpha$$
$$= (z-a)^2 W_{\alpha+2} + (C+Dz)W_{\alpha+1}$$
$$+ [\alpha(\alpha+1) - (2\alpha+2-D)(\alpha+1)]W_\alpha$$
$$= (W_1(z-a)^2 - ((2\alpha+2-D)z - C - 2a(\alpha+1))W)_{\alpha+1}$$
$$= \left(f_{-\alpha-1} + \left(f_{-\alpha-1}\frac{1}{(z-a)^2}e^{Q(z)}\right)_{-1}\right.$$
$$\cdot e^{-Q(z)}[(2\alpha+2-D)z-(C+2a(\alpha+1))]$$
$$-((2\alpha+2-D)z - C - 2a(\alpha+1))$$
$$\left.\cdot\left(f_{-\alpha-1}\frac{1}{(z-a)^2}e^{Q(z)}\right)_{-1} e^{-Q(z)}\right)_{\alpha+1}$$
$$= (f_{-\alpha-1})_{\alpha+1} = f.$$

This completes the proof of Theorem 1.

THEOREM 2. *The homogeneous second order linear ordinary differential equation*

$$(1.24) \qquad (z-a)(z-b)\varphi_2 + (C+Dz)\varphi_1 + E\varphi = 0 \qquad (z \neq a,\ z \neq b)$$

*has solutions of the form*

$$(1.25) \qquad \varphi = M(((z-a)(z-b))^{1+\alpha}(z-a)^{-(C+aD)/(a-b)}(z-b)^{(C+bD)/(a-b)})_\alpha$$

*for* $a \neq b$

*and*

(1.26) $$\varphi = M((z-a)^{2\alpha+2-D}e^{(C+aD)/(z-a)})_\alpha \qquad \textit{for}\ \ a=b,$$

*where* $a$, $b$, $C$, $D$ *and* $E$ *are constants,* $M$ *is an arbitrary integral constant,* $\varphi = \varphi(z)$,

(1.27) $$\alpha = \frac{(D-3) \pm \sqrt{(D-1)^2 - 4E}}{2} \quad \textit{or} \quad (\alpha+1)(D-\alpha-2) = E$$

*with* $(D-1)^2 \geq 4E$.

Proof. For $a \neq b$, define $\alpha$ and $\varphi$ as in (1.4) and (1.6) respectively. Then (1.24) becomes

(1.28)
$$(z-a)(z-b)W_{\alpha+2} + (C+Dz)W_{\alpha+1} + [\alpha(\alpha+1) - (2\alpha+2-D)(\alpha+1)]\,W_\alpha = 0.$$

It follows from (1.8), (1.9) and (1.10) that

$$(W_1(z-a)(z-b))_{\alpha+1} - (2\alpha+2-D)(Wz)_{\alpha+1} + (C+(a+b)(\alpha+1))W_{\alpha+1} = 0$$

that is

(1.29) $$W_1 - \frac{(2\alpha+2-D)z - C - (a+b)(\alpha+1)}{(z-a)(z-b)}W = 0.$$

A solution of the differential equation (1.29) is given by

(1.30) $$W = Me^{-P(z)}$$

where $P(z)$ is defined as (1.13) and $M$ is an arbitrary integral constant. Therefore,

(1.31) $$\begin{aligned}\varphi &= W_\alpha = (Me^{-P(z)})_\alpha \\ &= M\,(\,((z-a)(z-b))^{1+\alpha}\cdot(z-a)^{-(C+aD)/(a-b)}(z-b)^{(C+bD)/(a-b)})_\alpha.\end{aligned}$$

Conversely, substituting (1.31) into L.H.S. of (1.24), we have

$$\begin{aligned}\text{L.H.S. of (1.24)} &= (W_1(z-a)(z-b))_{\alpha+1} - (2\alpha+2-D)(Wz)_{\alpha+1} \\ &\quad + (C+(a+b)(\alpha+1))\,W_{\alpha+1} \\ &= (Me^{-P(z)}\frac{(2\alpha+2-D)z - (C+(a+b)(\alpha+1))}{(z-a)(z-b)}(z-a)(z-b) \\ &\quad - [(2\alpha+2-D)z - (C+(a+b)(\alpha+1)]\,Me^{-P(z)})_{\alpha+1} \\ &= (0)_{\alpha+1} = 0.\end{aligned}$$

Finally, for $a = b$, the proof of (1.26) is similar to the proof of (1.25).

