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J-groups of the quaternionic spherical space forms

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1. Introduction

Let $J(X)$ be the J-group of CW-complex X of finite dimension. Then by J. F. Adams [2] and D. Quillen [17], it is shown that

$$J(X) = KO(X)/\mathrm{Ker}\, J, \quad \mathrm{Ker}\, J = \textstyle\sum_k (\bigcap_e k^e(\Psi^k - 1)KO(X)), \tag{1.1}$$

where $KO(X)$ is the KO-group of $X, J: KO(X) \to J(X)$ is the natural epimorphism and Ψ^k is the Adams operation.

Let Q_r $(r = 2^{m-1} \geqq 2)$ be the generalized quaternion group of order $4r$ given by

$$Q_r = \{x, y: x^r = y^2, xyx = y\},$$

the group generated by two elements x and y with the relations $x^r = y^2$ and $xyx = y$, that is, Q_r is the subgroup of the unit sphere S^3 in the quaternion field H generated by the two elements

$$x = \exp(\pi \boldsymbol{i}/r) \quad \text{and} \quad y = \boldsymbol{j}.$$

In this paper, we study the J-group of the quaternionic spherical space form:

$$N^n(m) = S^{4n+3}/Q_r \qquad (r = 2^{m-1} \geqq 2),$$

which is the orbit manifold of the unit sphere S^{4n+3} in the quaternion $(n+1)$-space H^{n+1} by the diagonal action of Q_r. In the case $m = 2$ and 3, the reduced J-group $\tilde{J}(N^n(m))$ is determined by H. Ōshima [15], T. Kobayashi [12], respectively.

Throughout this paper, we identify the orthogonal representation ring $RO(Q_r)$ with the subring $c(RO(Q_r))$ of the unitary represetation ring $R(Q_r)$ through the complexification $c: RO(Q_r) \to R(Q_r)$, since c is a ring monomorphism (cf. (2.1)).

Consider the complex representation a_0, a_1 and b_1 of Q_r given by

$$\begin{cases} a_0(x) = \ \ \, 1 \\ a_0(y) = -1, \end{cases} \begin{cases} a_1(x) = -1 \\ a_1(y) = \ \ \, 1, \end{cases} b_1(x) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \ b_1(y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and the elements in $\tilde{R}(Q_r)$ defined by

$$\alpha_i = a_i - 1 \ (i = 0, 1), \quad \beta = b_1 - 2 \qquad \text{(cf. (2.4))},$$
$$\beta(0) = \beta, \quad \beta(s) = \beta(s-1)^2 + 4\beta(s-1) \qquad \text{(cf. (2.8))}.$$

Then

$$\alpha_i \ (i = 0, 1), \quad 2^{\varepsilon(2^s)}\beta(s) \in \widetilde{RO}(Q_r) \qquad \text{(cf. Prop. 2.7)},$$

where $\varepsilon(i) = 0$ if i is even, $= 1$ if i is odd.
Furthermore, consider the elements in $\widetilde{KO}(N^n(m))$:

(1.2) $$\alpha_i = \xi(\alpha_i) \ (i = 0, 1), \ 2^{\varepsilon(2^s)}\beta(s) = \xi(2^{\varepsilon(2^s)}\beta(s)) \qquad \text{(cf. (3.2))},$$

where ξ is the natural ring homomorphism of $\widetilde{RO}(Q_r)$ to $\widetilde{KO}(N^n(m))$. Then, the main purpose of this paper is to prove the following

THEOREM 1.3. *Let $m \geqq 2$ and put $\bar{n} = 2n + \varepsilon(n)$. Then, the order of the J-image*

$$\gamma_v = J(2^{\varepsilon(2^v)}\beta(v)) \qquad (v \geqq 0)$$

of the element $2^{\varepsilon(2^v)}\beta(v)$ is equal to $2^{f(n,m;v)}$. Here, in the case $m = 2$,

$$f(n, 2; 0) = \bar{n}, \ f(n, 2; 1) = n + \varepsilon(n), \ f(n, 2; v) = 0 \qquad (v \geqq 2);$$

and in the case $m \geqq 3$,

$$f(n, m; 0) = \max\{2n + 1, s - 1 + 2^s[\bar{n}/2^s] : 0 \leqq s < m, \ 2^s \leqq \bar{n}\},$$
$$f(n, m; 1) = \max\{n + 1, s - 1 + 2^{s-1}[\bar{n}/2^s] : 1 \leqq s < m, \ 2^s \leqq \bar{n}\},$$
$$f(n, m; v) = \max\{s - v + 2^{s-v}[\bar{n}/2^s] : v \leqq s < m, \ 2^s \leqq \bar{n}\} \quad (v \geqq 2),$$

where we define $\max\{\ \} = 0$, *if* $\{s : v \leqq s < m, \ 2^s \leqq \bar{n}\} = \emptyset$.

Some partial results for the order of γ_0 are obtained by H. Ōshima [15], T. Kobayashi [12] and K. Komatsu [14], and are applied to get the information about the stable homotopy types of the stunted spaces of $N^n(m)$ (cf. also [13]).

On the group structure of the reduced J-group $\tilde{J}(N^n(m))$ $(m \geqq 2)$, we have the following theorem, where

(1.4) $$a_s = [\bar{n}/2^s], \quad b_s = \bar{n} - 2^s[\bar{n}/2^s], \quad \bar{n} = 2n + \varepsilon(n), \quad \varepsilon(n) = (1 - (-1)^n)/2,$$

(1.5) $$X(d, v) = \sum_{j \in Z} (-1)^{j(2^v+1)} \binom{2d}{d + 2^v j},$$

(1.6) $$Y(d, v) = \sum_{j \in Z} \binom{2d - 1}{d + 2^v(2j + 1)}.$$

THEOREM 1.7. (i) *$\tilde{J}(N^n(m))$ is generated by the J-images*

$$J\alpha_i \ (i=0,1) \quad \text{and} \quad \gamma_v = J(2^{\varepsilon(2^v)}\beta(v)) \qquad (0 \leqq v < m)$$

of the elements in (1.2).

(ii) ([15, Th. 6.1]) $J: \widetilde{KO}(N^n(2)) \cong \tilde{J}(N^n(2))$.

(iii) *The relations of $\tilde{J}(N^n(m))$ for $m \geqq 3$ are given as follows*:

(a) *The case $n \equiv 0 \bmod 2$: Let $[2^k] \leqq n < 2^{k+1}$ for $k \geqq -1$.*

$$(1.7.1) \qquad 2^{n+2}J\alpha_i = 0 \qquad (i=0,1)$$

$$(1.7.2) \qquad 2^{m-s-1+a_s-\varepsilon(2^s)}\gamma_s = 0 \qquad (0 \leqq s < m).$$

$$(1.7.3) \qquad \textstyle\sum_{v=0}^{s} 2^{m-s-3+2^{s-v}(a_s+1)-\varepsilon(2^v)}\gamma_v = 0 \qquad (2 \leqq s \leqq \min\{k, m-1\}).$$

$$(1.7.4) \qquad \textstyle\sum_{v=0}^{s} (-1)^{2^{s-v}} 2^{m-s-4+2^{s+1-v}(a_{s+1}+\delta)+\varepsilon(d)-\varepsilon(2^v)} X(d,v)\gamma_v = 0$$
$$(1 \leqq s \leqq \min\{k, m-2\},\ 0 < d < 2^s,\ 2^s + d < N),$$

where $\delta = 0$ if $2d > b_{s+1}$, $=1$ otherwise and $N = \min\{2^{m-1}, n\}$.

$$(1.7.5) \qquad 2^{2i-3+\varepsilon(i)}\gamma_0 - \textstyle\sum_{v=1}^{t} 2^{\varepsilon(i)} Y(i,v)\gamma_v = 0 \qquad (N < i \leqq 2^{m-1}),$$

where $2^t \leqq i < 2^{t+1}$.

(b) *The case $n \equiv 1 \bmod 2$: The relations in* (a) *replaced n with $n+1$, and in addition,*

$$(1.7.6) \qquad 2^{n+1}J\alpha_i = 0 \qquad (i=0,1),$$

$$(1.7.7) \qquad 2^{2n-1}\gamma_0 - \textstyle\sum_{v=1}^{l} Y(n+1,v)\gamma_v = 0, \quad \text{where} \quad 2^l \leqq n+1 < 2^{l+1}.$$

For the special case $n = 2^{m-2}a$, we can reduce the relations of $\tilde{J}(N^n(m))$ in (iii) of the above theorem to more simple ones, and $\tilde{J}(N^n(m))$ is given by the following explicit form, where $Z_h\langle x\rangle$ denotes the cyclic group of order h generated by the element x.

THEOREM 1.8. *If $n = 2^{m-2}a$ ($m \geqq 3$, $a \geqq 2$), then $\tilde{J}(N^n(m))$ is the direct sum*

$$Z_{2^{n+2}}\langle J\alpha_0\rangle \oplus Z_{2^{n+2}}\langle J\alpha_1\rangle \oplus Z_{2^{m-2+a_0}}\langle\gamma_0\rangle \oplus Z_{2^{a_1}}\langle\gamma_1 - 2^{a_0-a_1}\gamma_0\rangle \oplus$$
$$\oplus_{v=2}^{m-1} Z_{2^{a_{v-1}}}\langle\gamma_v - 2^{a_{v-1}-a_v+1}\gamma_{v-1}\rangle,$$

where $a_v = 2^{m-1-v}a$ in (1.4).

By using the above theorem, we can determine the kernel of the homomorphism

$$(1.9) \qquad j^*: \tilde{J}(N^n(m)) \longrightarrow \tilde{J}(N^{n-1}(m))$$

induced by the inclusion $j\colon N^{n-1}(m) \subset N^n(m)$ as follows:

PROPOSITION 1.10. *j^* in (1.9) is epimorphic, and* $\operatorname{Ker} j^*$ *is given by*

$$\operatorname{Ker} j^* = \begin{cases} Z_8\langle J(2\beta^n)\rangle & \text{if } n \text{ is odd,} \\ Z_4\langle 2^n J\alpha_0\rangle \oplus Z_4\langle 2^n J\alpha_1 \pm 2^{m-3+2n}\gamma_0\rangle \oplus Z_{2^{h_m}}\langle J(\beta^n)\rangle & \text{if } n \text{ is even,} \end{cases}$$

where $h_m = \min\{m+1, \nu_2(n)+3\}$ and $\nu_2(n)$ is the exponent of 2 in the prime power decomposition of n, and the term $\pm 2^{m-3+2n}\gamma_0$ does not appear if $m = 2$.

By this proposition, we see immediately the following

THEOREM 1.11. *The order of the reduced J-group $\tilde{J}(N^n(m))$ is equal to*

$$2^{\varphi(n,m)},\ \varphi(n, m) = \textstyle\sum_{s=0}^{m-1} (s+2)[(a_s+1)/2] + 4\varepsilon(n+1),$$

where $\varepsilon(i)$ is the integer in (1.4).

By using Proposition 1.10 and Theorem 1.8, we can prove Theorem 1.3 by the induction on n and m.

We prepare in §2 some results on the orthogonal representation ring $RO(Q_r)$ ($r = 2^{m-1}$). In §3, we recall the additive structure of the KO-ring of $N^n(m)$ given in [10]. We study the behavior of the Adams operations on $\widetilde{KO}(N^n(m))$ and determine the generators of $\operatorname{Ker} J$ in (1.1) for $X = N^n(m)$ explicitely in §4. In §5, we give the key relations of $\tilde{J}(N^n(m))$ in Lemma 5.6, and the defining relations of $\tilde{J}(N^n(m))$ in Proposition 5.7, which are useful in the proof of Theorem 1.7. Some lemmas for the coefficients $X(d, v)$ and $Y(d, v)$ in (1.5–6) are prepared in §6.

By using these results, we prove Theorem 1.7 (i), (ii) and (iii) (a) in §7, and Theorem 1.8 in §8. In §9, we prove Proposition 1.10 in Corollary 9.10, and Theorem 1.11 in Proposition 9.8 (ii) by using the results on $\operatorname{Ker}\{j^*\colon \widetilde{KO}(N^{k+1}) \to \widetilde{KO}(N^k)\}$ ([10, §4]), where N^k is the k-skeleton of the CW-complex $N^n(m)$ ([6, Lemma 2.1]). In §10, Theorem 1.3 is proved first, and then Theorem 1.7 (iii) (b) is shown by using Proposition 1.10, Theorems 1.3 and 1.7 (iii) (a).

In the final section, we study the relation between $\tilde{J}(N^n(m+1))$ and $\tilde{J}(N^n(m))$ for $n < 2^{m-1}$.

2. The representation rings of Q_r ($r = 2^{m-1}$)

We denote the unitary (resp. orthogonal) representation ring of the group G by $R(G)$ (resp. $RO(G)$). By the natural inclusions

$$O(n) \subset U(n) \quad \text{and} \quad U(n) \subset O(2n),$$

the following group homomorphisms are defined:

$$RO(G) \xrightarrow{c} R(G), \quad R(G) \xrightarrow{r} RO(G).$$

The following facts are well known (cf., e.g. [3]).

(2.1) These representation rings are free over Z, and c is a ring homomorphism. Also

$$rc = 2, \quad cr = 1 + t,$$

(t denotes the conjugation), and c is monomorphic.

Hence throughout this paper, we identify $RO(G)$ with the subring $c(RO(G))$ of $R(G)$.

We regard the generalized quaternion group Q_r ($r = 2^{m-1} \geqq 2$) of order $4r$ as the subgroup of the unit sphere S^3 in the quaternion field H generated by the two elements

$$x = \exp(\pi \boldsymbol{i}/r) \quad \text{and} \quad y = \boldsymbol{j}.$$

Consider the complex representations a_i ($i = 0, 1, 2$) and b_j ($j \in Z$) of Q_r given by

$$\text{(2.2)} \quad \begin{gathered} \begin{cases} a_0(x) = 1 \\ a_0(y) = -1, \end{cases} \begin{cases} a_i(x) = -1 \\ a_i(y) = (-1)^{i-1} \end{cases} (i = 1, 2), \\ b_j(x)\begin{pmatrix} x^j & 0 \\ 0 & x^{-j} \end{pmatrix}, \quad b_j(y) = \begin{pmatrix} 0 & (-1)^j \\ 1 & 0 \end{pmatrix}. \end{gathered}$$

Then, we see easily the following

PROPOSITION 2.3 (cf. [4, §47.15, Example 2]). *$R(Q_r)$ is a free Z-module with basis* 1, a_i ($i = 0, 1, 2$) *and* b_j ($1 \leqq j < r$) *and the multiplicative structure is given as follows*:

$$a_0^2 = 1,\ a_1^2 = 1,\ a_2 = a_0 a_1,\ b_0 = 1 + a_0,\ b_r = a_1 + a_2,$$
$$b_{r+i} = b_{r-i},\ b_{-i} = b_i,\ b_i b_j = b_{i+j} + b_{i-j},\ a_0 b_i = b_i,\ a_1 b_i = b_{r-i}.$$

Let

$$\text{(2.4)} \quad \alpha_i = a_i - 1 \ (i = 0, 1, 2) \quad \text{and} \quad \beta_j = b_j - 2 \ (j \in Z)$$

be the elements in the reduced representation ring $\tilde{R}(Q_r)$.

