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Nonstationary flows of nonsymmetric fluids with thermal convection

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Introduction

The time behaviour of one-component isotropic isothermal fluid is usually described by a Navier-Stokes system (cf., [5], [6], [14]). In contrast to the classical point of view, the balance of angular momentum must be taken into account if the stress tensor is not assumed to be symmetric (cf., [3]). In this case the angular momentum can be decomposed into external and internal parts. The internal part represents the rotational motion of the fluid particles. If the fluid does not accompany any intrinsic motion, the stress tensor is symmetric (cf., [2]). However this situation is relevant for only fluids comprising spherical molecules or those characterized by very low mass density and, in general, the antisymmetric components cannot be neglected.

In this paper we consider the general case and treat the two-term representation of the total stress whose components stand for the scalar equilibrium stress as well as viscous stresses. This setting leads us to an equation of angular momentum balance of the form

$$(0.1)\quad \omega_t - (c_a + c_d)\Delta\omega - (c_0 + c_d - c_a)\nabla \operatorname{div}\omega + u\cdot\nabla\omega + 4\nu_r\omega = 2\nu_r \operatorname{curl} u + g(\theta),$$

where $\omega = (\omega_1, \omega_2, \omega_3)$ denotes the angular velocity vector of the constituent fluid particles, g represents the momentum density of exterior forces, positive constants ν_r, c_0, c_d, c_a are viscosity coefficients associated with the non-symmetricity of the stress tensor. In particular, the rotational viscous motion of the fluid may make both internal and external friction effects. These effects, together with the heat conduction and convection, yield variation of temperature θ. The energy is understood to the the sum of internal energy and kinetic energy. For the viscous flows under consideration, we employ the energy balance equation

$$(0.2)\qquad \theta_t - \kappa\Delta\theta + u\cdot\nabla\theta = \Phi(u, \omega) + h,$$

where the positive constant κ is the heat conductivity, h denotes the heat source and Φ is a dissipation function. The dissipation function can be written in the following form which is derived from the energy balance:

$$\Phi = \sum_{i=1}^{5} \Phi_i, \tag{0.3}$$

where

$$\Phi_1(u) = \frac{1}{2} \nu \sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2, \tag{0.4}$$

$$\Phi_2(u, \omega) = 4\nu_r \left(\frac{1}{2} \operatorname{curl} u - \omega \right)^2, \tag{0.5}$$

$$\Phi_3(\omega) = c_0 (\operatorname{div} \omega)^2, \tag{0.6}$$

$$\Phi_4(\omega) = (c_a + c_d) \sum_{i,j=1}^{3} \left(\frac{\partial \omega_i}{\partial x_j} \right)^2, \tag{0.7}$$

$$\Phi_5(\omega) = (c_d - c_a) \sum_{i,j=1}^{3} \frac{\partial \omega_i}{\partial x_j} \frac{\partial \omega_j}{\partial x_i}. \tag{0.8}$$

In the case that the stress tensor is nonsymmetric, the mechanical motion of the fluid is governed by the equation of linear momentum balance

$$u_t - (\nu + \nu_r) \Delta u + (u \cdot \nabla) u + \nabla p = 2\nu_r \operatorname{curl} \omega + f(\theta), \tag{0.9}$$

where u denotes the velocity vector, ν the usual kinematic Newtonian viscosity and f stands for the volumetric force. Mass density ρ of the fluid does not explicitly appear in equations (0.1)–(0.9). This is because ρ can be regarded as a constant function so far as the variation of θ remains small. We here assume that $\rho = \text{constant} = 1$. Then, it is natural to assume that the fluid is incompressible and the mass balance is formulated as

$$\operatorname{div} u = 0. \tag{0.10}$$

Equations (0.1), (0.2), (0.9) and (0.10) constitute a Navier-Stokes system describing the motion of viscous, incompressible, isotropic and heat convective fluids with nonsymmetric stress tensor (cf. [13]). Thus, an initial-boundary value problem can be formulated for the system (0.1), (0.2), (0.9) and (0.10) under the initial and boundary conditions

$$\begin{aligned} u|_{t=0} &= u_0, & u|_{\partial\Omega} &= 0, \\ \omega|_{t=0} &= \omega_0, & \omega|_{\partial\Omega} &= 0, \\ \theta|_{t=0} &= \theta_0, & \theta|_{\partial\Omega} &= 0, \end{aligned}$$

where Ω is a bounded domain in $\mathbf{R}^3$ with smooth boundary $\partial\Omega$. In what follows, the problem for (0.1), (0.2), (0.9) and (0.10) is simply called Problem (P).

Łukaszewicz [10] discussed the initial and boundary value problem for (0.1), (0.9) and (0.10) in the case where f and g do not depend upon θ. He gave a sufficient condition for the existence of global solutions of the problem (0.1), (0.9) and (0.10). If equations (0.1), (0.9) and (0.10) are coupled with temperature θ, the dissipation function Φ must be taken into account in equation (0.2). Łukaszewicz and Waluś [12] treated the stationary problem of (0.1), (0.2), (0.9) and (0.10) and proved the existence and uniqueness under the assumptions that f, g are bounded and ν, $c_a + c_d$ are sufficiently large.

Our objective here is to treat the nonstationary problem for the system (0.1), (0.2), (0.9) and (0.10) and discuss the existence and uniqueness of strong solutions of Problem (P). In the theory of weak solutions of the Navier-Stokes equations usual energy estimates are given in terms of L^2-estimates for u and ∇u. However, the dissipation function Φ contains quadratic nonlinearity of ∇u and $\nabla \omega$ as stated in (0.4) through (0.8), and so we also necessitate estimating the second order derivatives of u and ω. Therefore only strong solutions are considered in our argument. Using these a priori estimates, we show the local existence of strong solutions via the Banach fixed point argument. The Navier-Stokes equations in three space dimension admit only strong solutions in a local sense for general initial data. Our result corresponds to this well-known state. If we restrict ourselves to small initial data, we can obtain the global existence of strong solutions for the Navier-Stokes equations in three space dimension. In a way similar to this argument, we also discuss the global existence of strong solutions of Problem (P).

The present paper consists of six sections. In Section 1 we introduce the notion of strong solutions of Problem (P) and state our main results. In Section 2 we discuss the linearized problem for (0.9) and (0.10). In Sections 3 and 4 we treat equations (0.1) and (0.2) for ω and θ, respectively. In Section 5 we prove the existence theorem of strong solutions of Problem (P). Finally, in Section 6, we show the uniqueness of strong solutions.

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1. Main Results

We begin by introducing function spaces (functions from Ω into $\mathbf{R}^3$ and those from Ω into $\mathbf{R}$) and notation which are used in this paper. The usual L^p space is denoted by $L^p = L^p(\Omega)$ and the norm is denoted by $|\cdot|_p$. The closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_1 = |\nabla\cdot|_2$ is denoted by H_0^1 and the dual of the space H_0^1 is denoted by H^{-1}. The symbol $H^2 = H^2(\Omega)$

stands for the usual L^2 Sobolev space of order 2 with norm $\|\cdot\|_2$. We write H and V for the closures of $\{u \in C_0^\infty(\Omega)^3;\, \operatorname{div} u = 0\}$ in L^2 and H_0^1, respectively. Given a Banach space X with norm $\|\cdot\|_X$, we write $L^p(0, T; X)$ for the space of functions v on $(0, T)$ with values in X such that the real valued function $t \to \|v(t)\|_X$ belongs to $L^p(0, T)$. The space of continuous functions from $[0, T]$ into X is denoted by $C(0, T; X)$. We do not distinguish between the spaces of scalar functions and those of vector functions if there is no confusion. By $(.,.)$ and $((.,.))$ we mean the inner products on L^2 and H_0^1, respectively. The dual pairing between H_0^1 and H^{-1} is denoted by $\langle\cdot,\cdot\rangle$. Furthermore, we define the trilinear form

$$b(u, \omega, v) = (u \cdot \nabla \omega, v).$$

We often use the Gagliardo-Nirenberg inequality (cf. [4])

$$|u|_3 \leq C|\nabla u|_2^{\frac{1}{2}}|u|_2^{\frac{1}{2}}, \tag{1.1}$$

the Sobolev inequality (cf. [4])

$$|u|_6 \leq C|\nabla u|_2 \tag{1.2}$$

and Poincaré's inequality (cf. [1])

$$|\theta|_2 \leq d\,\|\theta\|_1, \tag{1.3}$$

where C means various positive constants depending only on Ω and d is the diameter of Ω.

