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# Lie algebras whose proper subalgebras are either semisimple, abelian or almost-abelian

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## Introduction

A number of papers published in recent years have actively studied the relationship between the structure of a Lie algebra to that of the lattice of its subalgebras. In these studies, Lie algebras whose proper subalgebras are either semisimple, abelian or almost-abelian (that we shall call $X$-algebras for short) have occurred frequently (c.f. [9], [10], [13]). For instance, Lie algebras with a relatively complemented lattice of subalgebras are $X$-algebras (Gein and Muhin [10]). Of special interest are the Lie algebras in which every subalgebra of dimension $> 1$ is simple (supersimple Lie algebras).

The purpose of this paper is first to investigate the structure of an $X$-algebra; and secondly to study upper semi-modular, relatively complemented, supersimple, and minimal non-modular Lie algebras; and thirdly to determine the Lie algebras with a subalgebra lattice of length 3 as well as their corresponding subalgebra lattices.

In section 1, we consider Lie algebras $L$ having an element $x$ such that $C_L(x)$ is abelian and $\dim N_L(C_L(x))/C_L(x) \leq 1$. This class of Lie algebras contains all $X$-algebras and the Lie algebras having a self-centralizing ad-nilpotent element (which have been determined in [4]). If $N_L(C_L(x)) = C_L(x)$, then we show that $x$ lies in the center of $L$. If $\dim N_L(C_L(x))/C_L(x) = 1$, then we get that either $N_L(C_L(x))$ is nilpotent, $C_L(x) \lhd L$, or $L/Z(L)$ has a self-centralizing ad-nilpotent element.

Moreover, we prove that the Engel subalgebras of a simple Lie algebra of dimension $> 3$ are neither almost-abelian nor 3-dimensional simple. This section finishes with two criteria for an element of a Lie algebra to be ad-semisimple.

In section 2, we study the structure of a nonsolvable $X$-algebra. Solvable $X$-algebras have been studied in Gein [12, Theorem 3].

In section 3, we use results in the previous sections to study upper semi-modular and relatively complemented Lie algebras. It is known that

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every upper semimodular Lie algebra is relatively complemented (see [17]). Gein [9] proved that an upper semimodular Lie algebra is either abelian, almost-abelian or supersimple. We obtain this same result for relatively complemented Lie algebras of characteristic $\neq 2, 3$. If the ground field $F$ is perfect with char $(F) \neq 2, 3$, then every supersimple Lie algebra is 3-dimensional non-split ([21]). An example of a supersimple Lie algebra of dimension 7 over a perfect field of characteristic 3 is given in [12, Example 2], every proper subalgebra of this algebra is either 1-dimensional or 3-dimensional non-split simple.

The existence of supersimple Lie algebras over a field $F$ of characteristic $> 3$, other than the 3-dimensional non-split, is an interesting open problem. By contrast with the characteristic 3 case, we obtain that such Lie algebras cannot contain 3-dimensional non-split proper subalgebras. Therefore, there exists a supersimple Lie algebra over $F$ of dimension greater than three if and only if there is over $F$ a Lie algebra of dimension $> 3$ whose non-trivial subalgebras are 1-dimensional.

In section 4, we assume that F is perfect with char $(F) \neq 2, 3$. We prove that if $L$ is simple and minimal non-modular then either $L \cong \mathrm{sl}(2)$, or $L$ has only abelian subalgebras, or $L$ is 3-dimensional non-split simple over its centroid. This result is a slight refinement of a Gein's result [12].

In section 5, we determine the Lie algebras whose lattice of subalgebras has length 3 and the structure of the corresponding lattices of subalgebras. In particular, we show that the lattice-theoretical characterization of the algebra $\mathrm{sl}(2, F)$ given by Gein in [11], when $F$ has characteristic 0, remains true when $F$ is any perfect field of characteristic $p \neq 2, 3$.

A Lie algebra $L$ is called almost-abelian if there exists a basis $a_1, \ldots, a_n$, $x$ for $L$ with product $[a_i, a_j] = 0$, $[a_i, x] = a_i$ for $i, j = 1, \ldots, n$.

Throughout $L$ will denote a finite dimensional Lie algebra over a field $F$, and $\Omega$ will be an algebraic closure of $F$.

## 1. On Lie algebras containing an abelian centralizer

If $L$ contains a self-centralizing ad-nilpotent element $x$, then $N_L(C_L(x))$ is 2-dimensional almost-abelian (see [4]). It is easy to see that if $L$ is a semisimple $X$-algebra, then $N_L(C_L(x))$ is either abelian or almost-abelian for every $x \in L$-(0). This leads us to study Lie algebras having an element $x$ such that $C_L(x)$ is abelian and $\dim N_L(C_L(x))/C_L(x) \leq 1$.