From now on, we denote β instead of β_1 for simplicity. Then, we have

PROPOSITION 2.5 (cf. [6, Prop. 3.3]). *$\tilde{R}(Q_r)$ is a free Z-module with basis*

α_i $(i=0, 1, 2)$ and β_j $(1 \leqq j < r)$, and the multiplicative structure is given as follows:

$$\alpha_0^2 = -2\alpha_0,\ \alpha_1^2 = -2\alpha_1,\ \alpha_2 = \alpha_0\alpha_1 + \alpha_0 + \alpha_1,$$

$$\beta_0 = \alpha_0,\ \beta_r = \alpha_1 + \alpha_2,\ \beta_{r+i} = \beta_{r-i},\ \beta_{-i} = \beta_i,$$

$$\beta_i\beta_j = \beta_{i+j} + \beta_{i-j} - 2(\beta_i + \beta_j),\ \alpha_0\beta_i = -2\alpha_0,\ \alpha_1\beta_i = \beta_{r-i} - \beta_i - 2\alpha_1.$$

These show that the ring $\tilde{R}(Q_r)$ is generated by α_0, α_1 and β.

Regarding $RO(Q_r)$ as the subring of $R(Q_r)$ under the complexification $c\colon RO(Q_r) \to R(Q_r)$ in (2.1), we have

PROPOSITION 2.6 (cf. [5, (3.5) and (12.3)]). *$RO(Q_r)$ is a free Z-module with basis 1, a_i $(i=0, 1, 2)$, b_{2j} and $2b_{2j+1}$ $(1 \leqq 2j, 2j+1 < r)$.*

By (2.4), Propositions 2.5 and 2.6, we have

PROPOSITION 2.7 (cf. [10, Prop. 2.7]). *The reduced representation ring $\widetilde{RO}(Q_r)$ is a free Z-module with basis α_i $(i=0, 1, 2)$, β_{2j} and $2\beta_{2j+1}$ $(1 \leqq 2j, 2j+1 < r)$. Also, the ring $\widetilde{RO}(Q_r)$ is generated by α_0, α_1, 2β and β^2.*

Let $m \geqq 2$, and define $\beta(s)$ in $\tilde{R}(Q_r)$ inductively as follows:

$$\beta(0) = \beta, \quad \beta(s) = \beta(s-1)^2 + 4\beta(s-1) \quad (s \geqq 1). \tag{2.8}$$

Then, we have the following

LEMMA 2.9 (cf. [10, Lemma 2.16], [9, Lemmas 5.3 and 5.4]). *The following relations hold in $\tilde{R}(Q_r)$*:

(i) $P_{m,s} = \beta(s)\prod_{t=s-1}^{m-2}(2+\beta(t)) = 0$ *and* $\beta(s) = 0 \quad (s \geqq m)$.

(ii) $\alpha_1\beta^n = (\beta^n - (-2)^n)\sum_{s=1}^{m-2}\beta(s)\prod_{t=s+1}^{m-2}(2+\beta(t)) + (-2)^n\alpha_1$

for any positive integer n.

PROOF. (i) The first relation is proved in [10, Lemma 2.16]. By the first relation, $\beta(m) = P_{m,m} = 0$, and so $\beta(s) = 0$ $(s \geqq m)$ follows from the definition of $\beta(s)$ in (2.8).

(ii) Since $\alpha_1\beta = \beta_{r-1} - \beta - 2\alpha_1$ by Proposition 2.5,

$$\alpha_1\beta^n = \beta^{n-1}(\beta_{r-1} - \beta) - 2\alpha_1\beta^{n-1}$$
$$= \left(\sum_{i=0}^{n-1}(-2)^i\beta^{n-1-i}\right)(\beta_{r-1} - \beta) + (-2)^n\alpha_1.$$

On the other hand,

$$\beta_{r-1} - \beta = (2+\beta)\sum_{s=1}^{m-2}\beta(s)\prod_{t=s+1}^{m-2}(2+\beta(t))$$

by [9, Lemma 5.3]. Therefore, we have the desired relation. q.e.d.

By the definition of $\beta(s)$, $P_{m,s}$, Lemma 2.9 and Proposition 2.7, we have

LEMMA 2.10 (cf. [10, Lemma 2.17]). *$2P_{m,1}=0$, $\beta P_{m,1}=0$, $P_{m,s}=0$ $(2 \leqq s \leqq m)$ and $\beta(s)=0$ $(s \geqq m)$ hold in $\widetilde{RO}(Q_r)$.*

3. The structure of $\widetilde{KO}(N^n(m))$

The generalized quaternion group Q_r $(r=2^{m-1} \geqq 2)$ acts on the unit sphere S^3 in the quaternion $(n+1)$-space H^{n+1} by the diagonal action

$$q(q_0,\dots,q_n)=(qq_0,\dots,qq_n) \qquad \text{for } q \in Q_r,\ q_i \in H,$$

and we have the quaternionic spherical space form

$$N^n(m)=S^{4n+3}/Q_r \qquad \text{of dimension } 4n+3.$$

Then the natural projection $S^{4n+3} \to N^n(m)$ define the ring homomorphism (cf. [11, Ch. 12, 5.4])

$$\xi\colon \widetilde{RO}(Q_r) \longrightarrow \widetilde{KO}(N^n(m)), \tag{3.1}$$

and by using the same letter, we define the elements

$$\begin{aligned} &\alpha_i=\xi(\alpha_i)\ (i=0,1), \quad 2^{\varepsilon(j)}\beta^j=\xi(2^{\varepsilon(j)}\beta^j)\ (j \geqq 1), \\ &2\beta(0)=\xi(2\beta(0)),\ \beta(s)=\xi(\beta(s))\ (s \geqq 1) \text{ in } \widetilde{KO}(N^n(m)), \end{aligned} \tag{3.2}$$

where $\varepsilon(j)$ is the integer in (1.4).

For the ring homomorphism $\xi\colon \widetilde{RO}(Q_r) \to \widetilde{KO}(N^n(m))$ in (3.1),

(3.3) (cf. [16, Th. 2.5], [7, Th. 1.1 and Cor. 1.2]) *ξ is an epimorphism, and*

$$\operatorname{Ker}\xi=\begin{cases} \langle \beta^{n+1} \rangle & \text{if } n \text{ is odd,} \\ \langle 2\beta^{n+1},\ \beta^{n+2} \rangle & \text{if } n \text{ is even,} \end{cases}$$

where $\langle S \rangle$ means the ideal generated by the set S.

Consider the following integers $u(i)$ and the elements δ_i and $\bar{\alpha}_1$ in $\widetilde{KO}(N^n(m))$ $(m \geqq 2)$, where α_i, $2\beta=2\beta(0)$ and $\beta(s)$ are the ones in (3.2). For $i=2^s+d \leqq N=\min\{2^{m-1}, n\}$ with $0 \leqq s < m$ and $0 \leqq d < 2^s$, put

$$\bar{n}=2n+\varepsilon(n)=2^s a_s+b_s, \qquad 0 \leqq b_s < 2^s;$$

$$u(1)=2^{m-2+a_0},\ \delta_1=2\beta \qquad \text{if } i=1;$$

$$u(2)=\begin{cases} 2^{m-3+a_1} & (n\colon \text{odd}), \\ 2^{m-2+a_1} & (n\colon \text{even}), \end{cases}$$

$$\delta_2 = \begin{cases} \beta(1) - 2^{1+a_1}\beta(0) - R_0(1, 0; a_1+1) & (n\colon \text{odd}), \\ \beta(1) & (n\colon \text{even}) \quad \text{if } i = 2; \end{cases}$$

$$u(i) = 2^{m-s-2+a_s},$$

(3.4)

$$\delta_i = \begin{cases} \sum_{t=0}^{s}(-1)^{2^t+1}2^{(2^t-1)(a_s+1)}\beta(s-t) - R_0(s, 0; a_s+1) \ (n\colon \text{odd}), \\ \sum_{t=0}^{s} 2^{(2^t-1)(a_s+1)}\beta(s-t) \ (n\colon \text{even}) \quad \text{if } i = 2^s \ (2 \leqq s < m); \end{cases}$$

$$u(i) = 2^{m-s-3+a(i)}, \ a(i) = \begin{cases} a_{s+1}+1 & \text{for } 2d \leqq b_{s+1}, \\ a_s & \text{for } 2d > b_{s+1}, \end{cases}$$

$$\delta_i = \begin{cases} 2\beta^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t)) + \sum_{t=0}^{s}(-1)^{2^t}2^{(2^{t+1}-1)a(i)}\beta^d\beta(s-t) \\ \qquad (d\colon \text{odd}), \\ \beta^{d-2}\beta(2)\prod_{t=1}^{s-1}(2+\beta(t)) + R(s, d; a(i)) \quad (n\colon \text{odd}, \ d\colon \text{even}), \\ \beta^{d-2}\beta(2)\prod_{t=1}^{s-1}(2+\beta(t)) + \sum_{t=0}^{s}(-1)^{2^t+a(i)}2^{(2^{t+1}-1)a(i)-1}\beta^d\beta(s-t) \\ \qquad (n\colon \text{even}, \ d\colon \text{even}) \quad \text{if } i = 2^s + d \geqq 3, \ d \geqq 1; \end{cases}$$

$$\bar{\alpha}_1 = \begin{cases} \alpha_1 & (n\colon \text{even or } m = 2), \\ \alpha_1 \pm 2^{m-2+n}\beta & (n\colon \text{odd and } m \geqq 3), \end{cases}$$

where $\varepsilon(n)$ is the integer in (1.4), and $R_0(s, 0; k)$ and $R(s, d; k)$ are the elements given in [10, Props. 7.1 and 7.2].

Then, the additive structure of $\widetilde{KO}(N^n(m))$ is given by the following theorem:

THEOREM 3.5 (cf. [10, Th. 1.6]). *$\widetilde{KO}(N^n(m))$ $(m \geqq 2)$ is additively generated by the elements α_0, α_1 and $2^{\varepsilon(i)}\beta^i$ $(1 \leqq i \leqq 2^{m-1})$, and the additive structure is given by the following relations*:

(3.5.1) $\qquad 2^{n+2-\varepsilon(n)}\alpha_0 = 0, \qquad 2^{n+2-\varepsilon(n)}\bar{\alpha}_1 = 0,$

(3.5.2) $\qquad u(i)\delta_i = 0 \qquad (1 \leqq i \leqq N = \min\{2^{m-1}, n\},$

(3.5.3) $\qquad 2^{\varepsilon(i)}\beta^i = 0 \qquad (N < i \leqq 2^{m-1}),$

where $\varepsilon(i)$ is the integer in (1.4).

REMARK 3.6. *δ_i is the linear combination of $2^{\varepsilon(j)}\beta^j$ $(1 \leqq j \leqq i)$ such that the coefficient of $2^{\varepsilon(i)}\beta^i$ is odd. Also, by the definition of $\beta(s)$ in* (2.8), *β^i is the linear combination of the monomials*

$$\beta(I) = \beta(i_1)\cdots\beta(i_t) \quad (0 \leqq i_1 < \cdots < i_t < m, \ |I| \leqq i)$$

such that the coefficient of $\beta(I)$ with $|I| = i$ is equal to 1, *where $|I| = 2^{i_1} + \cdots + 2^{i_t}$ for $I = (i_1, \ldots, i_t)$.*

4. The Adams operations on $\widetilde{KO}(N^m(m))$

Now, consider the complex representation rings $R(S^3)$, $R(S^1)$, $R(Q_r)$ ($r = 2^{m-1} \geqq 2$) and the ring homomorphisms

$$(4.1) \qquad i^*: R(S^3) \longrightarrow R(S^1) \quad \text{and} \quad j^*: R(S^3) \longrightarrow R(Q_r)$$

induced by the natural inclusions $S^1 \subset S^3$ and $Q_r \subset S^3$.

The following lemmas are well known:

LEMMA 4.2 (cf. [11, Ch. 13, Th. 3.1]). *$R(S^3)$ is the polynomial ring $Z[\zeta]$, where ζ is given by the representation*

$$\zeta(z_1 + jz_2) = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \quad \text{for } z_1 + jz_2 \in S^3.$$

LEMMA 4.3 (cf. [11, Ch. 13, Th. 3.1]). *$R(S^1)$ is the polynomial ring $Z[x, x^{-1}]$, where $x(z) = z$ ($z \in S^1$) and x^{-1} is the conjugation of x.*

Define the elements $\zeta(s)$ in $R(S^3)$ inductively as follows:

$$(4.4) \qquad \zeta(0) = \zeta - 2, \quad \zeta(s) = \zeta(s-1)^2 + 4\zeta(s-1) \quad (s \geqq 1).$$

Then, we have the following lemmas.

LEMMA 4.5. *$i^*: R(S^3) \to R(S^1)$ in (4.1) is monomorphic, and*

$$i^*(\zeta(s)) = x^{2^s} + x^{-2^s} - 2 \qquad (s \geqq 0).$$

PROOF. The first half follows from [11, Ch. 9, Th. 9.3]. We see easily that $i^*(\zeta) = x + x^{-1}$, and so $i^*(\zeta(0)) = x + x^{-1} - 2$. Hence, the second half is shown by the induction on s. q.e.d.