On the functions f, g and h, we put the following conditions: there exist positive constants M_f and M_g such that

$$|f(\theta_1) - f(\theta_2)| \leq M_f|\theta_1 - \theta_2|, \quad |g(\theta_1) - g(\theta_2)| \leq M_g|\theta_1 - \theta_2| \tag{1.4}$$

for $\theta_1, \theta_2 \in \mathbf{R}$,

$$f(0) = g(0) = 0$$

and

$$h \in L^2(0, T; L^2).$$

Furthermore, we suppose that the initial values u_0, ω_0, θ_0 belong to the following function spaces

$$u_0 \in V, \quad \omega_0 \in H_0^1, \quad \theta_0 \in L^2.$$

We now introduce a notion of strong solution to Problem (P).

DEFINITION. A triple of functions (u, ω, θ) is called a *strong solution*, if it satisfies

$$(1.5)\qquad \begin{aligned} &u\in L^\infty(0, T; V)\cap L^2(0, T; H^2), && u_t\in L^2(0, T; H),\\ &\omega\in L^\infty(0, T; H_0^1)\cap L^2(0, T; H^2), && \omega_t\in L^2(0, T; L^2),\\ &\theta\in L^\infty(0, T; L^2)\cap L^2(0, T; H_0^1), && \theta_t\in L^2(0, T; H^{-1}), \end{aligned}$$

the initial conditions

$$(1.6)\qquad u(0)=u_0,\quad \omega(0)=\omega_0,\quad \theta(0)=\theta_0,$$

the identities

$$(1.7)\qquad \int_0^T (u_t-(\nu+\nu_r)\Delta u, \varphi)\,dt+\int_0^T b(u, u, \varphi)\,dt=\int_0^T (2\nu_r \operatorname{curl}\omega+f(\theta), \varphi)\,dt,$$

$$(1.8)\qquad \begin{aligned} &\int_0^T (\omega_t, \psi)\,dt-(c_a+c_d)\int_0^T (\Delta\omega, \psi)\,dt-(c_0+c_d-c_a)\int_0^T (\nabla \operatorname{div}\omega, \psi)\,dt\\ &\quad+\int_0^T b(u, \omega, \psi)\,dt+4\nu_r\int_0^T (\omega, \psi)\,dt=\int_0^T (2\nu_r \operatorname{curl} u+g(\theta), \psi)\,dt \end{aligned}$$

and

(1.9)

$$\int_0^T \langle\theta_t, \phi\rangle\,dt+\kappa\int_0^T (\nabla\theta, \nabla\phi)\,dt+\int_0^T (u\cdot\nabla\theta, \phi)\,dt=\int_0^T (\Phi(u, \omega)+h, \phi)\,dt$$

for $\varphi\in L^2(0, T; H)$, $\psi\in L^2(0, T; L^2)$ and $\phi\in L^2(0, T; H_0^1)$.

By condition (1.5) and standard interpolation theorems (cf. [8]) it is seen that $u\in C(0, T; V)$, $\omega\in C(0, T; H_0^1)$ and $\theta\in C(0\ T; L^2)$. Hence, condition (1.6) makes sense. We now state our main results.

THEOREM 1.1. (i) *For each triple of initial values* $(u_0, \omega_0, \theta_0)$ *there exist a positive number* T_* *and a unique strong solution* (u, ω, θ) *to* Problem (P) *on* $[0, T_*]$, *where* T_* *depends only on* $\nu, \nu_r, c_a, c_d, c_0, \kappa, M_f, M_g, \Omega, (u_0, \omega_0, \theta_0)$ *and* h.

(ii) *If* $\nu_r, M_f, M_g, (u_0, \omega_0)$ *and* h *are sufficiently small, the strong solution* (u, ω, θ) *exists on all of* $[0, \infty)$.

2. Existence and uniqueness for the linear momentum equation

In this section we deal with the following problem which is a linearized version of (0.9) subject to (0.10).

PROBLEM 1. *Given* $u_0\in V$ *and* $F\in L^2(0, T; L^2)$, *find* u *satisfying*

$$u \in C(0, T; V) \cap L^2(0, T; H^2), \quad u_t \in L^2(0, T; H), \quad u(0) = u_0$$

and

$$(2.1) \qquad \int_0^T (u_t - (v + v_r)\Delta u, \varphi)\,dt = \int_0^T (F, \varphi)\,dt \qquad \textit{for } \varphi \in L^2(0, T; H).$$

For this initial-boundary value problem we use the following existence and uniqueness results (cf. [10]).

PROPOSITION 2.1. *Problem 1 admits a unique solution satisfying*

$$(2.2) \qquad \|u\|^2_{C(0,T;V)} + (v + v_r)\|Au\|^2_{L^2(0,T;L^2)} \le \|u_0\|_1^2 + \frac{1}{(v + v_r)}\|F\|^2_{L^2(0,T;L^2)},$$

where the operator A is defined on $(H^2)^3 \cap V$ by $A = -P \circ \Delta$ and P is the orthogonal projection of L^2 onto H.

PROOF. The existence and uniqueness results for Problem 1 are obtained directly by applying the Faedo-Galerkin approximations and energy estimates for linear parabolic equations (cf., [7], [15]). We then show a priori estimates (2.2) (cf., [10]), which are derived through the approximation argument.

Let u be a solution to Problem 1. By (2.1) we have the identity

$$(2.3) \qquad (u_t - (v + v_r)\Delta u, \varphi) = (F, \varphi)$$

for all $\varphi \in H$ in the sense of distributions on $(0, T)$. To show (2.2), we put $\varphi = Au(t)$ as test function in (2.3). Then the identity

$$(2.4) \qquad (u_t - (v + v_r)\Delta u, Au) = (F, Au)$$

holds in the sense of distributions on $(0, T)$. Since

$$(u_t, Au) = \frac{1}{2}\frac{d}{dt}\|u\|_1^2 \quad \text{and} \quad -(v + v_r)(\Delta u, Au) = (v + v_r)|Au|_2^2$$

for all $u \in H^2 \cap V$, we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_1^2 + (v + v_r)|Au|_2^2 \le \frac{1}{2(v + v_r)}|F|_2^2 + \frac{(v + v_r)}{2}|Au|_2^2,$$

and, consequently,

$$(2.5) \qquad \frac{d}{dt}\|u\|_1^2 + (v + v_r)|Au|_2^2 \le \frac{1}{(v + v_r)}|F|_2^2.$$

The desired estimate (2.2) follows immediately from (2.5). This completes the proof.

3. Existence and uniqueness for the angular momentum equation

In this section we study the initial-boundary value problem for ω. For this purpose, we introduce the operator L:

$$L\omega = -(c_a + c_d)\Delta\omega - (c_0 + c_d - c_a)\nabla \operatorname{div}\omega \qquad \text{for } \omega \in (H^2)^3 \cap (H_0^1)^3.$$

The operator L is strongly elliptic and satisfies

$$|L^{1/2}\omega|_2^2 = (L\omega, \omega) \geq \tilde{c}|\nabla\omega|_2^2$$

and

$$C^{-1}|L\omega|_2 \leq \|\omega\|_2 \leq C|L\omega|_2,$$

where $\tilde{c} = \min(c_a + c_d, c_0 + 2c_d)$ and C is a positive constant depending only on c_0, c_a, c_d and Ω.