With the help of Theorem 1 and Theorem 2, we have

THEOREM 3. *If* $f \in \mathscr{F}$ *and* $f_{-\alpha-1} \neq 0$ *then the fractional differintegrated functions*

(1.32)

$$\begin{aligned}\varphi = &((f_{-\alpha-1}((z-a)(z-b))^{-(\alpha+2)}(z-a)^{(C+aD)/(a-b)}(z-b)^{-(C+bD)/(a-b)})_{-1}\\ &\cdot((z-a)(z-b))^{\alpha+1}(z-a)^{-(C+aD)/(a-b)}(z-b)^{(C+bD)/(a-b)})_{\alpha}\\ &+M([(z-a)(z-b)]^{\alpha+1}(z-a)^{-(C+aD)/(a-b)}(z-b)^{(C+bD)/(a-b)})_{\alpha}\end{aligned}$$

*satisfies* (1.1) *for* $a \neq b$. *And*

(1.33)

$$\begin{aligned}\varphi = &((f_{-\alpha-1}(z-a)^{-(2\alpha+4-D)}e^{-(C+aD)/(z-a)})_{-1}(z-a)^{2\alpha+2-D}e^{(C+aD)/(z-a)})_{\alpha}\\ &+M((z-a)^{2\alpha+2-D}e^{(C+aD)/(z-a)})_{\alpha}\end{aligned}$$

*satisfies* (1.17). *Here* $\varphi = \varphi(z)$, *a, b, C, D and E are constants,* $\alpha$ *is defined as* (1.5), *and M is an arbitrary integral constant.*

If we take $a = -b = k$, $C = 0$, $D = 1$ and $E = -P^2$ in Theorem 1, then (1.1) becomes Chebyshev's equation of order $P$ and we have the following corollary.

COROLLARY (Theorem 1 in [13]). *If* $f \in \mathscr{F}$ *and* $f_{-P} \neq 0$, *then the generalized second-order nonhomogeneous Chebyshev's equation of order P*

$$(z^2 - k^2)\varphi_2 + z\varphi_1 - P^2\varphi = f \qquad (z \neq k,\ z \neq -k) \tag{1.34}$$

*has a particular solution of the form*

$$\varphi = ((f_{-P}(z^2-k^2)^{-(2P+1)/2})_{-1}(z^2-k^2)^{(2P-1)/2})_{P-1},$$

*where P and k are constants and* $P \in \mathbf{R}$.

PROOF. Take $\alpha = P - 1$ in (1.2).

## § 2. Examples

EXAMPLE 1. The nonhomogeneous second order differential equation

$$(z^2 - z)\varphi_2 - 2z\varphi_1 + 2\varphi = (z-1)^3 \qquad (z \neq 0,\ z \neq 1) \tag{2.1}$$

has a particular solution of the form

$$\varphi = \frac{1}{2}(z-1)^3. \tag{2.2}$$

If we set $a = C = 0$, $b = 1$, $D = -2$, $E = 2$, and $f = (z-1)^3$ in Theorem 1, we obtain $\alpha = -2$ or $\alpha = -3$. Here we take $\alpha = -2$ then by (1.2), we have

$$\varphi = \left( (f_1 (z-1)^{-2})_{-1} \cdot \frac{z-1}{z} \right)_{-2} = (3(z-1))_{-2} = \frac{1}{2}(z-1)^3.$$

EXAMPLE 2. The nonhomogeneous second order differential equation

(2.3)

$$(z^2 - 2z)\varphi_2 + \left(\frac{5}{2}z - 2\right)\varphi_1 + \frac{1}{2}\varphi = \left( \left( \frac{z-2}{z} \right)_1 \cdot z^{1/2} \right)_{1/2} \qquad (z \neq 0,\ z \neq 2),$$

has a particular solution of the form

$$\varphi = (z^{-3/2})_{1/2} = i \frac{\Gamma(1)}{\Gamma(3/2)} z^{-1}. \tag{2.4}$$