By the definitions of $\beta(s)$ in (2.8) and $\zeta(s)$ in (4.4), we see easily the following

LEMMA 4.6. *For the homomorphism $j^*: R(S^3) \to R(Q_r)$ in (4.1), we have the equality*

$$j^*(\zeta(s)) = \beta(s) \qquad (s \geqq 0).$$

Let Ψ^l be the Adams operation on $R(S^3)$, and

$$(4.7) \qquad W = [(\Psi^{2k+1} - 1)2^{\varepsilon(i)}\zeta(0)^i : k \in Z,\ i \geqq 1]$$

be the subgroup of $R(S^3)$ generated by the elements in the bracket.

Then, we have the following lemmas.

LEMMA 4.8. (i) *$i^*(W)$ is the subgroup of $R(S^1)$ generated by*

$$(\Psi^{2k+1}-1)2^{\varepsilon(2^v)}(x^{2^v}+x^{-2^v}) \qquad (k \geqq 0,\ v \geqq 0),$$

where Ψ^l *is the Adams operation on* $R(S^1)$.

(ii) *W is the subgroup of* $R(S^1)$ *generated by*

$$2^{\varepsilon(2^v)}(2+\zeta(v))\zeta(V) \qquad (0 \leqq v < v_1 < \cdots < v_t,\ t \geqq 1),$$

where $\zeta(V)=\zeta(v_1)\cdots\zeta(v_t)$ *for* $V=(v_1,\dots,v_t)$.

PROOF. (i) Since $\zeta(0)=\zeta-2$, we see that

$$W=[(\Psi^{2k+1}-1)2^{\varepsilon(i)}\zeta^i\colon k\in Z,\ i\geqq 1].$$

By the naturality of the Adams operations $i^*\Psi^l=\Psi^l i^*$ and Lemma 4.5, we have

$$i^*((\Psi^{2k+1}-1)\xi^i)=(\Psi^{2k+1}-1)i^*(\zeta^i)=(\Psi^{2k+1}-1)(x+x^{-1})^i$$
$$=\sum_{j=0}^{[(i-1)/2]}\binom{i}{j}(\Psi^{2k+1}-1)(x^{i-2j}+x^{-i+2j}).$$

Here, we set $i-2j=2^v q$ (q: odd). Then

$$(\Psi^{2k+1}-1)(x^{i-2j}+x^{-i+2j})=(\Psi^{(2k+1)q}-1)(x^{2^v}+x^{-2^v})$$
$$-(\Psi^q-1)(x^{2^v}+x^{-2^v}).$$

On the other hand, $x^{2^v}+x^{-2^v}$ is the linear combination of $(x+x^{-1})^i$ ($i\geqq 1$), and $\Psi^{-l}(x^{2^v}+x^{-2^v})=\Psi^l(x^{2^v}+x^{-2^v})$. Therefore, we have (i).

(ii) Let $k\geqq 0$ and $v\geqq 0$. By the equality

$$(\Psi^{2k+1}-1)(x^{2^v}+x^{-2^v})=(x^{2^v}+x^{-2^v})^{2k+1}-2^{2k}(x^{2^v}+x^{-2^v})$$
$$-\sum_{j=1}^{k}\binom{2k+1}{j}(\Psi^{2(k-j)+1}-1)(x^{2^v}+x^{-2^v}),$$

we see easily that

$$i^*(W)=[2^{\varepsilon(2^v)}\{(x^{2^v}+x^{-2^v})^{2k+1}-2^{2k}(x^{2^v}+x^{-2^v})\}\colon k\geqq 0,\ v\geqq 0].$$

Since $i^*(\zeta(v))=x^{2^v}+x^{-2^v}-2$ and i^* is monomorphic by Lemma 4.5,

$$W=[2^{\varepsilon(2^v)}\{(2+\zeta(v))^{2k+1}-2^{2k}(2+\zeta(v))\}\colon k\geqq 0,\ v\geqq 0].$$

Put

$$\zeta(v,k)=(2+\zeta(v))^{2k+1}-2^{2k}(2+\zeta(v)) \qquad (k\geqq 0,\ v\geqq 0),$$
$$\bar{\zeta}(v,k)=(2+\zeta(v))^{2k-1}\zeta(v+1) \qquad (k\geqq 1,\ v\geqq 0).$$

Then, we have

$$\zeta(v, k) = \bar{\zeta}(v, k) + 4\zeta(v, k-1) \qquad (k \geqq 1,\ v \geqq 0) \text{ and } \zeta(v, 0) = 0,$$

and so

$$W = [2^{\varepsilon(2^v)}(2+\zeta(v))^{2k+1}\zeta(v+1) \colon k \geqq 0,\ v \geqq 0].$$

Let $k \geqq 0$, $v \geqq 0$ and set

$$k + 1 = 2^{i_1} + \cdots + 2^{i_l} \qquad (0 \leqq i_1 < \cdots < i_l).$$

By the definition of $\zeta(v)$ in (4.4),

$$(2+\zeta(v))^{2k+1}\zeta(v+1) = (2+\zeta(v))\zeta(v+1)(\zeta(v+1)+4)^k,$$

and $\zeta(v+1)(\zeta(v+1)+4)^k$ is the polynomial of degree $k+1$ in $\zeta(v+1)$. Thus, we have the equality

$$(2+\zeta(v))^{2k+1}\zeta(v+1) = (2+\zeta(v))\zeta(v+1+i_1)\cdots\zeta(v+1+i_l) + \cdots$$
$$+ 4^k(2+\zeta(v))\zeta(v+1).$$

This equality implies (ii). q.e.d.

LEMMA 4.9. *$j^*(W)$ is the subgroup of $\widetilde{RO}(Q_r)$ ($r = 2^{m-1} \geqq 2$) generated by*

$$2^{\varepsilon(2^v)}\{\beta(v)\beta(V) - (-2)^t\beta(v_t)\} \qquad (0 \leqq v < v_1 < \cdots < v_t \leqq m-2,\ t \geqq 1),$$

where $\beta(V) = \beta(v_1)\cdots\beta(v_t)$ for $V = (v_1, \ldots, v_t)$.

PROOF. Since $\widetilde{RO}(Q_r)$ is identified with the subring $c(\widetilde{RO}(Q_r))$ of $\tilde{R}(Q_r)$, $j^*(W) \subset \widetilde{RO}(Q_r)$ and $j^*(W)$ is generated by

$$2^{\varepsilon(2^v)}(2+\beta(v))\beta(V) \qquad (0 \leqq v < v_1 < \cdots < v_t,\ t \geqq 1)$$

by Lemmas 2.7, 4.6 and 4.8 (ii). On the other hand, by Lemma 2.10, $\beta(s) = 0$ $(s \geqq m)$ and

$$2^{\varepsilon(2^v)}P_{m,v+1} = 2^{\varepsilon(2^v)}\{(2+\beta(v))\beta(m-1) + \textstyle\sum_I a(I)(2+\beta(v))\beta(I)\} = 0$$

hold in $\widetilde{RO}(Q_r)$, where $P_{m,v+1} = \beta(v+1)\prod_{t=v}^{m-2}(2+\beta(t))$ and $I = (i_1, \ldots, i_j)$ runs over $\{(i_1, \ldots, i_j) \colon v < i_1 < \cdots < i_j \leqq m-2)$ in $\sum$. Hence, $j^*(W)$ is generated by

$$2^{\varepsilon(2^v)}(2+\beta(v))\beta(V) \qquad (0 \leqq v < v_1 < \cdots < v_t \leqq m-2,\ t \geqq 1),$$

and also by

$$2^{\varepsilon(2^v)}\{\beta(v)\beta(V) - (-2)^t\beta(v_t)\} \qquad (0 \leqq v < v_1 < \cdots < v_t \leqq m-2,\ t \geqq 1).$$

q.e.d.

Here, we notice the following

Remark 4.10. *For* $j^*: R(S^3) \to R(Q_r)$ *in* (4.1) *and* $\xi: \widetilde{RO}(Q_r) \to \widetilde{KO}(N^n(m))$ *in* (3.1), *we have*

$$\xi j^*(W) = [(\Psi^{2k+1} - 1)2^{\varepsilon(i)}\beta^i: k \in Z, i \geqq 1],$$

where Ψ^l *is the Adams operation on* $\widetilde{KO}(N^n(m))$.

In fact, the above assertion follows from the facts that $\xi j^*(\zeta(0)^i) = \beta^i$ by Lemma 4.6, (2.8) and (3.2), and the Adams operations on $R(S^3)$, $R(Q_r)$, $\widetilde{RO}(Q_r)$ and $\widetilde{KO}(N^n(m))$ are natural with respect to j^*, the complexification $c: RO(Q_r) \to R(Q_r)$ and ξ.

Now, consider the J-homomorphism

$$J: \widetilde{KO}(N^n(m)) \longrightarrow \tilde{J}(N^n(m)) \qquad (m \geqq 2),$$

where J is an epimorphism and

$$\text{(4.11)} \qquad \operatorname{Ker} J = \textstyle\sum_k L_k, \quad L_k = \bigcap_e k^e(\Psi^k - 1)\widetilde{KO}(N^n(m))$$

by (1.1). Then, we have

Lemma 4.12. Ker $J = \xi j^*(W)$ *holds, and* Ket J *is generated by*

$$2^{\varepsilon(2^v)}\{\beta(v)\beta(V) - (-2)^t\beta(v_t)\} \qquad (0 \leqq v < v_1 < \cdots < v_t \leqq m-2, t \geqq 1),$$

where $\beta(V) = \beta(v_1)\cdots\beta(v_t)$ *for* $V = (v_1, \ldots, v_t)$.

Proof. By Lemma 4.9 and (3.2), it is sufficient to show that Ker $J = \xi j^*(W)$. Since $\widetilde{KO}(N^n(m))$ is a 2-group by Theorem 3.5, L_k in (4.11) is 0 if $k \equiv 0 \bmod 2$. Also, the group $\widetilde{KO}(N^n(m))$ is generated by α_i $(i = 0, 1)$ and $2^{\varepsilon(i)}\beta^i$ $(1 \leqq i \leqq 2^{m-1})$ by Theorem 3.5. We see easily that $(\Psi^k - 1)\alpha_i = 0$ $(i = 0, 1)$ if $k \equiv 1 \bmod 2$ by [1, Th. 5.1], (2.4), Proposition 2.3, (3.2) and the naturality of the Adams operations with respect to ξ in (3.1). Therefore, Ker J is generated by the elements

$$(\Psi^{2k+1} - 1)2^{\varepsilon(i)}\beta^i \qquad (k \in Z, i \geqq 1),$$

and so Ker $J = \xi j^*(W)$ holds by Remark 4.10. q.e.d.

5. Some relations in $\tilde{J}(N^n(m))$

For any non-negative integers a, b, and s with $(a, b) \neq (0, 0)$, consider the integers c_i $(i \geqq 0)$ satisfying

$$\text{(5.1)} \qquad (x + x^{-1} - 2)^a(x^{2^s} + x^{-2^s} - 2)^b = c_0 + \textstyle\sum_{i \geqq 1} c_i(x^i + x^{-i})$$

in the ring $Z[x, x^{-1}]$. Then, we have the following

LEMMA 5.2. (i) $c_i = \sum_{l=0}^{2b}(-1)^{l+a+i+2^s(b-l)}\binom{2a}{2^s(b-l)+a+i}\binom{2b}{l}$.

(ii) $c_0 + 2\sum_{i\geqq 1} c_i = 0,$

$c_0 + 2(\sum_{j\geqq 1} c_{2j} - \sum_{j\geqq 1} c_{2j-1}) = (-4)^a(2(-1)^{2^s} - 2)^b,$

$c_0 + 2\sum_{j\geqq 1} c_{2^v j} = 0 \qquad (b \geqq 1 \textit{ and } s \geqq v \geqq 0).$

PROOF. (i) follows from the fact that c_i is the coefficient of $x^{a+2^s b+i}$ in $x^{a+2^s b}(x + x^{-1} - 2)^a(x^{2^s} + x^{-2^s} - 2)^b = (x-1)^{2a}(x^{2^s} - 1)^{2b}$.

(ii) Set $x = 1$ (resp. $x = -1$) in (5.1). Then the first (resp. second) equality follows. Also, consider (5.1) modulo $x^{2^s} - 1$, and compair the constant terms of both sides. Then, the last equality is easily seen. q.e.d.

Let $v \geqq 0$, and define the integers

$$\theta(a, b; s, v) = \sum_{j\geqq 0} c_{2^v(2j+1)}, \tag{5.3}$$

where c_i are the integers in (5.1). Then, we have

LEMMA 5.4. *$\theta(a, b; s, v) = 0$ if $b \geqq 1$ and $s > v$, or $a + 2^s b < 2^v$, and*

$$\theta(a, b; s, 0) = (-4)^{a-1}(2(-1)^{2^s} - 2)^b.$$

PROOF. In case $b \geqq 1$ and $s > v$, by Lemma 5.2 (ii)

$$\sum_{j\geqq 1} c_{2^{v+1}j} = -c_0/2 = \sum_{j\geqq 1} c_{2^v j}.$$

Also, in case $a + 2^s b < 2^v$, $c_{2^v j} = 0$ $(j \geqq 1)$ by the definition of c_i in (5.1). Therefore, $\theta(a, b; s, v) = 0$ in these cases. The second equality follows from the first two equalities of Lemma 5.2 (ii) and (5.3). q.e.d.

Consider the elements

$$\gamma_0 = J(2\beta(0)),\ \gamma_v = J(\beta(v))\ (v \geqq 1) \text{ in } \tilde{J}(N^n(m)). \tag{5.5}$$

Then, we have the following

LEMMA 5.6. *For any non-negative integers a, b and s with $a + 2^s b > 0$, the relation*

$$J(2^{\varepsilon(a+2^s b)}\beta^a\beta(s)^b) = \sum_{v=0}^{m-1} 2^{\varepsilon(a+2^s b)-\varepsilon(2^v)}\theta(a, b; s, v)\gamma_v$$

holds in $\tilde{J}(N^n(m))$, where $\varepsilon(i)$ is the integer in (1.4).

PROOF. First, we notice that the coefficient of γ_v in the right hand side is an integer, since $\theta(a, b; s, v) \equiv 0 \bmod 2$ if $\varepsilon(a + 2^s b) = 0$ by Lemma 5.4.