We now pose the following problem for the angular momentum equation.

Problem 2. *Given $\omega_0 \in H_0^1$, $\bar{u} \in C(0, T; V) \cap L^2(0, T; H^2)$, $\bar{\omega} \in C(0, T; H_0^1) \cap L^2(0, T; H^2)$ and $\bar{\theta} \in C(0, T; L^2) \cap L^2(0, T; H_0^1)$, find ω satisfying $\omega \in C(0, T; H_0^1) \cap L^2(0, T; H^2)$, $\omega_t \in L^2(0, T; L^2)$, $\omega(0) = \omega_0$,*

$$\begin{aligned}(3.1)\qquad &\int_0^T (\omega_t, \psi)\,dt + \int_0^T (L\omega, \psi)\,dt + 4\nu_r \int_0^T (\omega, \psi)\,dt \\ &= \int_0^T (2\nu_r \operatorname{curl}\bar{u} + g(\bar{\theta}), \psi)\,dt - \int_0^T b(\bar{u}, \bar{\omega}, \psi)\,dt\end{aligned}$$

for all $\psi \in L^2(0, T; L^2)$.

Proposition 3.1. *There exists a unique solution ω of* Problem 2. *This solution satisfies*

$$\begin{aligned}(3.2)\qquad &\|L^{1/2}\omega\|_{C(0,T;L^2)}^2 + \|L\omega\|_{L^2(0,T;L^2)}^2 \\ &\leq |L^{1/2}\omega_0|_2^2 + C(T\|\bar{u}\|_{C(0,T;V)}^2 + T\|\bar{\theta}\|_{C(0,T;L^2)}^2 \\ &\quad + T^{1/2}\|\bar{u}\|_{C(0,T;V)}^2\|L^{1/2}\bar{\omega}\|_{C(0,T;L^2)}\|L\bar{\omega}\|_{L^2(0,T;L^2)}),\end{aligned}$$

where $C > 0$ is a constant depending only on ν_r, c_0, c_a, c_d, M_g and Ω.

Proof. The existence and uniqueness results for Problem 2 follow directly from the general theory of linear parabolic equations (cf., [7], [9]). Therefore, it remains to show the estimate (3.2).

Let ω be a solution to Problem 2. By (3.1),

$$(3.3)\qquad (\omega_t, v) + (L\omega, v) + 4\nu_r(\omega, v) = (2\nu_r \operatorname{curl}\bar{u} + g(\bar{\theta}), v) - b(\bar{u}, \bar{\omega}, v)$$

for all $v \in L^2$ in the sense of distributions on $(0, T)$. To show (3.2), we take $v = L\omega$ in (3.3) as test function. Then,

(3.4)
$$\frac{1}{2}\frac{d}{dt}|L^{1/2}\omega|_2^2 + |L\omega|_2^2 \le 2\nu_r|(\mathrm{curl}\,\bar{u}, L\omega)| + |(g(\bar{\theta}), L\omega)| + |b(\bar{u}, \bar{\omega}, L\omega)|.$$

We estimate the right-hand side of (3.4) term by term: The first term is estimated as

$$|(2\nu_r \mathrm{curl}\,\bar{u}, L\omega)| \le C\nu_r^2\|\bar{u}\|_1^2 + \frac{1}{6}|L\omega|_2^2,$$

the second term can be estimated as

$$|(g(\bar{\theta}), L\omega)| \le C|g(\bar{\theta})|_2^2 + \frac{1}{6}|L\omega|_2^2 \le CM_g^2|\bar{\theta}|_2^2 + \frac{1}{6}|L\omega|_2^2$$

and the Gagliardo-Nirenberg inequality (1.1) implies

$$\begin{aligned}|b(\bar{u}, \bar{\omega}, L\omega)| &\le |\bar{u}|_6|\nabla\bar{\omega}|_3|L\omega|_2 \le C\|\bar{u}\|_1|L^{1/2}\bar{\omega}|_2^{1/2}|L\bar{\omega}|_2^{1/2}|L\omega|_2 \\ &\le \frac{1}{6}|L\omega|_2^2 + C\|\bar{u}\|_1^2|L^{1/2}\bar{\omega}|_2|L\bar{\omega}|_2.\end{aligned}$$

We thus obtain

$$\frac{d}{dt}\|\omega\|_1^2 + |L\omega|_2^2 \le C(\|\bar{u}\|_1^2 + |\bar{\theta}|_2^2 + \|\bar{u}\|_1^2|L^{1/2}\bar{\omega}|_2|L\bar{\omega}|_2),$$

and (3.2) follows. The proof is complete.

4. Existence and uniqueness for the energy balance equation

In this section we consider the following initial-boundary value problem for θ.

PROBLEM 3. *Given $\bar{u} \in C(0, T; V) \cap L^2(0, T; H^2)$, $\bar{\omega} \in C(0, T; H_0^1) \cap L^2(0, T; H^2)$, $\bar{\theta} \in C(0, T; L^2) \cap L^2(0, T; H_0^1)$, $\theta_0 \in L^2$ and $h \in L^2(0, T; L^2)$, find θ such that*

$$\theta \in C(0, T; L^2) \cap L^2(0, T; H_0^1), \quad \theta_t \in L^2(0, T; H^{-1}), \quad \theta(0) = \theta_0,$$

(4.1)
$$\begin{aligned}&\int_0^T \langle\theta_t, \phi\rangle\, dt + \kappa\int_0^T (\nabla\theta, \nabla\phi)\, dt \\ &= \int_0^T (\Phi(\bar{u}, \bar{\omega}), \phi)\, dt - \int_0^T (\bar{u}\cdot\nabla\bar{\theta}, \phi)\, dt + \int_0^T (h, \phi)\, dt\end{aligned}$$

for all $\phi \in L^2(0, T; H_0^1)$.

PROPOSITION 4.1. Problem 3 *admits a unique solution* θ. *This solution satisfies*

$$
\begin{aligned}
&\|\theta\|^2_{C(0,T;L^2)} + \kappa\|\theta\|^2_{L^2(0,T;H_0^1)} \\
&\le |\theta_0|_2^2 + \frac{6d^2}{\kappa}\|h\|^2_{L^2(0,T;L^2)} \\
&+ C(T^{1/2}\|\bar{u}\|^3_{C(0,T;V)}\|A\bar{u}\|_{L^2(0,T;L^2)} + T\|L^{1/2}\bar{\omega}\|^4_{C(0,T;L^2)} \\
&+ T^{1/2}\|L^{1/2}\bar{\omega}\|^3_{C(0,T;L^2)}\|L\bar{\omega}\|_{L^2(0,T;L^2)} \\
&+ T^{1/2}\|\bar{u}\|^2_{C(0,T;V)}\|\bar{\theta}\|_{C(0,T;L^2)}\|\bar{\theta}\|_{L^2(0,T;H_0^1)}),
\end{aligned} \tag{4.2}
$$

where C is a positive constant depending only on ν, ν_r, c_0, c_a, c_d, κ and Ω.
We first show the following

LEMMA 4.2. *Let* ε *be a positive number. Then the inequality*

$$|(\Phi(u, \omega), \varphi)| \le \varepsilon\|\varphi\|_1^2 + C_\varepsilon(\nu^2+\nu_r^2)\|u\|_1^3|Au|_2 + C'_\varepsilon(\nu_r^2|L^{1/2}\omega|_2^4 + |L^{1/2}\omega|_2^3|L\omega|_2)$$

holds for $u \in (H^2)^3 \cap V$, $\omega \in (H^2)^3 \cap (H_0^1)^3$ *and* $\varphi \in H_0^1$. *Here* C_ε *is a positive constant depending only on* ε *and* Ω, *and* C'_ε *is a positive constant depending only on* ε, c_0, c_a, c_d *and* Ω.