Let $x$ be an element of a Lie algebra $L$. We will denote by $E_L(x)$ the Engel subalgebra of $L$ relative to $x$; that is $E_L(x)$ is the Fitting null-component of $L$ relative to the linear transformation ad $x$.

LEMMA 1.1. Let $x \in L$ such that $C_L(x)$ is abelian. Then the following holds:

1) $N_L(C_L(x)) = \mathrm{Ker}\,(\mathrm{ad}\,x)^2$ is a subalgebra of $L$.

2) If in addition $N_L(C_L(x))/C_L(x)$ is abelian, then $\mathrm{Ker}\,(\mathrm{ad}\,x)^3$ is a subalgebra too.

PROOF. (1): Clearly, $N_L(C_L(x)) \subseteq \mathrm{Ker}\,(\mathrm{ad}\,x)^2$. Let $y \in \mathrm{Ker}\,(\mathrm{ad}\,x)^2$. Then $[yx] \in C_L(x)$. For every $c \in C_L(x)$ we have $[[yc]x] = [[yx]c] = 0$, so that $[yc] \in C_L(x)$. This means $y \in N_L(C_L(x))$.

(2): Let $u, v \in \mathrm{Ker}\,(\mathrm{ad}\,x)^3$. Since $[ux], [vx] \in \mathrm{Ker}\,(\mathrm{ad}\,x)^2 = N_L(C_L(x))$, we have

$$[[ux], [vx]) \in N_L(C_L(x))' \leq C_L(x).$$

This yields, $[(\mathrm{ad}\,x)^2(u), [vx]] = -[[ux], (\mathrm{ad}\,x)^2(v)]$. Then, by using Leibniz's rule we obtain $(\mathrm{ad}\,x)^3[uv] = 0$. So that, $[uv] \in \mathrm{Ker}\,(\mathrm{ad}\,x)^3$.

PROPOSITION 1.2. Let $x \in L$ such that $N_L(C_L(x))$ is abelian. Then $E_L(x) = C_L(x)$.

PROOF. As $N_L(C_L(x))$ is abelian, we have $N_L(C_L(x)) = C_L(x)$. On the other hand, $N_L(C_L(x)) = \mathrm{Ker}\,(\mathrm{ad}\,x)^2$ by Lemma 1.1. This yields, $E_L(x) = C_L(x)$.

LEMMA 1.3. Lex $x$ be an ad-nilpotent element of $L$ such that $C_L(x)$ is abelian and $\dim N_L(C_L(x))/C_L(x) = 1$. Then the following holds:

1) The simple elementary divisor polynomials of $\mathrm{ad}\,x$ have the form: $\lambda^r, \lambda, \ldots, \lambda$ where $r > 1$.

2) Either $C_L(x) \lhd L$ or $C_L(x) = Z(N_L(C_L(x)) + Fx$.

PROOF. (1): By Lemma 1.1, $N_L(C_L(x)) = \mathrm{Ker}\,(\mathrm{ad}\,x)^2$. Then, (1) follows from $\dim N_L(C_L(x))/C_L(x) = 1$ and a Jordan matrix argument.

(2): Assume $N_L(C_L(x)) \neq L$. Then, there exists $u \in \mathrm{Ker}\,(\mathrm{ad}\,x)^3\text{-}\mathrm{Ker}\,(\mathrm{ad}\,x)^2$. Put $N_L(C_L(x)) = C_L(x) + Fy$. Since $[ux] \in N_L(C_L(x))$, we can decompose $[ux] = c_0 + \alpha y$ where $c_0 \in C_L(x)$, $\alpha \in F$. If $\alpha = 0$, then we have $[[ux]x] = [c_0 x] = 0$ and so $u \in \mathrm{Ker}\,(\mathrm{ad}\,x)^2$, which is a contradiction. Therefore, $\alpha \neq 0$. For every $c \in C_L(x)$, we find

$$[[uc]x] = [[ux]c] = [c_0, c] + \alpha[yc] = \alpha[yc] \in N_L(C_L(x))' \leq C_L(x).$$

Then, for every $c' \in C_L(x)$, $[[[uc]c']x] = [[[uc]x]c'] = 0$ since $C_L(x)$ is abelian. This yields, $[uc] \in N_L(C_L(x))$. Then we can write $[uc] = c_1 + \beta y$ where $c_1 \in C_L(x)$ and $\beta \in F$. So, $[[uc]x] = \beta[yx]$. This yields, $\alpha[yc] = \beta[yx]$ and so $[y, \alpha c - \beta x] = 0$. Therefore, $\alpha c - \beta x \in Z(N_L(C_L(x))$. From this it follows (2).