Put $e = 2^{\varepsilon(a+2^s b)}$, and consider the i^*-image of $e\zeta(0)^a\zeta(s)^b$ in Lemma 4.5. Then, we have

$$
\begin{aligned}
i^*(e\zeta(0)^a\zeta(s)^b) &= e(x+x^{-1}-2)^a(x^{2^s}+x^{-2^s}-2)^b && \text{(by Lemma 4.5)}\\
&= e(c_0+\textstyle\sum_{i\geqq 1}c_i(x^i+x^{-i})) && \text{(by (5.1))}\\
&= e\textstyle\sum_{i\geqq 1}c_i(x^i+x^{-i}-2) && \text{(by Lemma 5.2 (ii))}\\
&= e\textstyle\sum_{v\geqq 0}\sum_{j\geqq 0}c_{2^v(2j+1)}(x^{2^v(2j+1)}+x^{-2^v(2j+1)}-2)\\
&\equiv \textstyle\sum_{v\geqq 0}e(\sum_{j\geqq 0}c_{2^v(2j+1)})(x^{2^v}+x^{-2^v}-2) \bmod i^*(W)\\
& && \text{(by Lemma 4.8 (i))}\\
&= i^*(\textstyle\sum_{v\geqq 0}e\theta(a,b;s,v)\zeta(v)) && \text{(by (5.3) and Lemma 4.5).}
\end{aligned}
$$

Since i^* is monomorphic, we see that

$$(*)\qquad e\zeta(0)^a\zeta(s)^b\equiv\sum_{v\geqq 0}e\theta(a,b;s,v)\zeta(v) \bmod W \text{ in } \widetilde{RO}(Q_r).$$

Therefore, the desired relation follows from (3.2), Lemmas 4.6, 4.12 and (5.5) by considering the ξj^*-image of $(*)$. q.e.d.

By the above results, we have the following

PROPOSITION 5.7. *$\tilde{J}(N^n(m))$ is generated by*

$$J\alpha_i\ (i=0,1) \quad \textit{and} \quad \gamma_v\ (0\leqq v<m),$$

where $J\alpha_i$ is the J-image of α_i in (3.2) *and γ_v is the element of* (5.5). *Furthermore, $J\colon \widetilde{KO}(N^n(2))\cong\tilde{J}(N^n(2))$, and the relations between these generators for $m\geqq 3$ are given by the J-images*

$$(5.7.1)\qquad 2^{n+2-\varepsilon(n)}J\alpha_0=0,\quad 2^{n+2-\varepsilon(n)}J\bar{\alpha}_1=0,$$

$$(5.7.2)\qquad u(i)J(\delta_i)=0\qquad (1\leqq i\leqq N=\min\{2^{m-1},n\}),$$

$$(5.7.3)\qquad J(2^{\varepsilon(i)}\beta^i)=0\qquad (N<i\leqq 2^{m-1})$$

of the relations (3.5.1–3) *in $\widetilde{KO}(N^n(m))$. Here, the left hand sides of* (5.7.1–3) *can be written by $J\alpha_0$, $J\alpha_1$ and γ_v $(0\leqq v<m)$ by using Lemma* 4.6 *and the definition of $\bar{\alpha}_1$ in* (3.4).

PROOF. By Theorem 3.5, $\widetilde{KO}(N^n(m))$ is an abelian group generated by the elements

$$\alpha_0,\alpha_1 \quad\text{and}\quad 2^{\varepsilon(i)}\beta^i \qquad (1\leqq i\leqq 2^{m-1})$$

with the relations (3.5.1–3). Furthermore, by Remark 3.6, the subgroup generated by $2^{\varepsilon(i)}\beta^i$ $(1\leqq i\leqq 2^{m-1})$ coinsides with the one generated by

$$2^{\varepsilon(2^v)}\beta(v)\beta(V)\ (0\leqq v<v_1<\cdots<v_t<m-1,\ t\geqq 1) \quad\text{and}$$
$$2^{\varepsilon(2^v)}\beta(v)\ (0\leqq v<m);$$

and it contains Ker J, which is generated by

$$2^{\varepsilon(2^v)}\{\beta(v)\beta(V)-(-2)^t\beta(v_t)\} \qquad (0 \leqq v < v_1 < \cdots < v_t < m-1,\ t \geqq 1)$$

and is 0 if $m=2$ by Lemma 4.12, where $\beta(V)=\beta(v_1)\cdots\beta(v_t)$ for $V=(v_1,\dots,v_t)$. Thus, we see the proposition for $\tilde{J}(N^n(m))=\widetilde{KO}(N^n(m))/\mathrm{Ker}\,J$ by (5.5), Remark 3.6 and Lemma 5.6. q.e.d.

Remark 5.8. *Especialy for* $\tilde{J}(N^0(m))$, *the relations* (5.7.1–3) *are written as follows*:

$$2^2 J\alpha_0 = 0,\ 2^2 J\alpha_1 = 0 \quad \textit{and} \quad \gamma_v = 0\ (0 \leqq v < m).$$

In fact, the first two relations are the ones of (5.7.1), and $\gamma_v = 0$ $(0 \leqq v < m)$ are equivalent to (5.7.3) by the definitions of γ_v in (5.5) and $\beta(v)$.

We notice that there hold the relations

$$\text{(5.9)} \qquad 2^{m-s-1+a_s+\varepsilon(n)}\gamma_s = 0 \qquad (1 \leqq s < m) \quad \text{in } \tilde{J}(N^n(m)),$$

where a_s is the integer in (3.4). In fact, (5.9) is the J-images of the relations

$$2^{m-s-1+a_s+\varepsilon(n)}\beta(s) = 0 \qquad (1 \leqq s < m) \quad \text{in } \widetilde{KO}(N^n(m))$$

by [10, Lemmas 5.1 and 8.1]. In §7, we use these relations to represent the left hand sides of (5.7.1–3) by $J\alpha_0$, $J\alpha_1$ and γ_v $(0 \leqq v < m)$.

6. Some preliminary lemmas for binomial coefficients

In this section, we prepare some properties about the integers $\theta(a, b; s, v)$ in (5.3).

Let $d > 0$ and $v \geqq 0$, and define the integers

$$\text{(6.1)} \qquad X(d, v) = \sum_{j\in Z}(-1)^{j(2^v+1)}\binom{2d}{d+2^v j},$$

$$\text{(6.2)} \qquad Y(d, v) = \sum_{j\in Z}\binom{2d-1}{d+2^v(2j+1)}.$$

Then, we have

LEMMA 6.3. $\theta(d, 1; v, v) = (-1)^d X(d, v)$, $\theta(d, 0; s, v) = (-1)^{d+2^v} Y(d, v)$.

PROOF. By (5.3) and Lemma 5.2 (i), we see that

$$\theta(d, 1; v, v) = (-1)^d\left\{\binom{2d}{d+2^{v+1}+2^{v+1}j}\right.$$

$$+ 2(-1)^{2^v+1}\binom{2d}{d+2^v+2^{v+1}j} + \binom{2d}{d+2^{v+1}j}\Big\}.$$

Since $\binom{2d}{d+2^{v+1}+2^{v+1}j}=\binom{2d}{d-2^v(2j+2)}$ and $\binom{2d}{d+2^v+2^{v+1}j}=\binom{2d}{d-2^v(2j+1)}$, the right hand side of the above equality is equal to

$$(-1)^d \sum_{j\in Z}(-1)^{j(2^v+1)}\binom{2d}{d+2^v j}.$$

Thus, we have the first equality by (6.1). For the integer $\theta(d, 0; s, v)$, we see also that

$$\theta(d, 0; s, v) = (-1)^{d+2^v}\sum_{j\geqq 0}\binom{2d}{d+2^v(2j+1)}$$

by (5.3) and Lemma 5.2 (i). Here, we notice that

$$\binom{2d}{d+2^v(2j+1)} = \binom{2d-1}{d+2^v(2j+1)} + \binom{2d-1}{d-1+2^v(2j+1)}$$
$$= \binom{2d-1}{d+2^v(2j+1)} + \binom{2d-1}{d-2^v(2j+1)}.$$

Therefore, the second equality follows. q.e.d.

From now on, for any integer n, we denote by

$$\nu(n) = \nu_2(n) \quad \text{and} \quad \mu(n) = \mu_2(n)$$

the exponent of 2 in the prime power decomposition of n and the number of terms in the dyadic expansion of n, respectively. Also, we regard $\mu(0) = 0$.

Now, we state some properties of the integers $X(d, v)$.

LEMMA 6.4 (cf. [8, Lemmas 4.9 and 4.15]). *Put*

$$X(d, v) = 2^{\nu(d,v)}\xi(d, v) \qquad (\xi(d, v)\colon \textit{odd integer})$$

for the integer $X(d, v)$ $(d > 0, v \geqq 0)$ *in* (6.1). *Then,*

(i) $\nu(d, 0) = 2d$, $\xi(d, 0) = 1$;
(ii) $\nu(d, v) = [d/2^{v-1}] + \mu(d - 2^{v-1}[d/2^{v-1}])$ $(v > 0)$;
(iii) $\xi(2^{s-1}, v) = 2^{-1}\binom{2^s}{2^{s-1}}$ *for* $v \geqq s$, *and* $\xi(2^{s-1}, s) = 1$ *if* $s = 1$, $\equiv 3$ mod 8 *if* $s \geqq 2$;

(iv) $\xi(2^{s-1}, s-1) = 2^{-2}\left\{\binom{2^s}{2^{s-1}} - 2\right\} \equiv 1 \bmod 4$ *for* $s \geqq 2$.

LEMMA 6.5 (cf. [8, Lemma 4.16]). *Let* $0 < d < 2^s$. *Then*

$$\textstyle\sum_{i=0}^{s}(-1)^{2^i}2^{-i}X(d, s-i) = 0.$$

In the rest of this section, we shall study the divisibility of the integer $Y(d, v)$ in (6.2) by the powers of 2.

Let $d > 0$ and $v \geqq 0$. For the integers defined by

$$\bar{X}(d, v) = \textstyle\sum_{j \in Z}(-1)^j\binom{2d}{d+2^v j}, \tag{6.6}$$

$$\bar{Y}(d, v) = \textstyle\sum_{j \in Z}\binom{2d}{d+2^v j}, \tag{6.7}$$

we have the following

LEMMA 6.8. (i) $\bar{X}(d, v) + \bar{Y}(d, v) = 2\bar{Y}(d, v+1)$, $\bar{X}(d, v) = X(d, v)$ $(v \geqq 1)$, $\bar{X}(d, 0) = 0$, $\bar{X}(d, 1) = 2^d$.

(ii) $\bar{Y}(d, 0) = 2^{2d}$, $\bar{Y}(d, 1) = 2^{2d-1}$, $\bar{Y}(d, 2) = 2^{d-1}(1 + 2^{d-1})$.

PROOF. (i) The first three equalities are trivial. The last equality is easily seen by the following equality:

$$2^d = (1+\boldsymbol{i})^{2d}/\boldsymbol{i}^d = \textstyle\sum_{j \in Z}(-1)^j\binom{2d}{d+2j} + \sum_{j \in Z}(-1)^j\binom{2d}{d+2j+1}\boldsymbol{i}.$$

(ii) The first equality is obvious. The last two equalities are the immediate consequences of (i). q.e.d.

LEMMA 6.9. *Let* $v \geqq 0$ *and* $d \geqq 2^{v+1}$. *Then*

$$[d/2^v] + \mu(d - 2^v[d/2^v]) < [d/2^{v-1}] + \mu(d - 2^{v-1}[d/2^{v-1}]) - 1.$$

PROOF. The assertion is trivial in case $v = 0$. Suppose $v \geqq 1$ and put $d = 2^{i_1} + \cdots + 2^{i_t}$ $(i_1 > \cdots > i_t)$. Then, $\mu(d - 2^v[d/2^v]) = \mu(d - 2^{v-1}[d/2^{v-1}])$ and $[d/2^{v-1}] = 2[d/2^v]$ hold if $i_j \neq v-1$ for any j. On the other hand, $\mu(d - 2^v[d/2^v]) = \mu(d - 2^{v-1}[d/2^{v-1}]) + 1$ and $[d/2^{v-1}] = 2[d/2^v] + 1$ hold if $i_j = v-1$ for some j. Therefore, we obtain the desired inequality. q.e.d.

LEMMA 6.10. *Let* $d > 0$ *and* $v \geqq 0$. *Then*

$$\nu(\bar{Y}(d, v+1)) = [d/2^{v-1}] + \mu(d - 2^{v-1}[d/2^{v-1}]) - \varepsilon(d, v),$$

where $\varepsilon(d, v) = 0$ *if* $d < 2^v$, $= 1$ *if* $d \geqq 2^v$.

PROOF. If $d < 2^v$, then $\bar{Y}(d, v+1) = \binom{2d}{d}$, and so the desired equality follows from the equality $\nu_2(\binom{2d}{d}) = \mu(d)$ given in [8, Lemma 4.8]. Hence, we shall prove the equality

$$(*)_v \qquad \nu(\bar{Y}(d, v+1)) = [d/2^{v-1}] + \mu(d - 2^{v-1}[d/2^{v-1}]) - 1 \qquad \text{for } d \geqq 2^v$$

by the induction on v.

If $v = 0$, $(*)_0$ holds by Lemma 6.8 (ii). Thus, we assume that $(*)_v$ holds, and prove $(*)_{v+1}$. Suppose $d \geqq 2^{v+1}$. Then, $\nu(\bar{Y}(d, v+1)) = [d/2^{v-1}] + \mu(d - 2^{v-1}[d/2^{v-1}]) - 1$ by the inductive assumption, and $\nu(\bar{X}(d, v+1)) = \nu(X(d, v+1)) = [d/2^v] + \mu(d - 2^v[d/2^v])$ by Lemmas 6.8 (i) and 6.4 (ii). On the other hand, by Lemma 6.8 (i), $\bar{Y}(d, v+2) = \{\bar{Y}(d, v+1) + \bar{X}(d, v+1)\}/2$ holds. Hence, we see that

$$\begin{aligned}\nu(\bar{Y}(d, v+2)) &= \min\{\nu(\bar{Y}(d, v+1)), \nu(\bar{X}(d, v+1))\} - 1 \\ &= [d/2^v] + \mu(d - 2^v[d/2^v]) - 1\end{aligned}$$

by the above argument and Lemma 6.9. Thus, $(*)_{v+1}$ holds. q.e.d.