PROOF. Recall that

$$\Phi(u, \omega) = \sum_{i=1}^{5} \Phi_i$$

with

$$
\begin{aligned}
\Phi_1(u) &= \frac{1}{2}\nu\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2, \\
\Phi_2(u, \omega) &= 4\nu_r\left(\frac{1}{2}\operatorname{curl} u - \omega\right)^2, \\
\Phi_3(\omega) &= c_0(\operatorname{div}\omega)^2, \\
\Phi_4(\omega) &= (c_a + c_d)\sum_{i,j=1}^{3}\left(\frac{\partial \omega_i}{\partial x_j}\right)^2, \\
\Phi_5(\omega) &= (c_d - c_a)\sum_{i,j=1}^{3}\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_j}{\partial x_i}.
\end{aligned}
$$

Thus,

$$|(\Phi(u, \omega), \varphi)| \leq \sum_{i=1}^{5} |(\Phi_i(u, \omega), \varphi)|, \tag{4.3}$$

and so it suffices to estimate each term of the right-hand side of (4.3). Since the inequality

$$\|v\|_2 \leq C|Av|_2$$

holds for all $v \in (H^2)^3 \cap V$ with $C > 0$ depending only on Ω, we apply the Sobolev inequality (1.2) and the Gagliardo-Nirenberg inequality (1.1) to obtain the estimate

$$\begin{aligned}
|(\Phi_1(u), \varphi)| &\leq \nu \int |\nabla u|^2 |\varphi| \, dx \\
&\leq \nu \|u\|_1 |\nabla u|_3 |\varphi|_6 \\
&\leq C\nu \|u\|_1^{3/2} \|\varphi\|_1 |Au|_2^{1/2} \\
&\leq \frac{\varepsilon}{5} \|\varphi\|_1^2 + C_\varepsilon \nu^2 \|u\|_1^3 |Au|_2
\end{aligned}$$

with $C_\varepsilon = C_\varepsilon(\varepsilon, \Omega)$. We also have

$$\begin{aligned}
|(\Phi_2(u, \omega), \varphi)| &\leq \nu_r \int (|\nabla u|^2 |\varphi| + |\omega|^2 |\varphi|) \, dx \\
&\leq \frac{\varepsilon}{10} \|\varphi\|_1^2 + C_\varepsilon \nu_r^2 \|u\|_1^3 |Au|_2 + \nu_r |\omega|_2 |\omega|_3 |\varphi|_6 \\
&\leq \frac{\varepsilon}{5} \|\varphi\|_1^2 + C_\varepsilon \nu_r^2 \|u\|_1^3 |Au|_2 + C'_\varepsilon \nu_r^2 d^3 |L^{1/2}\omega|_2^4
\end{aligned}$$

with $C_\varepsilon = C_\varepsilon(\varepsilon, \Omega)$ and $C'_\varepsilon = C'_\varepsilon(\varepsilon, c_0, c_a, c_d, \Omega)$. Here we have used the Gagliardo-Nirenberg inequality (1.1) and the Poincaré inequality (1.3). Similarly, the other terms can be estimated as

$$|(\Phi_3(\omega), \theta)| \leq \frac{\varepsilon}{5} \|\varphi\|_1^2 + C'_\varepsilon |L^{1/2}\omega|_2^3 |L\omega|_2,$$

$$|(\Phi_4(\omega), \theta)| \leq \frac{\varepsilon}{5} \|\varphi\|_1^2 + C'_\varepsilon |L^{1/2}\omega|_2^3 |L\omega|_2,$$

$$|(\Phi_5(\omega), \theta)| \leq \frac{\varepsilon}{5} \|\varphi\|_1^2 + C'_\varepsilon |L^{1/2}\omega|_2^3 |L\omega|_2$$

with $C'_\varepsilon = C'_\varepsilon(\varepsilon, c_0, c_a, c_d, \Omega)$, and hence Lemma 4.2 immediately follows. This completes the proof.

PROOF OF PROPOSITION 4.1. The existence and uniqueness for Problem 3 follow directly from the general theory of linear parabolic equations (cf., [7], [9]). It remains to show the estimate (4.2).

Let θ be a solution to Problem 3. For all $v \in H_0^1$, (4.1) implies that

$$\langle \theta_t, v \rangle + \kappa(\nabla \theta, \nabla v) = (\Phi(\bar{u}, \bar{\omega}) + h, v) - (\bar{u} \cdot \nabla \bar{\theta}, v) \tag{4.4}$$

in the sense of distributions on $(0, T)$. Take $v = \theta(t)$ in (4.4) to get

$$\langle \theta_t, \theta \rangle + \kappa(\nabla \theta, \nabla \theta) = (\Phi(\bar{u}, \bar{\omega}) + h, \theta) - (\bar{u} \cdot \nabla \bar{\theta}, \theta).$$

Taking $\varepsilon = \kappa/6$ and $\varphi = \theta$ in Lemma 4.2, we then have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\theta|_2^2 + \kappa\|\theta\|_1^2 &\le |(\Phi(\bar{u}, \bar{\omega}), \theta)| + |(h, \theta)| + |(\bar{u} \cdot \nabla \bar{\theta}, \theta)| \\ &\le \frac{\kappa}{3}\|\theta\|_1^2 + \frac{3d^2}{\kappa}|h|_2^2 + |(\bar{u} \cdot \nabla \bar{\theta}, \theta)| \\ &\quad + C(\|\bar{u}\|_1^3 |A\bar{u}|_2 + |L^{1/2}\bar{\omega}|_2^4 + |L^{1/2}\bar{\omega}|_2^3 |L\bar{\omega}|_2).\end{aligned}$$

Since

$$\begin{aligned}|(\bar{u} \cdot \nabla \bar{\theta}, \theta)| = |(\bar{u} \cdot \nabla \theta, \bar{\theta})| &\le |\bar{u}|_6 \|\theta\|_1 |\bar{\theta}|_3 \le C\|\bar{u}\|_1 \|\theta\|_1 \|\bar{\theta}\|_1^{1/2} |\bar{\theta}|_2^{1/2} \\ &\le \frac{\kappa}{6}\|\theta\|_1^2 + C\|\bar{u}\|_1^2 \|\bar{\theta}\|_1 |\bar{\theta}|_2,\end{aligned}$$

we have

$$\begin{aligned}\frac{d}{dt}|\theta|_2^2 + \kappa\|\theta\|_1^2 &\le \frac{6d^2}{\kappa}|h|_2^2 \\ &\quad + C(\|\bar{u}\|_1^3 |A\bar{u}|_2 + |L^{1/2}\bar{\omega}|_2^4 + |L^{1/2}\bar{\omega}|_2^3 |L\bar{\omega}|_2 + \|\bar{u}\|_1^2 \|\bar{\theta}\|_1 |\bar{\theta}|_2).\end{aligned} \tag{4.5}$$

Inequality (4.2) immediately follows from (4.5). This completes the proof.

5. Proof of Theorem 1.1

We here prove Theorem 1.1 via the Banach fixed point argument.