THEOREM 1.4. Assume char $(F) \neq 2$. Let $x \in L$ such that $C_L(x)$ is abelian and $\dim N_L(C_L(x))/C_L(x) = 1$. Then one of the following holds:

1) $C_L(x) \lhd E_L(x)$.

2) $N_L(C_L(x))$ is nilpotent.

3) $E_L(x)/Z(E_L(x))$ is a simple Lie algebra having a self-centralizing ad-nilpotent element (the element $x + Z(E_L(x))$ is such an element).

PROOF. We may assume without loss of generality that $E_L(x) = L$. By Lemma 1.3, the simple elementary divisor polynomials of ad $x$ have the form: $\lambda^r, \lambda, \ldots, \lambda$ where $r \geq 1$. Thus, from the Jordan canonical matrix for ad $x$ it follows that there exits a basis $u_1, \ldots, u_r, v_1, \ldots, v_s$ for $L$ with $[u_i, x] = u_{i-1}$ for $2 \leq i \leq r$, and $[u_1, x] = [v_i, x] = 0$ for $1 \leq i \leq s$. Let $1 \leq m \leq s$, then we see that $\operatorname{Ker}(\operatorname{ad} x)^m$ is the span of $u_j, v_k$ for $j = 1, \ldots, m$; $k = 1, \ldots, s$.

Assume $C_L(x) \ntriangleleft L$. Let $Z := Z(N_L(C_L(x)))$. By Lemma 1.3, $C_L(x) = Z + Fx$. Thus we may decompose $u_1 = \alpha x + z_0$ where $\alpha \in F$, $z_0 \in Z$. Suppose $\alpha = 0$. Then we have $[u_2[u_2, x]] = 0$. This yields that $u_2$ acts nilpotently on $C_L(x)$, so $N_L(C_L(x))$ is nilpotent. Therefore, (2) holds. Now assume $\alpha \neq 0$. Then we claim that $Z = Z(L)$. Clearly, $[Z, u_2] = 0$. Let $2 \leq i < r$. We argue by induction. Suppose that $[Z, u_j] = 0$ for $j \leq i$ and let $z \in Z$. Then we find $[[z, u_{i+1}]x] = [z, u_i] + [[zx]u_{i+1}] = 0$, so $[z, u_{i+1}] \in C_L(x)$. Decompose $[z, u_{i+1}] = \beta x + z_1$ where $\beta \in F$, $z_1 \in Z$. Then we have,

$$[[u_{i+1}, u_2]z] = [u_2[z, u_{i+1}]] = \beta u_1 = \beta\alpha x + \beta z_0.$$

On the other hand, by using Leibniz's rule we obtain $(-1)^i(\operatorname{ad} x)^i[u_{i+1}, u_2] = (i-1)\alpha u_1$. It follows that $[u_{i+1}, u_2] \in \operatorname{Ker}(\operatorname{ad} x)^{i+1}$ and that $[u_{i+1}, u_2] \equiv (i-1)\alpha u_{i+1} \pmod{\operatorname{Ker}(\operatorname{ad} x)^i}$. Then, $[z[u_{i+1}, u_2]] = (i-1)\alpha[z, u_{i+1}] = (i-1)\alpha(\beta x + z_1)$. This yields, $(1-i)\alpha\beta = \beta\alpha$ and $(1-i)\alpha z_1 = \beta z_0$. Since $\alpha \neq 0$ and $i > 1$, we get $\beta = 0$ and then $z_1 = 0$. This yields, $[z, u_{i+1}] = 0$. It follows $Z = Z(L)$, and the claim is proved.

Now we consider the Lie algebra $\bar{L} = L/Z(L)$. Let $\bar{x} := x + Z(L)$. We see that $\bar{x}$ is a self-centralizing ad-nilpotent element of $\bar{L}$. So, Theorem 2.8 of [4] applies and $L/Z(L)$ is either 2-dimensional nonabelian or simple. In the former case, we have $C_L(x) \lhd L$ a contradiction. Therefore, $L/Z(L)$ is simple. The proof is complete.