If we set $a = 0$, $b = 2$, $C = -2$, $D = 5/2$, $E = 1/2$ and $f = \left( \left( \frac{z-2}{z} \right)_1 \cdot z^{1/2} \right)_{1/2}$ in Theorem 1, we obtain $\alpha = 0$ or $\alpha = -1/2$. Here, we take $\alpha = -1/2$, then by (1.2), we have

$$\varphi = ((f_{-1/2} \cdot z^{-1/2})_{-1} \cdot z^{-1/2}(z-2)^{-1})_{-1/2} = (z^{-3/2})_{-1/2} = i \frac{\Gamma(1)}{\Gamma(3/2)} z^{-1}.$$

Indeed, $\varphi = i \frac{\Gamma(1)}{\Gamma(3/2)} z^{-1}$ satisfies (2.3). Since

$$\varphi_1 = -i \frac{\Gamma(1)}{\Gamma(3/2)} z^{-2}, \qquad \varphi_2 = 2i \frac{\Gamma(1)}{\Gamma(3/2)} z^{-3},$$

substituting into the left hand side (L.H.S.) of (2.3) we obtain

$$\begin{aligned} \text{L.H.S. of (2.3)} &= i \frac{\Gamma(1)}{\Gamma(3/2)} \left( 2z^{-1} - 4z^{-2} - \frac{5}{2}z^{-1} + 2z^{-2} + \frac{1}{2}z^{-1} \right) \\ &= -2i \frac{\Gamma(1)}{\Gamma(3/2)} z^{-2}, \end{aligned}$$

and

$$f = \left( \left( \frac{z-2}{z} \right)_1 \cdot z^{1/2} \right)_{1/2} = 2(z^{-3/2})_{-1/2} = -2i \frac{\Gamma(1)}{\Gamma(3/2)} z^{-2}.$$

REMARK 1. Taking $a = b = 0$, $C = 0$, $D = 2\nu - k$, and $E = \nu(\nu - 1 - k)$ in Theorem 2, we obtain $\alpha = \nu - 1$. And the result coincides with Theorem 1 by Nishimoto ([1, vol II, p. 28]).

REMARK 2. By taking $a = 0$, $b = 1$, $C = \gamma - 1$, $D = \gamma$ and $E = k(\gamma - k - 1)$ in Theorem 1, the equation (1.1) becomes hypergeometric differential equation and we obtain $\alpha = k - 1$, this result coincides with Theorem 1 by Nishimoto ([1, vol II, p. 73]).

REMARK 3. By taking $a = 1$, $b = -1$, $C = 0$, $D = 2$, and $E = -\nu(\nu + 1)$ in Theorem 1, the equation (1.1) becomes Legendre equation of order $\nu$ and we obtain $\alpha = \nu$, this result coincides with Theorem 1 by Nishimoto ([1, vol II, p. 88] or [11]).

REMARK 4. Taking $a = 1$, $b = -1$, $C = -k$, $D = 2\nu$ and $E = \nu(\nu - 1)$ in Theorem 1, we obtain $\alpha = \nu - 1$, this result coincides with Theorem 1 by Nishimoto ([1, vol II, p. 99]).

REMARK 5. Taking $a = 0$, $b = 1$, $C = -\nu - 1$, $D = 2\nu$ and $E = \nu(\nu - 1)$ in Theorem 1, and we obtain $\alpha = \nu - 1$, this result coincides with Theorem 1 by Nishimoto ([5]).

REMARK 6. Taking $a = 0$, $b = 1$, $C = -\nu + 1$, $D = 2\nu$ and $E = \nu(\nu - 1)$ in Theorem 1, and we obtain $\alpha = \nu - 1$, this result coincides with Theorem 1 by Nishimoto ([12]).

REMARK 7. Taking $a = 0$, $b = \nu$, $C = -\nu^2 + \nu$, $D = 2\nu$ and $E = \nu(\nu - 1)$ in Theorem 1, and we obtain $\alpha = \nu - 2$, this result coincides with Theorem 1 by Nishimoto ([1, vol II, p. 110]).