PROOF OF THEOREM 1.1. Let $u_0 \in V$, $\omega_0 \in H_0^1$ and $\theta_0 \in L^2$. For $M > 0$ and $T > 0$, we denote $U = (u, \omega, \theta) \in \mathscr{L}(M, T)$ if $U = (u, \omega, \theta)$ has the following properties:

(i)
$$u \in C(0, T; V) \cap L^2(0, T; H^2), \quad \omega \in C(0, T; H_0^1) \cap L^2(0, T; H^2),$$
$$\theta \in C(0, T; L^2) \cap L^2(0, T; H_0^1);$$

$$\text{(ii)} \qquad \begin{aligned} |||U|||^2 \equiv{} & \|u\|^2_{C(0,T;V)} + \|L^{1/2}\omega\|^2_{C(0,T;L^2)} + \|\theta\|^2_{C(0,T;L^2)} \\ & + (\nu+\nu_r)\|Au\|^2_{L^2(0,T;L^2)} + \|L\omega\|^2_{L^2(0,T;L^2)} + \kappa\|\theta\|^2_{L^2(0,T;H_0^1)} \le M; \end{aligned}$$

$$\text{(iii)} \qquad u(0) = u_0, \quad \omega(0) = \omega_0, \quad \theta(0) = \theta_0.$$

We fix $M > 0$ so that

$$\|u_0\|_1^2 + |L^{1/2}\omega_0|_2^2 + |\theta_0|_2^2 + \frac{6d^2}{\kappa}\|h\|^2_{L^2(0,T;L^2)} \le M/2.$$

For each $\bar{U} = (\bar{u}, \bar{\omega}, \bar{\theta}) \in \mathscr{L}(M, T)$, we define a map Γ by $\Gamma(\bar{U}) = U$, where $U = (u, \omega, \theta)$ satisfies the system

$$u(0) = u_0, \quad \omega(0) = \omega_0, \quad \theta(0) = \theta_0,$$

$$\int_0^T (u_t - (\nu+\nu_r)\Delta u, \varphi)\,dt = \int_0^T (2\nu_r \operatorname{curl}\bar{\omega} + f(\bar{\theta}) - \bar{u}\cdot\nabla\bar{u}, \varphi)\,dt,$$

$$\begin{aligned} & \int_0^T (\omega_t, \psi)\,dt + \int_0^T (L\omega, \psi)\,dt + 4\nu_r\int_0^T (\omega, \psi)\,dt \\ & \quad = \int_0^T (2\nu_r \operatorname{curl}\bar{u} + g(\bar{\theta}), \psi)\,dt - \int_0^T b(\bar{u}, \bar{\omega}, \psi)\,dt \end{aligned}$$

and

$$\int_0^T (\theta_t, \phi)\,dt + \kappa\int_0^T (\nabla\theta, \nabla\phi)\,dt = \int_0^T (\Phi(\bar{u}, \bar{\omega}) + h, \phi)\,dt - \int_0^T (\bar{u}\cdot\nabla\bar{\theta}, \phi)\,dt$$

for all $\varphi \in L^2(0, T; H)$, $\psi \in L^2(0, T; L^2)$ and all $\phi \in L^2(0, T; H_0^1)$. As it follows from Propositions 2.1, 3.1 and 4.1 for each $\bar{U} = (\bar{u}, \bar{\omega}, \bar{\theta})$, there exists exactly one triple $U = (u, \omega, \theta)$ such that $u \in C(0, T; V) \cap L^2(0, T; H^2)$, $\omega \in C(0, T; H_0^1) \cap L^2(0, T; H^2)$, $\theta \in C(0, T; L^2) \cap L^2(0, T; H_0^1)$ and U satisfies the above-mentioned system.

Let $F = 2\nu_r \operatorname{curl}\bar{\omega} + f(\bar{\theta}) - \bar{u}\cdot\nabla\bar{u}$. Then we have the estimate (cf., [10])

$$\begin{aligned} |F|_2^2 & \le 12\nu_r^2\|\bar{\omega}\|_1^2 + 3|f(\bar{\theta})|_2^2 + C\|\bar{u}\|_1^3|A\bar{u}|_2 \\ & \le C(|L^{1/2}\bar{\omega}|_2^2 + |\bar{\theta}|_2^2 + \|\bar{u}\|_1^3|A\bar{u}|_2). \end{aligned}$$

This together with (2.2) yields

$$\text{(5.1)} \qquad \begin{aligned} & \|u\|^2_{C(0,T;V)} + (\nu+\nu_r)\|Au\|^2_{L^2(0,T;L^2)} \\ & \le \|u_0\|_1^2 + C(T\|L^{1/2}\bar{\omega}\|^2_{C(0,T;L^2)} + T\|\bar{\theta}\|^2_{C(0,T;L^2)} \\ & \quad + T^{1/2}\|\bar{u}\|^3_{C(0,T;V)}\|\bar{u}\|_{L^2(0,T;H^2)}). \end{aligned}$$

It follows from (3.2), (4.2) and (5.1) that if $\bar{U} \in \mathscr{L}(M, T)$, then the inequality

$$|||U|||^2 \le \frac{M}{2} + C(M^2 T^{1/2} + MT + M^2 T)$$

holds for some constant $C > 0$ depending only on ν, ν_r, c_0, c_a, c_d, κ and Ω. Thus, there exists a small $T_1 > 0$ such that Γ maps $\mathscr{L}(M, T_1)$ into $\mathscr{L}(M, T_1)$. We next show that Γ is a contraction map on $\mathscr{L}(M, T_2)$ for sufficiently small $T_2 > 0$. For this purpose, we consider $\bar{U}_i = (\bar{u}_i, \bar{\omega}_i, \bar{\theta}_i) \in \mathscr{L}(M, T)$, $i = 1, 2$. Set $U_i = \Gamma(\bar{U}_i)$, $\bar{U} = \bar{U}_1 - \bar{U}_2$ and $U = U_1 - U_2$. Then, by the same argument as in Sections 2–4 (cf., Section 6), we obtain

$$\begin{aligned}
&\|u\|^2_{C(0,T;V)} + (\nu + \nu_r)\|Au\|^2_{L^2(0,T;L^2)} \\
&\le C(T\|L^{1/2}\bar{\omega}\|^2_{C(0,T;L^2)} + T\|\bar{\theta}\|^2_{C(0,T;L^2)} \\
&\quad + T^{1/2}\|\bar{u}\|^2_{C(0,T;V)}\|A\bar{u}_1\|_{L^2(0,T;L^2)}\|\bar{u}_1\|_{C(0,T;V)} \\
&\quad + T^{1/2}\|\bar{u}_2\|^2_{C(0,T;V)}\|A\bar{u}\|_{L^2(0,T;L^2)}\|\bar{u}\|_{C(0,T;V)}),
\end{aligned}$$

$$\begin{aligned}
&\|L^{1/2}\omega\|^2_{C(0,T;L^2)} + \|L\omega\|^2_{L^2(0,T;L^2)} \\
&\le C(T\|\bar{u}\|^2_{C(0,T;V)} + T\|\bar{\theta}\|^2_{C(0,T;L^2)} \\
&\quad + T^{1/2}\|\bar{u}\|^2_{C(0,T;V)}\|L\bar{\omega}_1\|_{L^2(0,T;L^2)}\|L^{1/2}\bar{\omega}_1\|_{C(0,T;L^2)} \\
&\quad + T^{1/2}\|\bar{u}_2\|^2_{C(0,T;V)}\|L\bar{\omega}\|_{L^2(0,T;L^2)}\|L^{1/2}\bar{\omega}\|_{C(0,T;L^2)})
\end{aligned}$$

and

$$\begin{aligned}
&\|\theta\|^2_{C(0,T;L^2)} + \kappa\|\theta\|^2_{L^2(0,T;H^1_0)} \\
&\le C(T^{1/2}\|\bar{u}\|^2_{C(0,T;V)}\|\bar{\theta}_1\|_{L^2(0,T;H^1_0)}\|\bar{\theta}_1\|_{C(0,T;L^2)} \\
&\quad + T^{1/2}\|\bar{u}_2\|^2_{C(0,T;V)}\|\bar{\theta}\|_{L^2(0,T;H^1_0)}\|\bar{\theta}\|_{C(0,T;L^2)} \\
&\quad + T^{1/2}\|\bar{u}\|^2_{C(0,T;V)}\|\bar{u}_1 + \bar{u}_2\|_{C(0,T;V)}\|A(\bar{u}_1 + \bar{u}_2)\|_{L^2(0,T;L^2)} \\
&\quad + T\|L^{1/2}\bar{\omega}\|^2_{C(0,T;L^2)}\|L^{1/2}(\bar{\omega}_1 + \bar{\omega}_2)\|^2_{C(0,T;L^2)} \\
&\quad + T^{1/2}\|L^{1/2}\bar{\omega}\|^2_{C(0,T;L^2)}\|L^{1/2}(\bar{\omega}_1 + \bar{\omega}_2)\|_{C(0,T;L^2)}\|L(\bar{\omega}_1 + \bar{\omega}_2)\|_{L^2(0,T;L^2)}).
\end{aligned}$$

It follows that if $\bar{U}_i \in \mathscr{L}(M, T)$, then

$$|||U|||^2 \le C(T + MT^{1/2} + MT)|||\bar{U}|||^2.$$

Hence, Γ is a contraction map on $\mathscr{L}(M, T_2)$ for sufficiently small $T_2 > 0$. Therefore, if $T_* = \min(T_1, T_2)$, the Banach fixed point theorem implies that mapping Γ has a fixed point U in $\mathscr{L}(M, T_*)$ which solves our problem (P). The proof of (i) is complete.