COROLLARY 1.5. Assume char $(F) \neq 2$. For a Lie algebra $L$, the following are equivalent:

1) $L$ has an ad-nilpotent element $x$ such that $(\operatorname{ad} x)^2 \neq 0$, $C_L(x)$ is abelian, $N_L(C_L(x))$ is non-nilpotent and $\dim N_L(C_L(x))/C_L(x) = 1$,

2) $L/Z(L)$ is simple and has a self-centralizing ad-nilpotent element.

PROOF. (1) implies (2) follows from Theorem 1.4 and Lemma 1.1. To prove the converse, let $\bar{x} := x + Z(L)$ be a self-centralizing ad-nilpotent element of $\bar{L} = L/Z(L)$. Then, by Theorem 2.8 of [4] it follows that $N_{\bar{L}}(C_{\bar{L}}(\bar{x}))$ is 2-dimensional non-abelian. So, $N_L(C_L(x))$ is non-nilpotent. It is easy to verify that $x$ satisfies the remaining conditions in (1).

COROLLARY 1.6. Assume char $(F) \neq 2$. Let $x \in L$ such that $N_L(C_L(x))$ is almost-abelian. Then $E_L(x)$ is either almost-abelian, isomorphic to sl (2, $F$) or a form of an Albert-Zassenhaus algebra (in the latter two cases $C_L(x) = Fx$).

PROOF. Let $A$ denote the derived subalgebra of $N_L(C_L(x))$. Since $N_L(C_L(x))$ is almost-abelian, we have that $A$ is abelian, $\dim N_L(C_L(x))/A = 1$ and that every proper ideal of $N_L(C_L(x))$ is contained in $A$. This yields, $A = C_L(x)$. On the other hand, we have $Z(E_L(x)) \leq C_L(x)$ and thus $Z(E_L(x)) \leq Z(N_L(C_L(x))$. Since $N_L(C_L(x))$ is centerless, so is $E_L(x)$. Then, by Theorem 1.4 it follows that either $E_L(x) = N_L(C_L(x))$, and then $E_L(x)$ is almost-abelian, or $E_L(x)$ is simple and $C_L(x) = Fx$. Now, the result follows from Theorems 2.8, 7.1 and 8.1 of [4].

COROLLARY 1.7. Let $L$ be simple. Then the following are equivalent:

1) $L$ contains a self-centralizing ad-nilpotent element,

2) $L$ contains an ad-nilpotent element $x$ such that $N_L(C_L(x))$ is almost-abelian.

Next, we prove that none of the Engel subalgebras of a simple Lie algebra is almost-abelian or isomorphic to sl (2). For this, we will need the following lemma which will appear in [22]. We include it by completeness.

LEMMA 1.8. Let $L$ be a simple Lie algebra of dimension greater than 3 over an arbitrary field $F$ of characteristic $p \neq 2, 3$. Assume $x \in L$ is self-centralizing. Then one of the following holds:

1) $E_L(x) = Fx$ and $L$ is a form of an Albert-Zassenhaus algebra,

2) $E_L(x)$ is a form of an Albert-Zassenhaus algebra and $p \mid \dim L$.

PROOF. Let $\Gamma$ denote the centroid of $L$. Since $\Gamma x$ is an abelian subalgebra of $L$, we have that $\Gamma x \leq C_L(x) = Fx$. So $\Gamma = F$, thus $L$ is central-simple. Let $L_\Omega = L \otimes_F \Omega$. We have that $L_\Omega$ is simple over $\Omega$.

By using [4, Theorems 2.8, 7.1 and 8.1], we obtain that $E_L(x)$ is either 1-dimensional, 2-dimensional non-abelian, isomorphic to sl (2, $F$) or a form of an Albert-Zassenhaus algebra. First suppose that $\dim E_L(x) = 1$. Then we find that $\Omega x$ is a Cartan subalgebra of $L_\Omega$. So $L_\Omega$ is an Albert-Zassenhaus algebra by [6]. Therefore $L$ is as in (i).

In the remaining cases we have that $x \in E_L(x)'$. Then, we can write $x = \Sigma [a_i, b_i]$ where $a_i, b_i \in E_L(x)$. Let $L = E_L(x) + V$ be the Fitting decomposition of $L$ relative to ad $x$. By [14, p. 38], $V$ is invariant under ad $y$ for every $y \in E_L(x)$. Then, we obtain that ad $x$ has trace zero on $V$. So $p$ divides dim $V$ whenever $p > 0$. Therefore, if $E_L(x)$ is a form of an Albert-Zassenhaus algebra we obtain that $p \mid \dim L$.