## §3. Partial Differential Equation

THEOREM 4. *A partial differential equation of the second order*

$$\frac{\partial^2 \mu}{\partial z^2}(z - a)(z - b) + (C + Dz)\frac{\partial \mu}{\partial z} + \delta \cdot \mu(z, t) = A\frac{\partial^2 \mu}{\partial t^2} + B\frac{\partial \mu}{\partial t} \tag{3.1}$$

$$(z \neq a,\ z \neq b,\ a \neq b)$$

*has solutions of the form*

(3.2)

$$\mu(z, t) = M([(z - a)(z - b)]^{\alpha + 1}(z - a)^{-(C + aD)/(a - b)}(z - b)^{(C + bD)/(a - b)})_{\alpha} \cdot \exp\left[\frac{-B \pm \sqrt{B^2 + 4A(\delta - E)}}{2A}t\right] \quad \textit{for } AB \neq 0;$$

(3.3)
$$\mu(z, t) = M([(z-a)(z-b)]^{\alpha+1} \cdot (z-a)^{-(C+aD)/(a-b)} \cdot (z-b)^{(C+bD)/(a-b)})_\alpha \cdot \exp\left[\pm\left(\frac{\delta - E}{A}\right)^{1/2} t\right] \quad \textit{for } A \neq 0 \textit{ and } B = 0;$$

*and*

(3.4)
$$\mu(z, t) = M([(z-a)(z-b)]^{\alpha+1} \cdot (z-a)^{-(C+aD)/(a-b)} \cdot (z-b)^{(C+bD)/(a-b)})_\alpha \cdot \exp\left(\frac{\delta - E}{B} t\right) \quad \textit{for } A = 0 \textit{ and } B \neq 0,$$

*where $a, b, C, D$ and $\delta$ are given constants, $E = (\alpha + 1)(D - \alpha - 2)$ with $(D-1)^2 \geq 4E$, and $M$ is an arbitrary constant.*

PROOF. Let $\mu(z, t) = \varphi(z)e^{\lambda t}$ $(\lambda \neq 0)$ be a solution of (3.1). Since

$$\frac{\partial \mu}{\partial t} = \varphi \lambda e^{\lambda t}, \qquad \frac{\partial^2 \mu}{\partial t^2} = \varphi \lambda^2 e^{\lambda t},$$

$$\frac{\partial \mu}{\partial z} = \varphi_1 e^{\lambda t}, \qquad \frac{\partial^2 \mu}{\partial z^2} = \varphi_2 e^{\lambda t},$$

(3.1) becomes

$$\varphi_2(z-a)(z-b) + \varphi_1(C + Dz) + \varphi(\delta - A\lambda^2 - B\lambda) = 0. \tag{3.5}$$

Here we choose $\lambda$ as $\delta - A\lambda^2 - B\lambda = E$, that is,

$$\lambda = \frac{-B \pm \sqrt{B^2 + 4A(\delta - E)}}{2A} \qquad \text{for } AB \neq 0 \tag{3.6}$$

$$\lambda = \frac{\delta - E}{B} \qquad \text{for } A = 0 \text{ and } B \neq 0 \tag{3.7}$$

$$\lambda = \pm\left(\frac{\delta - E}{A}\right)^{1/2} \qquad \text{for } A \neq 0 \text{ and } B = 0. \tag{3.8}$$

Then (3.5) becomes

$$\varphi_2(z-a)(z-b) + \varphi_1(C + Dz) + E\varphi = 0, \qquad (z \neq a, z \neq b, a \neq b).$$

By Theorem 2, a solution is given by

$$\varphi = M([(z-a)(z-b)]^{\alpha+1}(z-a)^{-(C+aD)/(a-b)} \cdot (z-b)^{(C+bD)/(a-b)})_\alpha \quad \text{for } a \neq b.$$

Thus, for $AB \neq 0$, the partial differential equation (3.1) has a solution of the form (3.2). Moreover, for $B = 0$ and $A \neq 0$, a solution of (3.1) is given by (3.3) and for $B \neq 0$ and $A = 0$, (3.1) has a solution (3.4).