We now prove the assertion (ii). In view of the proof of (i), it suffices to give a priori bounds for (u, ω, θ) which satisfies the system

$$u_t + (\nu + \nu_r)Au + Pu \cdot \nabla u = P(2\nu_r \operatorname{curl}\omega + f(\theta)), \tag{5.2}$$

$$\omega_t + L\omega + u \cdot \nabla\omega + 4\nu_r\omega = 2\nu_r \operatorname{curl} u + g(\theta), \tag{5.3}$$

$$\theta_t - \kappa\Delta\theta + u \cdot \nabla\theta = \Phi(u, \omega) + h, \tag{5.4}$$

$$u(0) = u_0,\ \omega(0) = \omega_0,\ \theta(0) = \theta_0.$$

Taking the scalar product of (5.2) with Au, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_1^2 + (\nu + \nu_r)|Au|_2^2 = -(u \cdot \nabla u, Au) + 2\nu_r(\operatorname{curl}\omega, Au) + (f(\theta), Au). \tag{5.5}$$

By the Sobolev inequality (1.2) and the Gagliardo-Nirenberg inequality (1.1), we can estimate the right-hand side of (5.5) from above by

$$\begin{aligned}
&\le C\|u\|_1^{3/2}\|u\|_2^{1/2}|Au|_2 + 2\nu_r\|\omega\|_1|Au|_2 + |f(\theta)|_2|Au|_2 \\
&\le C\|u\|_1^{3/2}|Au|_2^{3/2} + 2\nu_r\|\omega\|_1|Au|_2 + M_f|\theta|_2|Au|_2 \\
&\le \frac{\nu + \nu_r}{4}|Au|_2^2 + C\left[\frac{\|u\|_1^6}{(\nu + \nu_r)^3} + \frac{\nu_r^2}{\nu + \nu_r}|L^{1/2}\omega|_2^2 + \frac{M_f^2 d^2}{\nu + \nu_r}\|\theta\|_1^2\right]
\end{aligned}$$

where C is a positive constant depending only on c_0, c_a, c_d and Ω. Thus, we have

$$\frac{d}{dt}\|u\|_1^2 + \frac{3(\nu + \nu_r)}{2}|Au|_2^2 \le C\left[\frac{\|u\|_1^6}{(\nu + \nu_r)^3} + \frac{\nu_r^2}{\nu + \nu_r}|L^{1/2}\omega|_2^2 + \frac{M_f^2 d^2}{\nu + \nu_r}\|\theta\|_1^2\right]$$

with $C > 0$ depending only on c_0, c_a, c_d and Ω, and so

$$\begin{aligned}
&\|u(t)\|_1^2 + \frac{3(\nu + \nu_r)}{2}\int_0^t |Au|_2^2\, ds \\
&\le \|u_0\|_1^2 + C_1\int_0^t\left(\frac{\|u\|_1^6}{(\nu + \nu_r)^3} + \frac{\nu_r^2}{\nu + \nu_r}|L^{1/2}\omega|_2^2 + \frac{M_f^2}{\nu + \nu_r}\|\theta\|_1^2\right)ds
\end{aligned} \tag{5.6}$$

with $C_1 > 0$ depending only on c_0, c_a, c_d and Ω. Similarly, we obtain

$$\begin{aligned}
&|L^{1/2}\omega(t)|_2^2 + \frac{3}{2}\int_0^t |L\omega|_2^2\, ds \\
&\le |L^{1/2}\omega_0|_2^2 + C_2\int_0^t (\|u\|_1^6 + |L^{1/2}\omega|_2^6 + \nu_r^2\|u\|_1^2 + M_g^2\|\theta\|_1^2)\, ds
\end{aligned} \tag{5.7}$$

with $C_2 > 0$ depending only on c_0, c_a, c_d and Ω. Taking the scalar product of (5.4) with θ, we have

$$\frac{1}{2}\frac{d}{dt}|\theta|_2^2 + \kappa\|\theta\|_1^2 = (\Phi(u, \omega), \theta) + (h, \theta).$$

It is easy to see from Lemma 4.2 that for any $\varepsilon > 0$ and any $\eta > 0$, the inequality

(5.8)
$$
\begin{aligned}
&|(\Phi(u, \omega), \theta)| \\
&\le \frac{\kappa}{8}\|\theta\|_1^2 + \frac{\varepsilon}{2}|Au|_2^2 + \frac{\eta}{2}|L\omega|_2^2 + C\left(\frac{\nu^4 + \nu_r^4}{\varepsilon\kappa^2}\|u\|_1^6 + \frac{1}{\eta\kappa^2}|L^{1/2}\omega|_2^6 + \frac{\nu_r^2}{\kappa}|L^{1/2}\omega|_2^4\right)
\end{aligned}
$$

holds for some constant $C > 0$ depending only on c_0, c_a, c_d and Ω. On the other hand,

$$
|(h, \theta)| \le |h|_2|\theta|_2 \le d|h|_2\|\theta\|_1 \le \frac{\kappa}{8}\|\theta\|_1^2 + \frac{2d^2}{\kappa}|h|_2^2.
$$

This together with (5.8) implies that

$$
\begin{aligned}
\frac{d}{dt}|\theta|_2^2 + \frac{3\kappa}{2}\|\theta\|_1^2 &\le \varepsilon|Au|_2^2 + \eta|L\omega|_2^2 + \frac{4d^2}{\kappa}|h|_2^2 \\
&\quad + C_3\left(\frac{\nu^4 + \nu_r^4}{\varepsilon\kappa^2}\|u\|_1^6 + \frac{1}{\eta\kappa^2}|L^{1/2}\omega|_2^6 + \frac{\nu_r^2}{\kappa}|L^{1/2}\omega|_2^4\right),
\end{aligned}
$$

where C_3 is a positive constant depending only on c_0, c_a, c_d and Ω. From this it follows that

(5.9)
$$
\begin{aligned}
|\theta(t)|_2^2 + \frac{3\kappa}{2}\int_0^t \|\theta\|_1^2\,ds &\le |\theta_0|_2^2 + \varepsilon\int_0^t |Au|_2^2\,ds + \eta\int_0^t |L\omega|_2^2\,ds + \frac{4d^2}{\kappa}\int_0^t |h|_2^2\,ds \\
&\quad + C_3\int_0^t\left(\frac{\nu^4+\nu_r^4}{\varepsilon\kappa^2}\|u\|_1^6 + \frac{1}{\eta\kappa^2}|L^{1/2}\omega|_2^6 + \frac{\nu_r^2}{\kappa}|L^{1/2}\omega|_2^4\right)ds.
\end{aligned}
$$

We take $\varepsilon > 0$ and $\eta > 0$ so that

$$
\frac{2\varepsilon}{3\kappa}\left(\frac{C_1 M_f^2}{\nu + \nu_r} + C_2 M_g^2\right) = \frac{\nu + \nu_r}{4}, \quad \frac{2\eta}{3\kappa}\left(\frac{C_1 M_f^2}{\nu + \nu_r} + C_2 M_g^2\right) = \frac{1}{4}.
$$