On the other hand, we have that $(E_L(x))_\Omega$ contains a Cartan subalgebra $C$ of $L_\Omega$ by [3]. Now assume tht $E_L(x)$ is either 2-dimensional non-abelian or isomorphic to sl(2, $F$). Then, as $C$ is also a Cartan subalgebra of $(E_L(x))_\Omega$, we find dim $C = 1$. This yields that $L_\Omega$ is an Albert-Zassenhaus algebra, by [6] again. In particular, we find that $\dim L = p^n$ for some $n$. But, from $p \mid \dim V$ and $L = E_L(x) + V$ it follows that $\dim L \equiv 2, 3 \pmod p$. This contradiction completes the proof.

PROPOSITION 1.9. Let $L$ be a simple Lie algebra of dimension $> 3$ over a field of characteristic $p \neq 2, 3$. Then, none of the Engel subalgebras of $L$ is almost-abelian or 3-dimensional simple.

PROOF. Let $x \in L$. First we note that $E_L(x)$ cannot be isomorphic to sl(2, $F$), since otherwise we would have $C_L(x) = Fx$ which contradicts Lemma 1.8. Now assume that $E_L(x)$ is almost-abelian. Then $C_L(x)$ is an abelian ideal of $E_L(x)$ and $E_L(x) = C_L(x) + Fy$ where ad $y$ acts as the identity map on $C_L(x)$. Let $\Gamma$ denote the centroid of $L$. We have that $\Gamma y$ is an abelian $F$-subalgebra of $L$ and $(\mathrm{ad}\, x)^2 (\Gamma y) = \Gamma [[yx]x] = 0$. This yields, $\Gamma y \leq E_L(x) \cap C_L(y) = Fy$ and so $\Gamma = F$. Therefore, $L_\Omega = L \otimes_F \Omega$ is simple. As $(E_L(x))_\Omega$ is the Engel subalgebra of $L_\Omega$ relative to $x$, $(E_L(x))_\Omega$ contains a Cartan subalgebra $H$ of $L_\Omega$ by Barnes [3]. We have $\dim H = 1$ since $(E_L(x))_\Omega$ is almost-abelian and $H$ is also a Cartan subalgebra of $(E_L(x))_\Omega$. Say $H = \Omega h$. We can decompose $h = c_0 + \alpha y$ where $c_0 \in (C_L(x))_\Omega$, $\alpha \in \Omega$. Let $c \in C_L(x)$. We have $[ch] = \alpha[cy] = \alpha c$ since $(C_L(x))_\Omega$ is abelian. So, $C_L(x)$ is contained in a root space of $L_\Omega$ relative to $H$. Then by Corollary 3.8 of [6] it follows that $C_L(x)$ is 1-dimensional. This yields, $C_L(x) = Fx$ which contradicts Lemma 1.8. The proof is now complete.

COROLLARY 1.10. Let $L$ be a simple Lie algebra of dimension greater than 3 over a field of characteristic $p \neq 2, 3$. Let $x \in L$ such that $N_L(C_L(x))$ is almost-abelian. Then $E_L(x)$ is a form of an Albert-Zassenhaus algebra and $p \mid \dim L$.

PROOF. It follows from Corollary 1.6, Proposition 1.9 and Lemma 1.8.

We finish this paragraph showing two criteria for an element $x$ of a Lie algebra to be ad-semisimple. We will need them in sections 2 and 3.

We will say that ad $x$ is separable if its minimum polynomial is separable.

LEMMA 1.11. Let $x$ be an element of a Lie algebra $L$ over a field $F$.

(1) If $\operatorname{ad} x$ is separable and $C_L(x)$ is a maximal subalgebra of $L$, then $\operatorname{ad} x$ is either nilpotent or semisimple.

(2) Assume char $(F) \neq 2, 3$. If $L$ is simple and $E_L(x) = Fx$, then $\operatorname{ad} x$ is semisimple and separable.

PROOF. (1): Suppose $C_L(x)$ is a maximal subalgebra of $L$. Since, $C_L(x) \leq E_L(x) \leq L$ it follows that either $C_L(x) = E_L(x)$ or $E_L(x) = L$. In the latter case, we have that $\operatorname{ad} x$ is nilpotent. Then suppose $C_L(x) = E_L(x)$. Let $K$ be a splitting field of the minimum polynomial of $\operatorname{ad} x$ over $F$, and let $L_K = L \otimes_F K$. Let $U$ denote the direct sum of all eigenspaces of $L_K$ relative to $\operatorname{ad} x$. As $E_L(x) \neq L$, $x$ acts non-nilpotently on $L$. So, $\operatorname{ad} x$ is not nilpotent on $L_K$. Therefore, $\operatorname{ad} x$ has a nonzero eigenvalue on $L_K$. This yields $U \neq (C_L(x))_K$.