LEMMA 6.11. *Let $d > 0$ and $v \geqq 0$. Then, we have the following*:

$Y(d, v) = 0 \quad (d < 2^v)$;

$Y(d, 0) = 2^{2d-2} \quad (d \geqq 1), \quad Y(d, 1) = 2^{d-2}(2^{d-1} - 1) \quad (d \geqq 2)$;

$\nu(Y(d, v)) = [d/2^{v-1}] + \mu(d - 2^{v-1}[d/2^{v-1}]) - 2 \quad (d \geqq 2^v)$.

PROOF. By (6.7),

$$\bar{Y}(d, v) - \bar{Y}(d, v+1) = \sum_{j \in Z} \binom{2d}{d + 2^v(2j+1)}$$

holds, and

$$\begin{aligned}\binom{2d}{d + 2^v(2j+1)} &= \binom{2d-1}{d + 2^v(2j+1)} + \binom{2d-1}{d - 1 + 2^v(2j+1)} \\ &= \binom{2d-1}{d + 2^v(2j+1)} + \binom{2d-1}{d - 2^v(2j+1)}.\end{aligned}$$

Therefore, we see easily that

$$(*) \qquad \bar{Y}(d, v) - \bar{Y}(d, v+1) = 2Y(d, v)$$

by the definition of $Y(d, v)$ in (6.2).

The first equality follows from Lemmas 5.4 and 6.3. The second and the third equalities are the immediate consequences of Lemma 6.8 (ii) and (*) above.

Since the last equality for $v=0$ follows from the second one, we prove the last equality under the assumption $d \geqq 2^v$ $(v \geqq 1)$. By Lemma 6.10, we have

$$\nu(\bar{Y}(d, v)) = [d/2^{v-2}] + \mu(d - 2^{v-2}[d/2^{v-2}]) - 1,$$
$$\nu(\bar{Y}(d, v+1)) = [d/2^{v-1}] + \mu(d - 2^{v-1}[d/2^{v-1}]) - 1.$$

On the other hand, we see that $\nu(\bar{Y}(d, v)) > \nu(\bar{Y}(d, v+1))$ by Lemma 6.9. Therefore, by (*) above, we have

$$\nu(Y(d, v)) = \nu(\bar{Y}(d, v+1)) - 1,$$

which is the desired equality. q.e.d.

7. Proof of Theorem 1.7 (i), (ii) and (iii) (a)

Throughout this section, we assume that $m \geqq 3$. By using Lemma 5.6 and the results obtained in the previous sections, $J(2^{\varepsilon(i)}\beta^i)$ in (5.7.3) and $u(i)J(\delta_i)$ in (5.7.2) can be represented by γ_v $(0 \leqq v \leqq m-1)$ as follows:

LEMMA 7.1. *If* $2^t \leqq i < 2^{t+1}$, *then*

$$(-1)^{i+1}J(2^{\varepsilon(i)}\beta^i) = 2^{2i-3+\varepsilon(i)}\gamma_0 - \textstyle\sum_{v=1}^{t} 2^{\varepsilon(i)}Y(i, v)\gamma_v \quad \text{in} \quad \tilde{J}(N^n(m)),$$

where $Y(i, v)$ *is the integer given in* (1.6) *or* (6.2).

PROOF. By Lemmas 5.6, 5.4 and 6.3, we see that

$$\begin{aligned} J(2^{\varepsilon(i)}\beta^i) &= \textstyle\sum_{v=0}^{t} 2^{\varepsilon(i)-\varepsilon(2^v)}\theta(i, 0; s, v)\gamma_v \\ &= (-1)^{i+1}2^{\varepsilon(i)-1}Y(i, 0)\gamma_0 + (-1)^i\textstyle\sum_{v=1}^{t} 2^{\varepsilon(i)}Y(i, v)\gamma_v. \end{aligned}$$

Here, $Y(i, 0) = 2^{2i-2}$ by Lemma 6.11. Thus, we have the desired relation. q.e.d.

In the following lemma, we use the relations

(7.2) $$2^{m-s-1+a_s}\gamma_s = 0 \quad (1 \leqq s < m) \quad \text{(cf. (5.9))}$$

in $\tilde{J}(N^n(m))$ for even n.

LEMMA 7.3. *Let* $2^s + d \leqq N = \min\{2^{m-1}, n\}$, $0 < d < 2^s$, $1 \leqq s \leqq m-2$ *and* $0 \leqq v \leqq s$. *Then,*

$$J(2^{m-s-4+2^{s+1}-va(i)}\beta^d\beta(v)) = (-1)^d 2^{m-s-4+2^{s+1}-va(i)-\varepsilon(2^v)}X(d, v)\gamma_v,$$

in $\tilde{J}(N^n(m))$ *for even n, where* $a(i)$ *is the integer given in* (3.4).

PROOF. By the assumption $2^s + d \leqq n$ and the definitions of a_{s+1}, b_{s+1} and $a(i)$ in (3.4), we see easily that $a_{s+1} \geqq 1$ if $2d \leqq b_{s+1}$ and $a_{s+1} \geqq 2$ otherwise. Thus, we have

$$(*) \qquad 2a(i) \geqq 2 + a_{s+1}.$$

Lemma 5.6 implies that the left hand side of the desired relation is equal to

$$\sum_{u=0}^{m-1} 2^{m-s-4+2^{s+1-v}a(i)-\varepsilon(2^u)}\theta(d, 1; v, u)\gamma_u$$
$$= \sum_{u=v}^{s} 2^{m-s-4+2^{s+1-v}a(i)-\varepsilon(2^u)}\theta(d, 1; v, u)\gamma_u \qquad \text{(by Lemma 5.4)}.$$

Here, we notice that

$$(7.4) \qquad a_v = 2^{t-v}a_t + [b_t/2^v] \leqq 2^{t-v}a_t + 2^{t-v} - 1 \qquad (t \geqq v)$$

holds by the definitions of a_t and b_t. Then, if $u > v$, we have

$$m - s - 4 + 2^{s+1-v}a(i) > m - s - 3 + a_u + 2^{s+1-u} \geqq m - u - 1 + a_u$$

by (∗) and (7.4), and so $2^{m-s-4+2^{s+1-v}a(i)}\gamma_u = 0$ $(u > v)$ by (7.2). On the other hand, $\theta(d, 1; v, v) = (-1)^d X(d, v)$ by Lemma 6.3. Thus, we have the lemma. q.e.d.

Now, consider the integers d_i and e_i satisfying

$$(7.5.1) \qquad (x + x^{-1} - 2)^{d-1}(x^2 + x^{-2} - 2)\prod_{t=0}^{s-1}(x^{2^t} + x^{-2^t}) = d_0 + \sum_{i\geqq 1} d_i(x^i + x^{-i}) \quad (0 < d < 2^s),$$

$$(7.5.2) \qquad (x + x^{-1} - 2)^{d-2}(x^4 + x^{-4} - 2)\prod_{t=1}^{s-1}(x^{2^t} + x^{-2^t}) = e_0 + \sum_{i\geqq 1} e_i(x^i + x^{-i}) \quad (1 < d < 2^s)$$

in the ring $Z[x, x^{-1}]$. Then, we see easily that $d_0 = 0 = e_0$ and $d_{2^v j} = 0 = e_{2^v j}$ if $v \geqq s+1$ and $j \geqq 1$, since left hand sides of (7.5.1–2) are equal to

$$(x-1)^{2d-1}(x+1)(x^{2^{s+1}} - 1)/x^{2^s+d} \quad \text{and} \quad (x-1)^{2d-4}(x^4-1)(x^{2^{s+1}} - 1)/x^{2^s+d},$$

respectively. Also, consider (7.5.1–2) modulo $x^{2^v} - 1$ in case $v < s+1$, and compair the constant terms of both sides. Then, we see that

$$d_0 + 2\sum_{j\geqq 1} d_{2^v j} = 0 = e_0 + 2\sum_{j\geqq 1} e_{2^v j}.$$

Therefore, we have

$$(7.6) \qquad d_0 = 0 = e_0 \quad \text{and} \quad \sum_{j\geqq 0} d_{2^v j} = 0 = \sum_{j\geqq 0} e_{2^v j} \qquad (v \geqq 0).$$

By using (7.6), we can prove the following lemma by the similar way to the proof of Lemma 5.6.

LEMMA 7.7. *In $\tilde{J}(N^n(m))$, the following relations hold*:

$$J(2^{\varepsilon(d)}\beta^{d-1}\beta(1)\prod_{t=0}^{s-1}(2+\beta(t))=0 \qquad \textit{for } 0<d<2^s,$$
$$J(2^{\varepsilon(d)}\beta^{d-2}\beta(2)\prod_{t=1}^{s-1}(2+\beta(t))=0 \qquad \textit{for } 1<d<2^s.$$

PROOF. Since the second relation can be proved in the same manner as the proof of the first one, we shall prove only the first relation.

Consider the i^*-mage of $2^{\varepsilon(d)}\zeta(0)^{d-1}\zeta(1)\prod_{t=0}^{s-1}(2+\zeta(t))$ in Lemma 4.5. Then, this is equal to

$$\begin{aligned}
&2^{\varepsilon(d)}(x+x^{-1}-2)^{d-1}(x^2+x^{-2}-2)\prod_{t=0}^{s-1}(x^{2^t}+x^{-2^t})\\
&=2^{\varepsilon(d)}\sum_{i\geqq 1}d_i(x^i+x^{-i}) \qquad \text{(by (7.6))}\\
&=2^{\varepsilon(d)}\sum_{v\geqq 0}\sum_{j\geqq 0}d_{2^v(2j+1)}(x^{2^v(2j+1)}+x^{-2^v(2j+1)})\\
&\equiv 2^{\varepsilon(d)}\sum_{v\geqq 0}(\sum_{j\geqq 0}d_{2^v(2j+1)})(x^{2^v}+x^{-2^v}) \bmod i^*(W) \qquad \text{(by Lemma 4.8 (i))}.
\end{aligned}$$

Here, $\sum_{j\geqq 0}d_{2^v(2j+1)}=\sum_{j\geqq 0}d_{2^vj}-\sum_{j\geqq 0}d_{2^{v+1}j}=0$ by (7.6). Hence, we see that

$$(*) \qquad 2^{\varepsilon(d)}\zeta(0)^{d-1}\zeta(1)\prod_{t=0}^{s-1}(2+\zeta(t))\equiv 0 \bmod W \quad \text{in} \quad \widetilde{RO}(Q_r),$$

since i^* is monomorphic by Lemma 4.5. Therefore, the desired relation follows from (3.2), Lemmas 4.6 and 4.12 by considering the ξj^*-image of $(*)$, where ξ and j^* are the homomorphisms in (3.1) and (4.1), respectively. q.e.d.

Now, we are ready to prove Theorem 1.7 (i), (ii) and (iii)(a).

PROOF OF THEOREM 1.7 (i), (ii) AND (iii)(a). Based on Proposition 5.7, we complete the proof of Theorem 1.7 (i), (ii) and (iii)(a) by combining (3.4), (7.2), Lemmas 7.1, 7.3, 7.7 and Remark 5.8. q.e.d.

8. Proof of Theorem 1.8

In this section, we assume that $m\geqq 3$ and n is a positive even integer with $n\geqq 4$ unless otherwise stated.

Let l be an integer such that

$$(8.1) \qquad n\geqq 2^l \quad \text{and} \quad 2\leqq l\leqq m-1.$$

Then, the following relations hold in $\tilde{J}(N^n(m))$ by Theorem 1.7 (iii)(a) and (7.4):

$$(8.2) \qquad 2^{n+2}J\alpha_0=0, \quad 2^{n+2}J\alpha_1=0,$$

$$(8.3) \qquad 2^{m-s-1+a_s-\varepsilon(2^s)}\gamma_s=0 \qquad (0\leqq s<m),$$

(8.4) $\qquad \sum_{v=0}^{s} 2^{m-s-3+a_v-[b_s/2^v]+2^{s-v}-\varepsilon(2^v)}\gamma_v = 0 \qquad (2 \leqq s \leqq l),$

(8.5.1) $\qquad \sum_{v=0}^{s}(-1)^{2^{s-v}}2^{m-s-4+a_v-[b_{s+1}/2^v]+2^{s+1-v}+\varepsilon(d)-\varepsilon(2^v)}X(d, v)\gamma_v = 0$

$$(1 \leqq s \leqq l-1,\ 0 < 2d \leqq b_{s+1}),$$

(8.5.2) $\qquad \sum_{v=0}^{s}(-1)^{2^{s-v}}2^{m-s-4+a_v-[b_{s+1}/2^v]+\varepsilon(d)-\varepsilon(2^v)}X(d, v)\gamma_v = 0$

$$(1 \leqq s \leqq l-1,\ b_{s+1} < 2d < 2^{s+1}),$$

where $X(d, v) = 2^{\nu(d,v)}\xi(d, v)$ is the integer in Lemma 6.4.

Here, we notice the following

REMARK 8.6. *In case* $n = 2^{m-2}a$ $(a \geqq 2)$, *the relations* (1.7.1–5) *in Theorem* 1.7(iii)(a) *are equivalent to* (8.2–4) *and* (8.5.2) *for* $l = m-1$ *above.*

In fact, $n \geqq 2^{m-1}$ and $b_{s+1} = 0$ $(1 \leqq s \leqq m-2)$ hold, and so the above remark follows.

Now, we shall reduce (8.3–4) and (8.5.2) to more simple ones under some condition in the following lemmas.

LEMMA 8.7. *In addition to* (8.1), *assume* $b_{t+1} = 0$ *for some integer* t *with* $1 \leqq t \leqq l-1$. *Then,* (8.3) *and* (8.5.2) *for* $s = t$ *imply*

(8.7.1) $\qquad 2^{m-3+a_1}\gamma_1 = \pm 2^{m-3+a_0}\gamma_0 \qquad$ *if* $t = 1$,

(8.7.2) $\qquad 2^{m-t-3+a_t}\gamma_t + 2^{m-t-2+a_{t-1}}\gamma_{t-1} + 2^{m-t+a_{t-2}-\varepsilon(2^{t-2})}\gamma_{t-2} = 0 \qquad$ *if* $t \geqq 2$.

PROOF. Let $t = 1$. Then, the relation (8.7.1) follows from (8.5.2) for $s = 1 = d$ and (8.3), since $X(1, 1) = 2$ and $X(1, 0) = 2^2$ by Lemma 6.4.