Then (5.6), (5.7) and (5.9) together imply

$$
\begin{aligned}
&\|u(t)\|_1^2 + |L^{1/2}\omega(t)|_2^2 + \frac{5(\nu + \nu_r)}{4}|Au|_2^2 + \frac{5(c_a + c_d)}{4}|L\omega|_2^2 \\
&\le \|u_0\|_1^2 + |L^{1/2}\omega_0|_2^2 + \frac{C_4}{\kappa}\left(\frac{M_f^2}{\nu + \nu_r} + M_g^2\right)|\theta_0|_2^2
\end{aligned}
$$

$$
(5.10)\qquad
\begin{aligned}
&+\frac{C_4}{\kappa^2}\left(\frac{M_f^2}{\nu+\nu_r}+M_g^2\right)\int_0^t |h|_2^2\,ds\\
&+\rho_1\int_0^t \|u\|_1^6\,ds+\gamma_1\int_0^t |L^{1/2}\omega|_2^6\,ds+\gamma_2\int_0^t |L^{1/2}\omega|_2^4\,ds\\
&+C_4\int_0^t\left(\nu_r^2\|u\|_1^2+\frac{\nu_r^2}{\nu+\nu_r}|L^{1/2}\omega|_2^2\right)ds,
\end{aligned}
$$

where the constants $\rho_1, \gamma_1, \gamma_2$ are defined by

$$
\begin{aligned}
\rho_1 &= C_4\left(1+\frac{1}{(\nu+\nu_r)^3}+\frac{(\nu+\nu_r)^3}{\kappa^4}\left(\frac{M_f^2}{\nu+\nu_r}+M_g^2\right)^2\right),\\
\gamma_1 &= C_4\left(1+\frac{1}{\kappa^4}\left(\frac{M_f^2}{\nu+\nu_r}+M_g^2\right)^2\right),\\
\gamma_2 &= \frac{C_4\nu_r^2}{\kappa^2}\left(\frac{M_f^2}{\nu+\nu_r}+M_g^2\right),
\end{aligned}
$$

and $C_4>0$ is a positive constant dpending only on c_0, c_a, c_d and Ω. Since there exist positive constants $c_1' = c_1'(\Omega)$ and $c_2' = c_2'(c_0, c_a, c_d, \Omega)$ such that

$$\|u\|_1 \le c_1'|Au|_2 \qquad \text{for } u\in(H^2)^3\cap V$$

and

$$|L^{1/2}\omega|_2 \le c_2'|L\omega|_2 \qquad \text{for } \omega\in(H^2)^3\cap(H_0^1)^3,$$

we can conclude with the aid of (5.10) that if

$$
(5.11)\qquad C_4c_1'^2\nu_r^2 \le \frac{\nu+\nu_r}{4} \quad\text{and}\quad C_4c_2'^2\nu_r^2 \le \frac{\nu+\nu_r}{4},
$$

then

$$
(5.12)\qquad
\begin{aligned}
&\|u(t)\|_1^2+|L^{1/2}\omega(t)|_2^2+(\nu+\nu_r)\int_0^t |Au|_2^2\,ds+\int_0^t |L\omega|_2^2\,ds\\
&\le \|u_0\|_1^2+|L^{1/2}\omega_0|_2^2+\frac{C_4}{\kappa}\left(\frac{M_f^2}{\nu+\nu_r}+M_g^2\right)|\theta_0|_2^2\\
&+\frac{C_4}{\kappa^2}\left(\frac{M_f^2}{\nu+\nu_r}+M_g^2\right)\int_0^t |h|_2^2\,ds\\
&+\rho_1\int_0^t \|u\|_1^6\,ds+\gamma_1\int_0^t |L^{1/2}\omega|_2^6\,ds+\gamma_2\int_0^t |L^{1/2}\omega|_2^4\,ds.
\end{aligned}
$$

Now, assume that

$$(5.13)\qquad \begin{aligned} &\|u_0\|_1^2 + |L^{1/2}\omega_0|_2^2 + \frac{C_4}{\kappa^2}\left(\frac{M_f^2}{\nu+\nu_r} + M_g^2\right)\int_0^\infty |h|_2^2\,dt \le c_3 \\ &\frac{C_4}{\kappa}\left(\frac{M_f^2}{\nu+\nu_r} + M_g^2\right)|\theta_0|_2^2 \le c_3, \end{aligned}$$

where $c_3 = 4^{-1}\min(c_3', c_3'')$, $c_3' = \left(\dfrac{\nu+\nu_r}{2\rho_1 c_1'^2}\right)^{1/2}$ and c_3'' is a positive root of $\gamma_1 y^2 + \gamma_2 y - 1/(2c_2'^2) = 0$. Then,

$$(5.14)\qquad \|u(t)\|_1^2 + |L^{1/2}\omega(t)|_2^2 < 4c_3 \qquad \text{for all } t.$$

Indeed, $\|u(t)\|_1^2 + |L^{1/2}\omega(t)|_2^2$ is a continuous function of t, and hence we have

$$(5.15)\qquad \|u(t)\|_1^2 + |L^{1/2}\omega(t)|_2^2 < 4c_3 \qquad \text{for small } t.$$

On the other hand, we can show that

$$\frac{(\nu+\nu_r)}{2}\int_0^t |Au(s)|_2^2\,ds - \rho_1\int_0^t \|u(s)\|_1^6\,ds > 0$$

and

$$\frac{1}{2}\int_0^t |L\omega(s)|_2^2\,ds - \gamma_1\int_0^t |L^{1/2}\omega(s)|_2^6\,ds - \gamma_2\int_0^t |L^{1/2}\omega(s)|_2^4\,ds > 0$$

whenever $\|u(s)\|_1^2 + |L^{1/2}\omega(s)|_2^2 < 4c_3$ for all $0 \le s \le t$. These inequalities follow from the estimates

$$\begin{aligned} \frac{(\nu+\nu_r)}{2}|Au(s)|_2^2 - \rho_1\|u(s)\|_1^6 &\ge \frac{\nu+\nu_r}{2c_1'^2}\|u(s)\|_1^2 - \rho_1\|u(s)\|_1^6 \\ &= \|u(s)\|_1^2\left(\frac{\nu+\nu_r}{2c_1'^2} - \rho_1\|u(s)\|_1^4\right) > 0 \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2}|L\omega(s)|_2^2 - \gamma_1|L^{1/2}\omega(s)|_2^6 - \gamma_2|L^{1/2}\omega(s)|_2^4 \\ &\qquad \ge \frac{1}{2c_2'^2}|L^{1/2}\omega(s)|_2^2 - \gamma_1|L^{1/2}\omega(s)|_2^6 - \gamma_2|L^{1/2}\omega(s)|_2^4 \\ &\qquad = |L^{1/2}\omega(s)|_2^2\left(\frac{1}{2c_2'^2} - \gamma_1|L^{1/2}\omega(s)|_2^4 - \gamma_2|L^{1/2}\omega(s)|_2^2\right) > 0. \end{aligned}$$

Thus, we deduce from (5.12) that

$$\|u(t)\|_1^2 + |L^{1/2}\omega(t)|_2^2 \le \|u_0\|_1^2 + |L^{1/2}\omega_0|_2^2 + \frac{C_4}{\kappa^2}\left(\frac{M_f^2}{\nu+\nu_r} + M_g^2\right)\int_0^t |h|_2^2\,ds$$
$$+ \frac{C_4}{\kappa}\left(\frac{M_f^2}{\nu+\nu_r} + M_g^2\right)|\theta_0|_2^2 \le 2c_3$$

provided that $\|u(s)\|_1^2 + |L^{1/2}\omega(s)|_2^2 < 4c_3$ for all $0 \le s \le t$. This together with (5.15) implies (5.14). We thus obtain the desired a priori bounds from (5.9), (5.12) and (5.14) provided that (5.11) and (5.13) are satisfied. This completes the proof.