Let $G$ be the Galois group of $K$ over $F$. Then, for each $\sigma \in G$ the $K$-linear map $\sigma' = 1 \otimes \sigma$ is a Lie-automorphism of $L_K$. As $K$ is a Galois extension of $F$, an element of $L_K$ lies in $L$ if and only if it is fixed by $\sigma'$ for every $\sigma \in G$. Let $z$ be an eigenvector of $L_K$ relative to $\operatorname{ad} x$, so $[z, x] = \alpha z$ for some $\alpha \in K$. We find,

$$[\sigma'(z), x] = [\sigma'(z), \sigma'(x)] = \sigma'[z, x] = \sigma(\alpha)\sigma'(z).$$

This yields, $\sigma'(U) \leq U$ for every $\sigma \in G$. Therefore, $(U \cap L)_K = U$ (see [7, p. 54]).

We have $(C_L(x))_K \leq U = (U \cap L)_K$, whence $C_L(x) \leq U \cap L$. Since $U \cap L$ is a subalgebra of $L$, from the maximality of $C_L(x)$ it follows either $U \cap L = C_L(x)$ or $U \cap L = L$. In the former case, we find $U = (U \cap L)_K = (C_L(x))_K$ which is a contradiction. Consequently, $U \cap L = L$. This yields, $U = L_K$. Therefore, $\operatorname{ad} x$ is semisimple on $L$.

(2): Suppose $L$ is simple and $E_L(x) = Fx$. Let $\Gamma$ denote the centroid of $L$. Since $[\gamma x, x] = \gamma[x, x] = 0$, we have $\Gamma x \leq E_L(x) = Fx$. This yields $\Gamma = F$ and so $L_\Omega$ is simple. Since $E_L(x)_\Omega$ is the Engel subalgebra of $L_\Omega$ relative to $\operatorname{ad} x$, we have that $\Omega x$ is a Cartan subalgebra of $L_\Omega$. The result now follows from Corollary 3.8 of [6].

## 2. Lie algebras all of whose proper subalgebras are either semisimple, abelian or almost-abelian

PROPOSITION 2.1. Assume char $(F) \neq 2$. For a Lie algebra $L$ the following statements are equivalent:

1) Every subalgebra (including $L$ itself) is either semisimple, abelian or almost-abelian,

2) For every $a \in L - (0)$, $N_L(a))$ is either abelian or almost-abelian.

PROOF. (1) implies (2): Let $a \in L-(0)$. Since $Fa \leq Z(C_L(a)) \trianglelefteq N_L(C_L(a))$, we have that $N_L(C_L(a))$ is not semisimple. Therefore, $N_L(C_L(a))$ is either abelian or almost-abelian.

(2) implies (1): Let $S$ be a subalgebra of $L$ of dimension $> 1$. Assume that $S$ is not semisimple. So that, $S$ has an abelian minimal ideal $A$. Pick $a \in A-(0)$. Then, ad $a$ is nilpotent on $S$ and so $S \leq E_L(a)$. Assume $N_L(C_L(a))$ is abelian. Then $E_L(a) = C_L(a)$ by Proposition 1.2, and so $S$ is abelian. Now assume that $N_L(C_L(a))$ is almost-abelian. Then, by Corollary 2.6 we have that either $E_L(a)$ is almost-abelian or $C_L(a) = Fa$. In the former case, we have that $S$ is either abelian or almost-abelian. In the latter case, by using Theorem 2.8 of [4] we obtain that $S$ is 2-dimensional nonabelian. The proof is complete.

THEOREM 2.2. Let $L$ be a nonsolvable Lie algebra over a field $F$ with char $(F) \neq 2, 3$. Assume that every proper subalgebra of $L$ is either semisimple, abelian or almost-abelian. Then one of the following holds:

1) $L \cong \mathrm{sl}(2, F)$.

2) $L = Fa \oplus L'$, $L'$ is simple, and $L'$ is the only proper subalgebra of $L$ which is not abelian.

3) $Z(L) \neq 0$, $L/Z(L)$ is simple, and $L$ has only abelian proper subalgebras.