Let $t \geqq 2$, and consider (8.5.2) for $s = t$ and $d = 2^{t-1}$:

(*) $\qquad \sum_{v=0}^{t}(-1)^{2^{t-v}}2^{m-t-4+a_v-\varepsilon(2^v)+\nu(2^{t-1},v)}\xi(2^{t-1}, v)\gamma_v = 0.$

Here, $\xi(2^{t-1}, v)$ is odd and

$$\nu(2^{t-1}, v) = 2^{t-v},\ \xi(2^{t-1}, v) \equiv \begin{cases} -1 \bmod 4 & \text{if } v = t, \\ \ \ 1 \bmod 4 & \text{if } v = t-1, \end{cases}$$

by Lemma 6.4. Thus, (8.3) and (*) imply (8.7.2), since $2^k > k+3$ if $k \geqq 3$.

q.e.d.

LEMMA 8.8. *Under the same assumption as Lemma* 8.7, (8.3) *and* (8.5.2) *for* $1 \leqq s \leqq t$ *imply*

(8.8.1) $\qquad 2^{m-t-2+a_1}\gamma_1 = 2^{m-t-2+a_0}\gamma_0 \qquad$ *if* $t \geqq 1$,

(8.8.2) $\qquad 2^{m-t-3+a_v}\gamma_v = 2^{m-t-2+a_{v-1}}\ \gamma_{v-1}\ (2 \leqq v \leqq t) \qquad$ *if* $t \geqq 2$.

PROOF. The assumption $b_{t+1} = 0$ implies $b_{s+1} = 0$ $(1 \leqq s \leqq t)$. Therefore, by the above lemma, there hold the relations

(8.7.1)′ $$2^{m-3+a_1}\gamma_1 = \pm 2^{m-3+a_0}\gamma_0,$$

(8.7.2)′ $$2^{m-s-3+a_s}\gamma_s + 2^{m-s-2+a_{s-1}}\gamma_{s-1} + 2^{m-s+a_{s-2}-\varepsilon(2^{s-2})}\gamma_{s-2} = 0 \qquad (1 < s \leqq t).$$

Thus, (8.8.1) for $t = 1$ is equal to (8.7.1)′.

Let $t = 2$. Then, (8.8.2) for $v = t$ follows easily from (8.7.1–2)′. Consider (8.5.2) for $s = 2$ and $d = 1$. Then

$$-2^{m-4+a_2}\gamma_2 + 2^{m-4+a_1}\gamma_1 + 2^{m-4+a_0}\gamma_0 = 0,$$

since $X(1, 2) = 2 = X(1.1)$ and $X(1, 0) = 2^2$ by Lemma 6.4. Thus, (8.8.1) for $t = 2$ is easily seen from (8.8.2) for $t = 2$ and the above relation. Therefore, (8.8.1–2) hold for $t = 2$.

Now, let $t \geqq 3$, and assume inductively that (8.8.1–2) hold for $t - 1$ ($\geqq 2$), i.e., that

(8.8.1)′ $$2^{m-t-1+a_1}\gamma_1 = 2^{m-t-1+a_0}\gamma_0,$$

(8.8.2)′ $$2^{m-t-2+a_v}\gamma_v = 2^{m-t-1+a_{v-1}}\gamma_{v-1} \qquad (2 \leqq v < t).$$

Then, (8.8.2) for $v = t$ follows easily from (8.7.2)′ for $s = t$, $2 \times$ (8.8.2)′ for $v = t - 1$ and (8.3) for $s = t - 1$. Let $2 \leqq k < t$ and assume inductively that (8.8.2) holds for any v with $k < v \leqq t$. Consider the relation

(*) $$\sum_{v=0}^{t}(-1)^{2^{t-v}}2^{m-t-4+a_v-\varepsilon(2^v)+\nu(d,v)}\xi(d, v)\gamma_v = 0$$

in (8.5.2) for $s = t$ and even integer d with $2^{k-1} \leqq d < 2^k$. Then, by (6.1), Lemma 6.4 and the condition $2^{k-1} \leqq d < 2^k$, we see that $\nu(d, v) = \nu(d, k)$ and $\xi(d, v) = \xi(d, k)$ for $k \leqq v \leqq t$, since $X(d, v) = \binom{2d}{d} = X(d, k)$. Therefore,

(a) $\sum_{v=k}^{t}$ in (*) is equal to $-2^{m-t-4+a_k+\nu(d,k)}\xi(d, k)\gamma_k$

by the inductive assumption (8.8.2) for $k < v \leqq t$. Furthermore, if $v < k$, then $\nu(d, v) \geqq 2^{k-v}$ ($\geqq k - v + 1$) by Lemma 6.4, and hence

$$m - t - 4 + a_v + \nu(d, v) - \varepsilon(2^v) \geqq \begin{cases} m - t - 3 + k - v + a_v & \text{if } 1 \leqq v < k, \\ m - t - 1 + a_0 & \text{if } v = 0. \end{cases}$$

Therefore,

(b) $\sum_{v=0}^{k-1}$ in (*) is equal to

$$\sum_{v=0}^{k-1} 2^{m-t-4+a_{k-1}-(k-1-v)+\nu(d,v)}\xi(d, v)\gamma_{k-1} \qquad \text{(by (8.8.1–2)′)}$$

$$= 2^{m-t-3+a_{k-1}+\nu(d,k)}\xi(d, k)\gamma_{k-1} \qquad \text{(by Lemma 6.5).}$$

Thus, (*) is written as

$$(**)\qquad 2^{m-t-4+a_k+\nu(d,k)}\gamma_k = 2^{m-t-3+a_{k-1}+\nu(d,k)}\gamma_{k-1}$$

for any even integer d with $2^{k-1} \leqq d < 2^k$, since $\xi(d, k)$ is odd. (**) for $d = 2^{k-1}$ is (8.8.2) for $v = k$, since $\nu(2^{k-1}, k) = 1$ by Lemma 6.4 (ii). Therefore, (8.8.2) holds for any v with $1 \leqq v \leqq t$ by the induction on v. Consider (8.5.2) for $s = t$ and $d = 1$. Then, we have

$$2^{m-t-2+a_0}\gamma_0 + \sum_{v=1}^{t}(-1)^{2^{t-v}}2^{m-t-2+a_v}\gamma_v = 0,$$

since $X(1, v) = 2^2$ if $v = 0$, $= 2$ if $v \geqq 1$ by Lemma 6.4. On the other hand, we have

$$\sum_{v=1}^{t}(-1)^{2^{t-v}}2^{m-t-2+a_v}\gamma_v = -2^{m-t-2+a_1}\gamma_1$$

by (8.8.2) for t. Therefore, (8.8.1) holds for t. Hence, (8.8.1–2) are shown by the induction on t. q.e.d.

LEMMA 8.9. *In addition to* (8.1), *assume* $b_l = 0$. *Then,* (8.3) *and* (8.4) *for* $s = l$ *imply*

$$2^{m-l-2+a_l}\gamma_l = \pm 2^{m-l-1+a_{l-1}}\gamma_{l-1}.$$

PROOF. By (8.4) for $s = l$, we have

$$\sum_{v=0}^{l} 2^{m-l-3+a_v+2^{l-v}-\varepsilon(2^v)}\gamma_v = 0.$$

If $0 \leqq v \leqq l-2$,

$$m - l - 3 + a_v + 2^{l-v} - \varepsilon(2^v) \geqq m - v - 1 + a_v - \varepsilon(2^v),$$

since $2^{l-v} \geqq l - v + 2$. Therefore, we have the desired result by (8.3).

q.e.d.

LEMMA 8.10. *In addition to* (8.1), *assume* $b_l = 0$. *Then,* (8.3), (8.4) *and* (8.5.2) *are equivalent to* (8.3),

$$(8.10.1)\qquad 2^{m-l-1+a_1}\gamma_1 = 2^{m-l-1+a_0}\gamma_0,$$

$$(8.10.2)\qquad 2^{m-l-2+a_v}\gamma_v = 2^{m-l-1+a_{v-1}}\gamma_{v-1} \qquad (2 \leq v \leq l).$$

PROOF. By Lemma 8.8 for $t = l - 1$ and Lemma 8.9, it is sufficient to show that (8.3) and (8.10.1–2) imply (8.4) and (8.5.2).

Let $2 \leqq s \leqq l$ and assume that (8.3) and (8.10.1–2). Since $b_s = 0$ and $2^{s-v} \geqq s - v + 2$ $(0 \leqq v \leqq s-2)$, we have

$$m - s - 3 + a_v - [b_s/2^v] + 2^{s-v} - \varepsilon(2^v) \geqq m - 1 - v + a_v - \varepsilon(2^v)$$
$$(0 \leqq v \leqq s-2).$$

Therefore, by (8.3), the left hand side of (8.4) is equal to

$$2^{m-s-2+a_s}\gamma_s+2^{m-s-1+a_{s-1}}\gamma_{s-1}=2^{m-s-1+a_s}\gamma_s=0 \qquad \text{(by (8.10.2) and (8.3))}.$$

Hence, (8.4) is shown.

Let $1\leqq s\leqq l-1$ and $2^{k-1}\leqq d<2^k$, and assume (8.3) and (8.10.1–2). Since $b_{s+1}=0$, the left hand side of (8.5.2) is equal to

$$(*) \qquad \sum_{v=0}^{s}(-1)^{2^{s-v}}2^{m-s-4+a_v+\varepsilon(d)-\varepsilon(2^v)+\nu(d,v)}\xi(d,v)\gamma_v.$$

Here, we see that $X(d,v)=\binom{2d}{d}=X(d,k)$ $(k\leqq v\leqq s)$ by (6.1) and the condition $2^{k-1}\leqq d<2^k$, and so $\nu(d,v)=\nu(d,k)\geqq 1$, $\xi(d,v)=\xi(d,k)$ for $k\leqq v\leqq s$ by Lemma 6.4. Therefore, $\sum_{v=k}^{s}$ in $(*)$ is equal to

$$-2^{m-s-4+a_k+\varepsilon(d)+\nu(d,k)}\xi(d,k)\gamma_k$$

by (8.10.2). If $v<k$, $\nu(d,v)\geqq 2^{k-v}\geqq k-v+1$ by Lemma 6.4, and so

$$\begin{aligned} &m-s-4+a_v+\varepsilon(d)-\varepsilon(2^v)+\nu(d,v)\\ &\qquad\geqq\begin{cases} m-s-3+(k-v)+a_v & (1\leqq v<k),\\ m-s-2+a_0 & (v=0).\end{cases}\end{aligned}$$

Thus, by (8.10.1–2), $\sum_{v=0}^{k-1}$ in $(*)$ is equal to

$$\begin{aligned}&\sum_{v=0}^{k-1}2^{m-s-4+a_{k-1}-(k-1-v)+\varepsilon(d)-\varepsilon(2^{k-1})+\nu(d,v)}\xi(d,v)\gamma_{k-1}\\ &=2^{m-s-3+a_{k-1}+\varepsilon(d)-\varepsilon(2^{k-1})+\nu(d,k)}\xi(d,k)\gamma_{k-1} \qquad \text{(by Lemma 6.5)}.\end{aligned}$$

Therefore, $(*)=0$ follows from (8.10.1–2), since $d=1$ if $k=1$, $\nu(d,k)\geqq 1$ and $\xi(d,k)$ is odd. Hence, (8.5.2) is shown. q.e.d.

Proof of Theorem 1.8. Let $n=2^{m-2}a$ $(m\geqq 3,\ a\geqq 2)$. Then, $b_{m-1}=0$. Thus, (8.2–4) and (8.5.2) for $l=m-1$ are equivalent to (8.2), (8.3) and (8.10.1–2) for $l=m-1$ by Lemma 8.10. Furthermore, (8.3) for $s=0$ and (8.10.1–2) for $l=m-1$ are equivalent to (8.3) and (8.10.1–2) for $l=m-1$. Therefore, Theorem 1.8 is proved by Theorem 1.7 and Remark 8.6. q.e.d.

Finally, we notice the following

Lemma 8.11. *If $n=2^{m-2}$ $(m\geqq 3)$, there hold the following relations in $\tilde{J}(N^n(m))$:*

$$2^{a_1}\gamma_1=2^{a_0}q_0\gamma_0,\quad 2^{a_v-1}\gamma_v=2^{a_{v-1}}q_{v-1}\gamma_{v-1} \qquad (2\leqq v\leqq m-1)$$

for some odd integer q_v $(0\leqq v\leqq m-2)$.

Proof. By (1.7.5) in Theorem 1.7(iii)(a), the relation

$$(*) \qquad 2^{2i-3+\varepsilon(i)}\gamma_0 = \sum_{v=1}^{t} 2^{\varepsilon(i)} Y(i, v)\gamma_v \qquad (2^{m-2} < i \leqq 2^{m-1})$$

holds in $\tilde{J}(N^n(m))$, where $2^t \leqq i < 2^{t+1}$. Let $v'(i, v)$ and $\xi'(i, v)$ be the integers such that

$$Y(i, v) = 2^{v'(i,v)}\xi'(i, v) \qquad (\xi'(i, v)\colon \text{odd integer})$$

for $i \geqq 2^v$.

The relation $(*)$ for $i = 2^{m-1}$ is equal to

$$\gamma_{m-1} = 2^{2m-3}\gamma_0 - \sum_{v=1}^{m-2} 2^{2m-v-2}\xi'(i, v)\gamma_v,$$

since $Y(i, m-1) = 1$ by (6.2), and $v'(i, v) = 2^{m-v} - 2$ by Lemma 6.11. Also, $2^{m-v} - 2 \geqq m - 1 - v + a_v$ $(0 \leqq v \leqq m-3)$ holds. Thus, by (1.7.2), we have $\gamma_{m-1} = -2^2\gamma_{m-2}$, which is the desired relation for $v = m-1$.