6. Uniqueness of strong solutions to Problem (P)

We here prove the uniqueness of strong solutions to Problem (P) we constructed in the previous sections.

Our uniqueness theorem is stated as follows.

THEOREM 6.1. *A strong solution* (u, ω, θ) *of* Problem (P) *is unique.*

PROOF. Let $(u_i, \omega_i, \theta_i)$, $i = 1, 2$ be two solutions of Problem (P) with the same initial value. For simplicity in notation, we put

$$u = u_1 - u_2, \quad \omega = \omega_1 - \omega_2, \quad \theta = \theta_1 - \theta_2.$$

In terms of these new variables we get the following inequality

(6.1)
$$\frac{1}{2}\frac{d}{dt}\|u\|_1^2 + (\nu+\nu_r)|Au|_2^2$$
$$\le |b(u, u_1, Au)| + |b(u_2, u, Au)| + |(2\nu_r \operatorname{curl}\omega, Au) + |(f(\theta_1) - f(\theta_2), Au)|.$$

We then estimate the right-hand side of (6.1) term by term. Using the Gagliardo-Nirenberg inequality (1.1) and Sobolev inequality (1.2) we have

$$|b(u, u_1, Au)| \le \frac{(\nu+\nu_r)}{8}|Au|_2^2 + C\|u\|_1^2\|u_1\|_1|Au_1|_2, \tag{6.2}$$

$$|b(u_2, u, Au)| \le \frac{(\nu+\nu_r)}{8}|Au|_2^2 + C\|u_2\|_1^4\|u\|_1^2. \tag{6.3}$$

Standard calculations under the assumption (1.4):

$$|f(\theta_1) - f(\theta_2)|_2 \le M_f|\theta_1 - \theta_2|_2$$

give the estimates

$$|(2\nu_r \operatorname{curl}\omega,\ Au)| \le \frac{(\nu+\nu_r)}{8}|Au|_2^2 + C|L^{1/2}\omega|_2^2 \tag{6.4}$$

and

$$|(f(\theta_1) - f(\theta_2),\ Au)| \le \frac{(\nu+\nu_r)}{8}|Au|_2^2 + M_f^2|\theta|_2^2. \tag{6.5}$$

Inequalities (6.1) through (6.5) together imply

$$\begin{aligned}&\frac{d}{dt}\|u\|_1^2 + (\nu+\nu_r)|Au|_2^2\\ &\le C(\|u\|_1^2\|u_1\|_1|Au_1|_2 + \|u\|_1^2\|u_2\|_1^4 + |\theta|_2^2 + |L^{1/2}\omega|_2^2).\end{aligned}$$

As in Sections 3 and 4, and by the assumption (1.4)

$$|g(\theta_1) - g(\theta_2)|_2 \le M_g|\theta_1 - \theta_2|_2,$$

we see that the functions ω and θ satisfy

$$\frac{d}{dt}|L^{1/2}\omega|_2^2 + |L\omega|_2^2 \le C(\|u\|_1^2|L^{1/2}\omega_1|_2|L\omega_1|_2 + \|u_2\|_1^4|L^{1/2}\omega|_2^2 + \|u\|_1^2 + |\theta|_2^2),$$

and

$$\frac{d}{dt}|\theta|_2^2 + \kappa\|\theta\|_1^2 \le |(H,\theta)|, \tag{6.6}$$

where

$$H = (u\cdot\nabla\theta_2, \theta) - \Phi(u_1,\omega_1) + \Phi(u_2,\omega_2).$$

In view of (0.4) through (0.8) which define Φ, the right-hand side of (6.6) is estimated in the following way:

$$\begin{aligned}|(u\cdot\nabla\theta_2,\theta)| = |(u\cdot\nabla\theta,\theta_2)| &\le |u|_6|\nabla\theta|_2|\theta_2|_3\\ &\le C\|u\|_1\|\theta\|_1\|\theta_2\|_1^{1/2}|\theta_2|_2^{1/2}\\ &\le \frac{\kappa}{12}\|\theta\|_1^2 + C\|u\|_1^2\|\theta_2\|_1|\theta_2|_2,\end{aligned}$$

$$\begin{aligned}|(\Phi_1(u_1) - \Phi_1(u_2),\theta)| &\le C\int_\Omega |\nabla u||\nabla(u_1+u_2)||\theta|\\ &\le C|\nabla u|_2|\nabla(u_1+u_2)|_3|\theta|_6\\ &\le \frac{\kappa}{12}\|\theta\|_1^2 + C\|u\|_1^2|\nabla(u_1+u_2)|_3^2\end{aligned}$$

$$\leq \frac{\kappa}{12} \|\theta\|_1^2 + C\|u\|_1^2 \|u_1 + u_2\|_1 |A(u_1 + u_2)|_2,$$

$$\begin{aligned}
&|(\Phi_2(u_1, \omega_1) - \Phi_2(u_2, \omega_2), \theta)| \\
&\quad \leq C(|\nabla u|_2 |\nabla(u_1 + u_2)|_3 |\theta|_6 + |\omega|_3 |\nabla(u_1 + u_2)|_2 |\theta|_6 \\
&\qquad + |\omega_1 + \omega_2|_3 |\nabla u|_2 |\theta|_6 + |\omega|_2 |\omega_1 + \omega_2|_3 |\theta|_6) \\
&\quad \leq \frac{\kappa}{12} \|\theta\|_1^2 + C(\|u\|_1^2 |A(u_1 + u_2)|_2 \|u_1 + u_2\|_1^2 + |L^{1/2}\omega|_2^2 \|u_1 + u_2\|_1^2 \\
&\qquad + |L^{1/2}(\omega_1 + \omega_2)|_2^2 \|u\|_1^2 + |L^{1/2}\omega|_2^2 |L^{1/2}(\omega_1 + \omega_2)|_2^2),
\end{aligned}$$

$$\begin{aligned}
\sum_{i=3}^{4} |(\Phi_i(\omega_1) - \Phi_i(\omega_2), \theta)| &\leq C \int_\Omega |\nabla \omega| |\nabla(\omega_1 + \omega_2)| |\theta| \\
&\leq C |\nabla \omega|_2 |\nabla(\omega_1 + \omega_2)|_3 |\theta|_6 \\
&\leq \frac{\kappa}{6} \|\theta\|_1^2 + C |L^{1/2}\omega|_2^2 |L^{1/2}(\omega_1 + \omega_2)|_2 |L(\omega_1 + \omega_2)|_2,
\end{aligned}$$

and

$$\begin{aligned}
&|(\Phi_5(\omega_1) - \Phi_5(\omega_2), \theta)| \\
&\quad \leq C\left(\int_\Omega |\nabla \omega| |\nabla \omega_1| |\theta| + \int_\Omega |\nabla \omega| |\nabla \omega_2| |\theta| \right) \\
&\quad \leq \frac{\kappa}{12} \|\theta\|_1^2 + C |L^{1/2}\omega|_2^2 (|L\omega_1|_2 |L^{1/2}\omega_1|_2 + |L\omega_2|_2 |L^{1/2}\omega_2|_2).
\end{aligned}$$

Finally, combining the above estimates gives

$$\frac{d}{dt}(\|u\|_1^2 + |L^{1/2}\omega|_2^2 + |\theta|_2^2) \leq S(t)(\|u\|_1^2 + |L^{1/2}\omega|_2^2 + |\theta|_2^2),$$

where $S(t)$ is an integrable function on $[0, T]$. Now the application of Gronwall's inequality implies

$$\|u(t)\|_1 = \|\omega(t)\|_1 = |\theta(t)|_2 = 0.$$

This completes the proof.

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