Conversely, for $AB \neq 0$, we shall show that (3.2) satisfies (3.1). Let

$$M([(z-a)(z-b)]^{\alpha+1} \cdot (z-a)^{-(C+aD)/(a-b)} \cdot (z-b)^{(C+bD)/(a-b)})_\alpha = \varphi(z)$$

and

$$\frac{-B \pm \sqrt{B^2 + 4A(\delta - E)}}{2A} = \lambda.$$

Then (3.2) becomes

$$\mu(z, t) = \varphi(z) e^{\lambda t}.$$

Since

$$\frac{\partial \mu}{\partial z} = \varphi_1 e^{\lambda t}, \qquad \frac{\partial^2 \mu}{\partial z^2} = \varphi_2 e^{\lambda t},$$

$$\frac{\partial \mu}{\partial t} = \lambda \varphi e^{\lambda t}, \qquad \frac{\partial^2 \mu}{\partial t^2} = \lambda^2 \varphi e^{\lambda t},$$

the left hand side of (3.1) $= e^{\lambda t}[\varphi_2 (z-a)(z-b) + (C + Dz)\varphi_1 + \delta\varphi]$

$$\begin{aligned} &= e^{\lambda t}(\delta - E)\varphi \qquad \text{(by Theorem 2)} \\ &= e^{\lambda t}\varphi(A\lambda^2 + B\lambda) \\ &= A\frac{\partial^2 \mu}{\partial t^2} + B\frac{\partial \mu}{\partial t}. \end{aligned}$$

The proofs of (3.3) and (3.4) are obvious.

Similarly, we can easily deduce the following result.

THEOREM 5. *A partial differential equation of the second order*

$$\frac{\partial^2 \mu}{\partial z^2}(z-a)^2 + (C + Dz)\frac{\partial \mu}{\partial z} + \delta \cdot \mu(z, t) = A\frac{\partial^2 \mu}{\partial t^2} + B\frac{\partial \mu}{\partial t} \qquad (z \neq a)$$

*has solutions of the form*

$$\mu(z, t) = M((z-a)^{2\alpha+2-D} \cdot e^{(C+aD)/(z-a)})_\alpha \exp\left(\frac{-B \pm \sqrt{B^2 + 4A(\delta - E)}}{2A} t\right)$$

*for* $AB \neq 0$;

$$\mu(z, t) = M((z-a)^{2\alpha+2-D} \cdot e^{(C+aD)/(z-a)})_\alpha \exp\left(\pm\left(\frac{\delta - E}{A}\right)^{1/2} t\right)$$

*for* $A \neq 0$ *and* $B = 0$;

*and*

$$\mu(z, t) = M((z-a)^{2\alpha+2-D} \cdot e^{(C+aD)/(z-a)})_\alpha \exp\left(\frac{\delta - E}{B} t\right)$$

*for* $A = 0$ *and* $B \neq 0$,

*where* $a, C, D$ *and* $\delta$ *are given constants,* $E = (\alpha+1)(D-\alpha-2)$ *with* $(D-1)^2 \geq 4E$ *and* $M$ *is an arbitrary constant.*

Remark 8. By taking $a = -b = K$, $C = 0$, $D = 2$, $\delta = 1$ and $E = -\alpha(\alpha+1)$ in Theorem 4 and Theorem 5, the equation (3.1) becomes Legendre's differential equation of Fuchs type and the results coincide with the main theorem by Nishimoto, Tu and Wu ([14]).

Example 3 [14]. A solution of the partial differential equation of Fuchs type

$$\frac{\partial^2 \mu}{\partial z^2}(z^2 - 1) + \frac{\partial \mu}{\partial z} \cdot 2z + \mu(z, t) = 2\frac{\partial^2 \mu}{\partial t^2} + \frac{\partial \mu}{\partial t}$$

is given by

$$\mu(z, t) = 4(3z^2 - 1)e^{((-1 \pm \sqrt{57})/4)t}.$$

Letting $a = 1$, $b = -1$, $C = 0$, $D = 2$, $\delta = 1$, $\alpha = 2$, $A = 2$ and $B = 1$ in Theorem 4, by (3.2) we obtain the solution directly

$$\mu(z, t) = M((z^2 - 1)^2)_2 \exp\left(\frac{-1 \pm \sqrt{57}}{4} t\right).$$

## References

[1] K. Nishimoto, Fractional Calculus (Volumes I, II, III), Descartes Press, Koriyama, 1984, 1987, and 1989.