Consider the case $i = 2^{m-2} + 2^k$ $(0 \leqq k \leqq m-3)$ in $(*)$. Since,

$$v'(i, v) = \begin{cases} a_v - 1 & \text{if } k < v \leqq m-2, \\ a_v + 2^{k-v+1} - 2 & \text{if } 0 \leq v \leq k, \end{cases}$$

by Lemma 6.11, $(*)$ is equal to

$$(**) \qquad \begin{aligned} &2^{a_0 + 2^{k+1} - 3 + \varepsilon(i)}\gamma_0 - \sum_{v=1}^{k} 2^{a_v + 2^{k-v+1} - 2 + \varepsilon(i)}\xi'(i, v)\gamma_v \\ &= \sum_{v=k+1}^{m-2} 2^{a_v - 1 + \varepsilon(i)}\xi'(i, v)\gamma_v. \end{aligned}$$

Moreover, let $k = 0$ in $(**)$. Then, we have

$$(***) \qquad 2^{a_0}\gamma_0 = \sum_{v=1}^{m-2} 2^{a_v}\xi'(i, v)\gamma_v,$$

and so the desired relation $2^{a_1}\gamma_1 = 2^{a_0}q_0\gamma_0$ for $m = 3$ is easily seen, since $\tilde{J}(N^n(m))$ is a 2-group. If $m \geqq 4$, we have

$$2^{a_v}\gamma_v = 2^{a_1 + v - 1}\gamma_1 \qquad (1 \leqq v \leqq m-2)$$

by Lemma 8.10 for $l = m-2$ and Theorem 1.7(iii)(a). Hence, the desired first relation for $m \geqq 4$ follows from $(***)$. Let $m \geqq 4$ and $1 \leqq k \leqq m-3$ in $(**)$. Then, by Lemma 8.10 for $l = m-2$ and Theorem 1.7(iii)(a),

$$\begin{aligned} &2^{a_0 + 2^{k+1} - 3}\gamma_0 = 2^{a_k + 2^{k+1} - 2 - k}\gamma_k, \\ &2^{a_v + 2^{k+1-v} - 2}\gamma_v = 2^{a_k + 2^{k+1-v} - 2 - k + v}\gamma_k \qquad (1 \leqq v \leqq k) \end{aligned}$$

hold. Therefore, by $(**)$, we have

$$\sum_{v=0}^{k} (-1)^{2^v + 1} 2^{a_k + 2^{k+1-v} - (k+2-v)}\xi'(i, v)\gamma_k = \sum_{v=k+1}^{m-2} 2^{a_v - 1}\xi'(i, v)\gamma_v,$$

where $\xi'(i, 0) = 1$. Here, $\sum_{v=0}^{k} (-1)^{2^v+1} 2^{2^{k+1-v} - (k+2-v)}\xi'(i, v)$ is an odd integer, and so the relation

$$(**)_k \qquad 2^{a_k}\gamma_k = \sum_{v=k+1}^{m-2} 2^{a_v-1}p_v\gamma_v \qquad (1 \leqq k \leqq m-3)$$

is obtained for some odd integers p_v. The desired second relation for $v = m-2$ is obtained from $(**)_{m-3}$. Now, we shall prove the desired second relations for $2 \leqq v \leqq m-1$ by the downward induction on v. Let $1 \leqq k \leqq m-3$ and assume that

$$2^{a_v-1}\gamma_v = 2^{a_{v-1}}q_{v-1}\gamma_{v-1}$$

holds for any v with $k+2 \leqq v \leqq m-2$ for some odd integer q_{v-1}. Then, by $(**)_k$ and the inductive assumption, it is easily seen that the relation

$$2^{a_{k+1}-1}\gamma_{k+1} = 2^{a_k}q_k\gamma_k$$

holds for some odd integer q_k, since $\tilde{J}(N^n(m))$ is a 2-group. Thus, the proof is completed. q.e.d.

9. The induced homomorphism on the J-groups of the inclusion $N^{n-1}(m) \subset N^n(m)$

Let N^k be the k-skeleton of the CW-complex $N^n(m)(m \geqq 2)$ in [6, Lemma 2.1], and $j: N^k \subset N^n(m)$ be the inclusion. For an element $a \in KO(N^n(m))$ (resp. $\tilde{J}(N^n(m))$), we denote its j^*-image $j^*(a) \in KO(N^k)$ (resp. $\tilde{J}(N^k)$) by the same letter a.

Consider the inclusion

$$(9.1) \qquad j_{k,l}: N^{8k+l-1} \subset N^{8k+l} \qquad (0 \leqq l \leqq 7).$$

Then, for the induced homomorphism $j_{k,l}{}^*: \widetilde{KO}(N^{8k+l}) \to \widetilde{KO}(N^{8k+l-1})$, we have

Proposition 9.2 (cf. [10, §4]). *$j_{k,l}{}^*$ is isomorphic if $l = 7, 6, 5$ or 3, and epimorphic otherwise. Furthermore,*

$$(9.3) \qquad \operatorname{Ker} j_{k,l}{}^* = \begin{cases} Z_{2^{m+1}}\langle 2\beta^{2k+1}\rangle & \text{if } l=4,\ k \geqq 0, \\ Z_2\langle 2\alpha_0\beta^{2k}\rangle \oplus Z_2\langle 2\alpha_1\beta^{2k}\rangle & \text{if } l=2,\ k \geqq 0, \\ Z_2\langle \alpha_0\beta^{2k}\rangle \oplus Z_2\langle \alpha_1\beta^{2k}\rangle & \text{if } l=1,\ k \geqq 0, \\ Z_{2^{m+1}}\langle \beta^{2k}\rangle & \text{if } l=0,\ k > 0. \end{cases}$$

Lemma 9.4. *Let $n \geqq 0$ be even. Then, the following relations hold:*

$$2\alpha_0\beta^n = 2^{n+1}\alpha_0,\ 2\alpha_1\beta^n = 2^{n+1}\alpha_1 \quad \text{in} \quad \widetilde{KO}(N^{4n+2}),$$
$$\alpha_0\beta^n = 2^n\alpha_0,\ \alpha_1\beta^n = 2^n\alpha_1 \pm 2^{n+m-3}\beta(1) \quad \text{in} \quad \widetilde{KO}(N^{4n+1}),$$

where the term $\pm 2^{n+m-3}\beta(1)$ in the last relation does not appear if $m = 2$.

PROOF. By Propositions 2.5 and 2.7, $\alpha_0\beta^i = 2^i\alpha_0$ holds in $\widetilde{RO}(Q_r)$ ($r = 2^{m-1}$) for any integer $i \geqq 0$, and so the first and the third relations also hold by (3.2). In $\widetilde{RO}(Q_r)$ ($r = 2^{m-1}$) and also in $\widetilde{KO}(N^n(m))$, the relation

$$\alpha_1\beta^n = (\beta^n - 2^n)\sum_{s=1}^{m-2}\beta(s)\prod_{t=s+1}^{m-2}(2+\beta(t)) + 2^n\alpha_1$$

holds by Propositions 2.5, 2.7, Lemma 2.9(ii) and (3.2). Hence, the second and the last relations are obtained. Let $m \geqq 3$. Then, the relations $\beta^n\beta(s) = 0$ ($s \geq 1$), $2^{n+i-2}\beta(m-i) = 0$ ($2 \leqq i \leqq m-2$) and $2^{n+m-2}\beta(1) = 0$ hold in $\widetilde{KO}(N^n(m)) = \widetilde{KO}(N^{4n+3})$ (cf. [10, Lemma 8.1]). Therefore,

$$\alpha_1\beta^n = 2^n\alpha_1 \pm 2^{n+m-3}\beta(1)$$

holds in $\widetilde{KO}(N^{4n+2})$ and also in $\widetilde{KO}(N^{4n+1})$. Thus, we complete the proof. q.e.d.

To study the induced homomorphism $j_{k,l}{}^*\colon \tilde{J}(N^{8k+l}) \to \tilde{J}(N^{8k+l-1})$, we use the following

(9.5) ([2, II, (3.12)] and [17]) *Let $X \xrightarrow{i} Y \xrightarrow{\pi} Z$ be a cofibering of finite connected CW-complexes and assume that the upper sequence in the commutative diagram*

$$\begin{array}{ccccccc}
\widetilde{KO}(Z) & \xrightarrow{\pi^*} & \widetilde{KO}(Y) & \xrightarrow{i^*} & \widetilde{KO}(X) & \longrightarrow & 0\\
\downarrow J & & \downarrow J & & \downarrow J & & \\
\tilde{J}(Z) & \xrightarrow{\pi^*} & \tilde{J}(Y) & \xrightarrow{i^*} & \tilde{J}(X) & \longrightarrow & 0
\end{array}$$

is exact. Then the lower sequence is also exact.

LEMMA 9.6. *Let Ψ^3 be the Adams operation on $\widetilde{KO}(N^k)$. Then*

$$(\Psi^3 - 1)(2^{\varepsilon(i)}\beta^i) = (3^{2i}-1)2^{\varepsilon(i)}\beta^i + \sum_{j=1}^{2i}\binom{2i}{j}3^{2i-j}2^{\varepsilon(i)}\beta^{i+j} \qquad (i \geqq 1),$$

and $3^{2i} - 1 \equiv 2^{\nu+3} \bmod 2^{\nu+4}$, *where* $\nu = \nu_2(i)$.

PROOF. For the monomorphism $i^*\colon R(S^3) \to R(S^1)$ in Lemma 4.5, we have

$$\begin{aligned} i^*\Psi^3(\zeta(0)) &= \Psi^3 i^*(\zeta(0)) = \Psi^3(x + x^{-1} - 2) = x^3 + x^{-3} - 2 \\ &= (x + x^{-1} - 2)(x + x^{-1} + 1)^2 = i^*(\zeta(0)(\zeta(0)+3)^2). \end{aligned}$$

Therefore, we have

$$\Psi^3(\zeta(0)) = \zeta(0)(\zeta(0)+3)^2 \quad \text{in} \quad \tilde{R}(S^3).$$

Also, by (3.2), Lemma 4.6 and the naturality of the Adams operation Ψ^3, we have

$$\Psi^3(2^{\varepsilon(i)}\beta^i) = 2^{\varepsilon(i)}\beta^i(\beta+3)^{2i}$$

in $\widetilde{KO}(N^n(m))$, and so in $\widetilde{KO}(N^k)$ $(4n+3 \geqq k)$. Thus, we have the first half. The second half is easily shown by the induction on ν (cf. [8, Lemma 7.8]). q.e.d.

LEMMA 9.7. *Let* $m \geqq 3$ *and* $k \geqq 0$. *Then, the relation*

$$2^{m-3+2k}\gamma_1 = 2^{m-3+4k}\gamma_0$$

holds in $\tilde{J}(N^{8k+1})$.

PROOF. It is sufficient to show that

$$2^{m-3+2k}\gamma_1 = 2^{m-3+4k}\gamma_0$$

holds in $\tilde{J}(N^{8k+3}) = \tilde{J}(N^{2k}(m))$. Since $\gamma_0 = \gamma_1 = 0$ if $k=0$ by Remark 5.8, we assume that $k>0$. In case $k=1$, we have $2^4\gamma_0 = 2^2 \cdot 3\gamma_1$ by (1.7.5) in Theorem 1.7 (iii) (a) and $Y(3,1)=6$ in (6.2). Thus, by (8.3), the desired relation for $k=1$ is obtained. In case $k>1$, the desired relation is (8.10.1) for $l=2$. q.e.d.

By using the above results and Theorem 1.8, we see the following proposition, where (ii) is Theorem 1.11:

PROPOSITION 9.8. (i) *The induced homomorphism*

$$j_{k,l}{}^*: \tilde{J}(N^{8k+l}) \longrightarrow \tilde{J}(N^{8k+l-1}) \quad (j_{k,l}: N^{8k+l-1} \subset N^{8k+l},\ m \geqq 2)$$

is isomorphic if $l = 7, 6, 5$ *or* 3, *epimorphic otherwise, and*

$$\text{(9.9)}\qquad \operatorname{Ker} j_{k,l}{}^* = \begin{cases} Z_8\langle J(2\beta^{2k+1})\rangle & \text{if } l=4,\ k\geqq 0,\\ Z_2\langle 2^{2k+1}J\alpha_0\rangle \oplus Z_2\langle 2^{2k+1}J\alpha_1\rangle & \text{if } l=2,\ k\geqq 0,\\ Z_2\langle 2^{2k}J\alpha_0\rangle \oplus Z_2\langle 2^{2k}J\alpha_1+\omega\rangle & \text{if } l=1,\ k\geqq 0,\\ Z_{2^h}\langle J(\beta^{2k})\rangle & \text{if } l=0,\ k>0, \end{cases}$$

where $h = \min\{m+1, \nu(4k)+2\}$, *and the term* $\omega = \pm 2^{m-3+4k}\gamma_0$ *does not appear if* $m=2$.

(ii) $\#\tilde{J}(N^n(m)) = 2^{\varphi(n,m)}$,

$$\varphi(n,m) = \textstyle\sum_{s=0}^{m-1}(s+2)[(a_s+1)/2] + (m+1)a_{m+1} + 4\varepsilon(n+1),$$

where $\#G$ *is the order of the group* G, a_s *and* $\varepsilon(i)$ *are the integers in* (1.4).

PROOF. Consider (9.5) for the cofibering $N^{i-1} \subset N^i \to N^i/N^{i-1}$ $(i=8k+l)$. Then, the first half of (i) is obvious by the first half of Proposition 9.2.

Furthermore, by (9.3), Lemmas 9.4 and 9.7, it is easy to see that $\operatorname{Ker} j_{k,l}{}^*$ is generated by the generators of the group given in the right hand side of (9.9).

Now, we can show that

$$(*) \qquad \# \operatorname{Ker} j_{k,l}{}^* \leqq \begin{cases} 8 & \text{if } l=4,\ k \geqq 0, \\ 4 & \text{if } l=2 \text{ or } 1,\ k \geqq 0, \\ 2^h & \text{if } l=0,\ k>0. \end{cases}$$

In fact, $(*)$ for the second case is easily seen by (9.3) and (9.5) for the cofibering $N^{i-1} \subset N^i \to N^i/N^{i-1}$ $(i=8k+l)$. Now, $\operatorname{Ker} j_{k,4}{}^*$ is the cyclic group generated by $J(2\beta^{2k+1})$. On the other hand, by Lemma 9.6, the first half of Proposition 9.2 and (3.5.3), $(\Psi^3-1)(2\beta^{2k+1})=(3^{4k+2}-1)(2\beta^{2k+1})=2^3 a(2\beta^{2k+1})$ (a: odd) in $\widetilde{KO}(N^{8k+4})$. Thus, $2^3 J(2\beta^{2k+1})=0$ in $\widetilde{J}(N^{8k+4})$ by (1.1), since $\widetilde{J}(N^i)$ is a 2-group by Theorem 3.5 and (1.1), and so $(*)$ for the first case is valid. Finally, $\operatorname{Ker} j_{k,0}{}^*$ is the cyclic group generated by $J(\beta^{2k})$. In the similar way to the case $l=4$ above, $(\Psi^3-1)(\beta^{2k})=(3^{4k}-1)\beta^{2k}=2^{\nu(4k)+2} b\beta^{2k}$ (b: odd) in $\widetilde{KO}(N^{8k})$. Therefore, $2^{\nu(4k)+2}J(\beta^{2k})=0$ in $\widetilde{J}(N^{8k})$ by (1.1), and also $2^{m+1}J(\beta^{2k})=0$ by (9.3). Thus $(*)$ for the last case follows.