4) $L$ is semisimple and the last term $L^{(\infty)}$ in the derived series of $L$ is simple. If $L^{(\infty)} \neq L$ then $L$ has no nonzero ad-nilpotent elements. If $L^{(\infty)} = L$, then for every $a \in L-(0)$ either $E_L(a)$ is abelian or $C_L(a) = Fa$ and $E_L(a)$ is a form of an Albert-Zassenhaus algebra (in particular, $L$ contains no almost-abelian subalgebra of dimension greater than two).

PROOF. Let us first suppose that $L$ is not semisimple. Then there exists a nonzero maximal abelian ideal $A$ of $L$. Since $A \leq Z(C_L(A))$, we have that $C_L(A)$ is neither semisimple nor almost-abelian. So, $C_L(A)$ is abelian. By the maximality of $A$, it follows $C_L(A) = A$, or $L$. Assume $C_L(A) = A$ and let $x, y \in L-A$. Then $A + Fx$ and $A + Fy$ are solvable but not abelian, so they are almost-abelian. Thus, there exist $\lambda, \mu \in F$ such that $[ax] = \lambda a$, $[ay] = \mu a$ for every $a \in A$. We find,

$$\lambda\mu a = \lambda[ay] = [[ax]y] = [a[xy]] + [[ay]x] = [a[xy]] + \mu\lambda a,$$

whence $[a[xy]] = 0$ for every $a \in A$. This yields, $L' \leq C_L(A) = A$ so $L$ is solvable, which is a contradiction. Therefore, $C_L(A) = L$. Thus $A = Z(L)$ and then $Z(L)$ is the unique maximal abelian ideal of $L$. This yields that $L/Z(L)$ is simple. Now suppose that there exists a nonabelian proper subalgebra $S$ of $L$. Then $S$ must be either semisimple or almost-abelian. Choose $z \in Z(L)$-(0). Then $S + Fz = L$, since $z \in Z(S + Fz) \neq S + Fz$. It follows that

$S = L'$, $Z(L) = Fz$ and $L/Z(L) \cong L'$. We deduce that $L'$ is simple and that it is the unique nonabelian proper subalgebra of $L$. Hence, $L$ is either as in (2) or (3).

We assume that $L$ is semisimple and $L \ncong \mathrm{sl}(2, F)$. Let $a \in L - (0)$. By Proposition 2.1, $N_L(C_L(a))$ is either abelian or almost-abelian. By using Proposition 1.2 and Corollary 1.6 we obtain that $E_L(a)$ is either abelian, almost-abelian, or simple with $C_L(a) = Fa$. Assume that $L$ is not simple an let $N$ be a minimal ideal of $L$. Pick $b \in N - (0)$. We have $L = N + E_L(b)$ since $N$ contains the Fitting 1-component of $L$ relative to $\mathrm{ad}\, b$. Thus, $L/N \cong E_L(b)/E_L(b) \cap N$. Since $b \in E_L(b) \cap N \lhd E_L(b)$, we have that $E_L(b)$ is not simple. So, $E_L(b)$ is either abelian or almost-abelian. This yields that $L/N$ is solvable. Thus $N = L^{(\infty)}$ and $N$ is the unique minimal ideal of $L$. Moreover, we have that $N$ cannot be abelian or almost-abelian, so $N$ is semisimple. From the minimality of $N$ it follows that $N$ is simple (see [23, p. 30]). Now suppose $L = L^{(\infty)}$. Then by Proposition 1.9 none of the Engel subalgebras of $L$ is almost-abelian. Let $S$ be an almost-abelian subalgebra of $L$. Pick $a \in S'$, $a \neq 0$. We have $S \leq E_L(a)$ and $S' = C_L(a)$. It follows that $E_L(a)$ cannot be abelian and so $C_L(a) = Fa$. Therefore $\dim S = 2$. This completes the proof.

A field $F$ is said to be a field of type $(C_1)$ if every homogeneous polynomial $f(\lambda_1, \ldots, \lambda_n)$ over $F$ of degree less than the number $n$ of variables has a nontrivial root in $F^n$.

Corollary 2.3. Let $F$ be a perfect field of type $(C_1)$ with $\mathrm{char}\,(F) \neq 2, 3$. Let $L$ be a non-solvable Lie algebra whose proper subalgebras are either semisimple, abelian or almost-abelian. Then, $L \cong \mathrm{sl}(2, F)$.