[2] K. Nishimoto and S. L. Kalla, Application of fractional calculus to ordinary differential equations of Fuchs type, Rev. Técn. Fac. Ingr. Univ. Zulia, **12** (1989), 43–46.

[3] H. M. Srivastava, S. Owa, and K. Nishimoto, Some fractional differintegral equations, J. Math. Anal. Appl., **106** (1985), 360–366.

[4] K. Nishimoto, Applications of fractional calculus to the solutions of linear second order differential equations of Fuchs type, Research Notes in Math., Vol. 138 (1985), 140–153, Pitman.

[5] K. Nishimoto, An application of fractional calculus to differential equatins of Fuchs type $\phi_2 \cdot (z^2 - z) + \phi_1(2\nu z - \nu - 1) + \phi \cdot \nu(\nu - 1) = f$, J. Coll. Engng. Nihon Univ., B-**27** (1986), 5–16.
[6] K. Nishimoto, An application of fractional calculus to differential equations of Fuchs type $\phi_2 \cdot z^3 + \phi_1\{2(1 + \alpha)z^2 + (\alpha + \beta + \gamma)z + \alpha\beta\gamma\} + \phi\{\alpha(1 + \alpha)z + \alpha(\alpha + \beta + \gamma)\} = f$, J. Coll. Engng. Nihon Univ., B-**28** (1987), 9–13.
[7] K. Nishimoto and Shih-Tong Tu, Applications of fractional calculus to third order ordinary and partial differential equations of Fuchs type, Coll. Engng. Nihon Univ. Fractional Calculus and Its Applications, International (Tokyo) Conference Proceedings (1990), 166–174.
[8] K. Nishimoto, An application of fractional calculus to the non-homogeneous Gauss equations, J. Coll. Engng. Nihon Univ., B-**28** (1987), 1–8.
[9] K. Nishimoto and Shih-Tong Tu, Applications of fractional calculus to ordinary and partial differential equations of Gauss type, Coll. Engng. Nihon Univ. Fractional Calculus and Its Applications, International (Tokyo) Conference Proceedings (1990), 159–165.
[10] K. Nishimoto and S. L. Kalla, Use of fractional calculus to solve certain linear differential equations, J. Coll. Engng. Nihon Univ., B-**30** (1989).
[11] K. Nishimoto, An application of the fractional calculus to a differential equation of Fuchs type $\varphi_2(z^2 - 1) + \varphi_1 2z - \varphi\nu(\nu + 1) = f$, J. Coll. Engng. Nihon Univ., B-**29** (1988), 9–19.
[12] K. Nishimoto, Application of fractional calculus to a differential equation of Fuchs type $\varphi_2(z^2 - z) + \varphi_1(2\nu z - \nu + 1) + \varphi\nu(\nu - 1) = f$, J. Coll. Engng. Nihon Univ., B-**27** (1986), 17–30.
[13] Shih-Tong Tu, S. J. Jaw, and S. D. Lin, An application of fractional calculus to Chebyshev's equation, Chung Yuan J., Vol. XIX (1990), 1–4.
[14] K. Nishimoto, Shih-Tong Tu, and Jyi Feng Wu, An application of fractional calculus to a partial differential equation of the second order, J. Coll. Engng. Nihon Univ., B–**32** (1991), 55–62.

*Department of Mathematics*
*Chung Yuan Christian University*
*Chung Li, Taiwan* 320
*R.O.C.*

*Department of Mathematics*
*College of Engineering*
*Nihon University*
*Koriyama, Fukushima* 963, *Japan*

*Department of Mathematics*
*Chung Yuan Christian University*
*Chung Li, Taiwan* 320
*R.O. C.*
*and*
*Department of Mathematics*
*Chung Yuan Christian University*
*Chung Li, Taiwan* 320
*R.O. C.*