Now, $(*)$ implies that

$$\prod_{k=1}^{[n/2]} \# \operatorname{Ker} j_{k,0}{}^* \leqq 2^{\psi(n,m)},\ \psi(n,m)=\sum_{s=2}^{m-1}(s+2)[(a_s+1)/2]+(m+1)a_m,$$

$$\prod_{k=0}^{[n/2]} \# \operatorname{Ker} j_{k,l}{}^* \leqq 2^{n+2-\varepsilon(n)} \quad (l=1 \text{ or } 2),$$

$$\prod_{k=0}^{[n/2]-1+\varepsilon(n)} \# \operatorname{Ker} j_{k,4}{}^* \leqq 2^{n+[n/2]+2\varepsilon(n)},$$

and hence we see by the routine calculations that

$(**)$ $(*)$ implies $\#\widetilde{J}(N^n(m)) \leqq 2^{\varphi(n,m)}$ and the equality holds if and only if the equality holds in $(*)$ for any k and l with $8k+l \leqq 4n+3$.

On the other hand, by Theorems 1.7 (ii), 3.5 and 1.8, we see easily that

$$\#\widetilde{J}(N^n(m))=2^{\varphi(n,m)} \qquad \text{for } n=2^{m-2}a \ (a \geqq 2).$$

Thus, we see the proposition by $(**)$. q.e.d.

Propositions 5.7 and 9.8 (i) imply immediately the following corollary, which is Proposition 1.10:

COROLLARY 9.10. *For the homomorphism*

$$j^*: \widetilde{J}(N^n(m)) \longrightarrow \widetilde{J}(N^{n-1}(m)) \qquad (j: N^{n-1}(m) \subset N^n(m),\ m \geqq 2),$$

j^* is epimorphic, and

$$\operatorname{Ker} j^* = \begin{cases} Z_8\langle J(2\beta^n)\rangle & \textit{if } n \textit{ is odd}, \\ Z_4\langle 2^n J\alpha_0\rangle \oplus Z_4\langle 2^n J\alpha_1+\omega\rangle \oplus Z_{2^{h_m}}\langle J(\beta^n)\rangle & \textit{if } n \textit{ is even}, \end{cases}$$

where $h_m = \min\{m+1, \nu_2(n)+3\}$ *and the term* $\omega = \pm 2^{m-3+2n}\gamma_0$ *does not appear if* $m=2$.

10. Proofs of Theorems 1.3 and 1.7 (iii) (b)

Let $f(n, m; v)$ be the non-negative integer such that

(10.1) $$\#\gamma_v = 2^{f(n,m;v)} \text{ in } \tilde{J}(N^n(m)) \qquad (n \geqq 0,\ m \geqq 2)$$

by Proposition 9.8 (ii), where $\#\gamma$ denotes the order of γ. Then, by the definition of γ_v in (5.5) and by Lemma 2.10, (3.2) and Remark 5.8,

(10.2) $$f(n, m; v) = 0 \qquad \text{if } n = 0 \text{ or } v \geqq m.$$

For the case $m = 2$, by Theorems 1.7 (ii), 3.5 (cf. [7, Th. 1.3]) and (5.5), we have

(10.3) $$f(n, 2; 0) = a_0, \quad f(n, 2; 1) = a_1 + \varepsilon(n).$$

LEMMA 10.4. *If* $n = 2^{m-2}a$ $(a \geqq 1)$ *and* $m \geqq 3$, *then*

$$f(n, m; 0) = m - 2 + a_0, \quad f(n, m; v) = m - 1 - v + a_v \qquad (1 \leqq v < m).$$

PROOF. Let $n = 2^{m-2}a$ $(a \geqq 1)$ and $m \geqq 3$. Then, by Corollary 9.10,

$$\#J(\beta^n) = 2^{m+1} \quad \text{in} \quad \tilde{J}(N^n(m)).$$

On the other hand, $2^m\beta^n = -2^{m+2n-3}(2\beta)$ in $KO(N^n(m))$ by [10, Lemma 8.1]. Thus, we obtain

$$f(n, m; 0) = m - 2 + a_0.$$

Furthermore, Theorem 1.8 and Lemma 8.11 imply immediately

$$f(n, m; v) = m - 1 - v + a_v \qquad (1 \leqq v < m).$$

q.e.d.

Let $G(n, m)$ be the subgroup of $\tilde{J}(N^n(m))$ generated by γ_v $(0 \leqq v < m)$, and define $K(n, m) = \operatorname{Ker} j^* \cap G(n, m)$, where $j^*: \tilde{J}(N^n(m)) \to \tilde{J}(N^{n-1}(m))$ is the homomorphism in Corollary 9.10. Then, by Lemma 5.6, Proposition 5.7 and Corollary 9.10, we have

(10.5) $$K(n, m) = \begin{cases} Z_8\langle J(2\beta^n)\rangle & \text{if } n \text{ is odd,} \\ Z_{2^{h_m}}\langle J(\beta^n)\rangle & \text{if } n > 0 \text{ is even,} \end{cases}$$

where $m \geqq 2$ and $h_m = \min\{m+1, \nu_2(n)+3\}$.

Let $m \geqq 3$ and $\pi: N^n(m-1) \to N^n(m)$ be the natural projection induced by the inclusion $i: Q_{2^{m-2}} \subset Q_{2^{m-1}}$ $(i(x)=x,\ i(y)=y)$. Then, we see easily that $\pi^*(\gamma_v)=\gamma_v$ $(0 \leqq v < m)$ by the definition of γ_v, where $\pi^*: \tilde{J}(N^n(m)) \to \tilde{J}(N^n(m-1))$ is the induced homomorphism of π. Therefore, we can define the homomorphism

$$\pi^*: G(n, m) \longrightarrow G(n, m-1) \qquad (m \geqq 3)$$

by the restriction of $\pi^*: \tilde{J}(N^n(m)) \to \tilde{J}(N^n(m-1))$. Also, we can define the restricted homomorphism

$$j^*: G(n, m) \longrightarrow G(n-1, m) \qquad (m \geqq 2)$$

of $j^*: \tilde{J}(N^n(m)) \to \tilde{J}(N^{n-1}(m))$ in Corollary 9.10, since $j^*(\gamma_v)=\gamma_v$ $(0 \leqq v < m)$ holds. For these homomorphisms, we have the commutative diagram $(m \geqq 3)$

$$\begin{array}{ccccc} K(n, m) = \operatorname{Ker} j^* & \subset & G(n, m) & \xrightarrow{j^*} & G(n-1, m) \\ & & \downarrow \pi^* & & \downarrow \pi^* \\ K(n, m-1) = \operatorname{Ker} j^* & \subset & G(n, m-1) & \xrightarrow{j^*} & G(n-1, m-1). \end{array} \tag{10.6}$$

LEMMA 10.7. *If* $n \not\equiv 0 \bmod 2^{m-2}$ $(m \geqq 3)$, *then*

$$\pi^* \mid K(n, m): K(n, m) \longrightarrow K(n, m-1)$$

is isomorphic.

PROOF. Since $\pi^*(J(2^{\varepsilon(n)}\beta^n)) = J(2^{\varepsilon(n)}\beta^n)$ holds, $\pi^* \mid K(n, m)$ is epimorphic by (10.5). On the other hand, we see easily that $\# K(n, m) = \# K(n, m-1)$ by the assumption $n \not\equiv 0 \bmod 2^{m-2}$ and (10.5). Therefore, $\pi^* \mid K(n, m)$ is isomorphic. q.e.d.

LEMMA 10.8. *If* $n \not\equiv 0 \bmod 2^{m-2}$ $(m \geqq 3)$, *then*

$$f(n, m; v) = \max\{f(n-1, m; v), f(n, m-1; v)\}.$$

PROOF. Consider the diagram (10.6). Then the definition of $f(n, m; v)$ in (10.1) implies that

$$f(n, m; v) \geqq \max\{f(n-1, m; v), f(n, m-1; v)\},$$

since $j^*(\gamma_v)=\gamma_v$ and $\pi^*(\gamma_v)=\gamma_v$. Moreover, if $f(n, m; v) > \max\{f(n-1, m; v), f(n, m-1; v)\}$, then the non-zero element $2^{f(n,m;v)-1}\gamma_v$ in $\tilde{J}(N^n(m))$ is mapped to 0 by j^* and π^*. This contradicts Lemma 10.7. Thus we have the lemma. q.e.d.

For the case $m=3$, by Lemmas 10.4, 10.8, (10.2) and (10.3), we see easily that

$$(10.9)\qquad \begin{aligned} &f(n, 3; 0) = \begin{cases} 1 + a_0 & \text{if } n > 0 \text{ is even,} \\ a_0 & \text{if } n \text{ is odd,} \end{cases} \\ &f(n, 3; v) = 2 - v + a_v \ (v = 1, 2) \ \text{ if } n > 0. \end{aligned}$$

PROOF OF THEOREM 1.3. The results for $m = 2$ and 3 are given in (10.2–3) and (10.9). By (10.2), it is sufficient to show that $f(n, m; v)$ $(0 \leqq v < m)$ is equal to the number given in Theorem 1.3 for the case $m \geqq 4$ and $n > 0$. By Lemma 10.4, Theorem 1.3 holds if $m \geqq 4$ and $n \equiv 0 \bmod 2^{m-2}$.

For the case $m \geqq 4$ and $2^{m-2}a < n < 2^{m-2}(a+1)$, assume that Theorem 1.3 holds for $(n-1, m; v)$ and $(n, m-1; v)$ instead of $(n, m; v)$. Then, we see easily that the right hand side of the equality in Lemma 10.8 is equal to

$$\begin{aligned} &f(n, m-1; v) && \text{if } a = 0, \\ &\begin{cases} \max\{f(n, m-1; 0), m - 2 + 2^{m-1}a\} & (v = 0) \\ \max\{f(n, m-1; v), m - 1 - v + 2^{m-1-v}a\} & (1 \leqq v < m) \end{cases} && \text{if } a > 0, \end{aligned}$$

and hence to the right hand side of the equality in Theorem 1.3. Thus, Lemma 10.8 implies Theorem 1.3 by the induction on n and m.

These complete the proof of Theorem 1.3. q.e.d.

PROOF OF THEOREM 1.7(iii)(b). Let $m \geqq 3$ and n is odd. By Corollary 9.10, $j^*: \tilde{J}(N^{n+1}(m)) \to \tilde{J}(N^n(m))$ is epimorphic, and $\operatorname{Ker} j^*$ is generated by the elements $2^{n+1}J\alpha_0$, $2^{n+1}J\alpha_1 \pm 2^{m-1+2n}\gamma_0$ and $J(\beta^{n+1})$. Thus, $\tilde{J}(N^n(m))$ is the abelian group generated by $J\alpha_i$ $(i = 0, 1)$ and γ_s $(0 \leqq s < m)$ with the relations in Theorem 1.7(iii)(a) replaced n with $n+1$, and in addition,

$$2^{n+1}J\alpha_0 = 0,\ 2^{n+1}J\alpha_1 \pm 2^{m-1+2n}\gamma_0 = 0 \text{ and } J(\beta^{n+1}) = 0.$$

On the other hand, it is easily seen that $f(n, m; 0) \leqq m - 2 + 2n$ by Theorem 1.3, and so $2^{m-1+2n}\gamma_0 = 0$ in $\tilde{J}(N^n(m))$. By Lemma 7.1, the relation $J(\beta^{n+1}) = 0$ is written as

$$2^{2n-1}\gamma_0 - \textstyle\sum_{v=1}^{l} Y(n+1, v)\gamma_v = 0 \quad \text{where} \quad 2^l \leqq n + 1 < 2^{l+1}.$$

Therefore, we complete the proof of Theorem 1.7(iii)(b). q.e.d.

11. The relation between $\tilde{J}(N^n(m+1))$ and $\tilde{J}(N^n(m))$ for $n < 2^{m-1}$

In this section, we present the relation between $\tilde{J}(N^n(m+1))$ and $\tilde{J}(N^n(m))$ for $n < 2^{m-1}$, which is stated as follows:

PROPOSITION 11.1. (i) $\tilde{J}(N^n(m))$ $(m \geqq 2)$ *is the direct sum*

$$Z_{2^{n+2-\varepsilon(n)}}\langle J\alpha_0\rangle \oplus Z_{2^{n+2-\varepsilon(n)}}\langle J\alpha_1\rangle \oplus G(n, m),$$

where $G(n, m)$ is the subgroup of $\tilde{J}(N^n(m))$ generated by γ_v $(0 \leqq v < m)$.

(ii) *Let $m \geqq 2$ and $n < 2^{m-1}$. Then there exists an isomorphism*

$$f: \tilde{J}(N^n(m+1)) \longrightarrow \tilde{J}(N^n(m)),$$

which is given by

$$f(J\alpha_i) = J\alpha_i \ (i = 0, 1) \quad \text{and} \quad f(\gamma_v) = \gamma_v \ (0 \leqq v < m+1). \tag{11.2}$$

PROOF. (i) follows immediately from Theorem 1.7 and [7, Th. 1.3].

(ii) The subgroups generated by $J\alpha_i$ $(i = 0, 1)$ of $\tilde{J}(N^n(m+1))$ and $\tilde{J}(N^n(m))$ are isomorphic via $f(J\alpha_i) = J\alpha_i$ $(i = 0, 1)$. The assumption $n < 2^{m-1}$ implies that $\#\tilde{J}(N^n(m+1)) = \#\tilde{J}(N^n(m))$ by Proposition 9.8 (ii). On the other hand, $\pi^*(\gamma_v) = \gamma_v$ for the homomorphism $\pi^*: G(n, m+1) \to G(n, m)$ in (10.6). Thus $G(n, m+1)$ and $G(n, m)$ are isomorphic via $f(\gamma_v) = \gamma_v$. Therefore, we obtain the desired isomorphism f by (11.2). q.e.d.

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