Proof. By Corollary 1.3 of [20], every Lie algebra over $F$ contains nonzero ad-nilpotent elements. Thus it is clear that (2) in Theorem 2.2 cannot occur. Next suppose $L$ is as in (3) in that theorem. Then, we have that every element of $L/Z(L)$ is ad-semisimple by Theorem 4.1 of [8]. This yields that $L/Z(L)$ contains no nonzero ad-nilpotent elements a contradiction again. Now suppose that $L$ is as (4) in Theorem 2.2. Then we have that $L$ must be simple and that every nonzero ad-nilpotent element must be self-centralizing. But then, we find that $L$ contains a solvable subalgebra which is neither abelian or almost-abelian (see Theorem 7.1 of [4] and Theorem 3.1 of [5]), which is a contradiction. Therefore $L \cong \mathrm{sl}(2, F)$ by Theorem 2.2.

The following example shows a simple Lie algebra in which every proper subalgebra is either 3-dimensional simple, 2-dimensional nonabelian or 1-dimensional.

EXAMPLE. Let $\lambda \in F$. We denote by $W_{F,1}(\lambda)$ the derivation algebra of $F[X]/(X^p - \lambda)$. It is well known that $W_{F,1}(\lambda)$ is simple. Clearly, $W_{F,1}(\lambda)$ contains self-centralizing ad-nilpotent elements, so that $W_{F,1}(\lambda)$ has almost-abelian subalgebras of dimension two.

Let $F$ be non-perfect and let $\lambda \in F - F^p$. Then we prove that every proper subalgebra of $W_{F,1}(\lambda)$ is either 3-dimensional simple, 2-dimensional nonabelian or 1-dimensional. Let $L = W_{F,1}(\lambda)$. We have that $L_\Omega$ is isomorphic to the Witt algebra over $\Omega$. So $L_\Omega$ has a unique subalgebra $(L_\Omega)_0$ of codimension one, and every proper subalgebra of $L_\Omega$ not contained in $(L_\Omega)_0$ is either 2-dimensional nonabelian, 1-dimensional or isomorphic to $\mathrm{sl}(2, \Omega)$ (see Corollary 3.10 of [4]). Let $S$ be maximal among subalgebras of $L$ with $S_\Omega \leq (L_\Omega)_0$. First assume that every element of $S$ is ad-nilpotent on $L$. Then by Engel's theorem $N_L(S) \neq S$. By our choice of $S$, we have $(N_L(S))_\Omega \nleq (L_\Omega)_0$. Pick $x \in (N_L(S))_\Omega$, $x \notin (L_\Omega)_0$. Since no eigenvector of $\mathrm{ad}\, x$ lies in $(L_\Omega)_0$ (see Lemma 3.7 of [4]), we have $S = 0$. Next assume that $S$ contains an element $y$ such that $\mathrm{ad}\, y$ is not nilpotent. By Proposition 1.9, $E_L(y)$ is neither almost-abelian nor 3-dimensional simple. It follows that $(E_L(y))_\Omega \leq (L_\Omega)_0$. Let $E_L(b)$ be the minimal Engel subalgebra of $L$ contained in $E_L(y)$. By [3], $E_L(b)$ is a Cartan subalgebra of $L$. So that $(E_L(b))_\Omega$ is a Cartan subalgebra of $L_\Omega$. Thus, $(E_L(b))_\Omega = \Omega b$. Since $\Omega b \leq (L_\Omega)_0$, we have that $(L_\Omega)_0$ is a direct sum of root spaces relaces relative to $\mathrm{ad}\, b$. On the other hand, we have that $\mathrm{ad}\, b$ is semisimple and separable over $F$ (Lemma 1.11). Let $K \leq \Omega$ be the splitting field of the minimal polynomial of $\mathrm{ad}\, b$ over $F$. Since every root space of $L_\Omega$ relative to $\mathrm{ad}\, b$ is 1-dimensional, we deduce that $L_K$ has a subalgebra $M$ such that $M_\Omega \cong (L_\Omega)_0$. We have that $M$ is invariant under all automorphisms of $L_K$ since it is the unique subalgebra of $L_K$ of codimension one. Since $K$ is a Galois extension of $F$, it follows that there exists a subalgebra $N$ of $L$ such that $N_K = M$. We have $\dim L/N = 1$. Then, since $L$ has a self-centralizing ad-nilpotent element, from Theorem 3.9 of [4] it follows that $L$ is isomorphic to the Witt algebra $W_{F,1}(0)$. This yields $\lambda \in F^p$, a contradiction. We conclude that $\Omega x \nleq (L_\Omega)_0$ for every $x \in L - (0)$. From this it follows that every proper subalgebra of $L$ of dimension greater than one is either 2-dimensional nonabelian or 3-dimensional simple, as desired.

For perfect fields $F$ of characteristic $p \neq 2, 3$, Gein [12, Proposition 3] has proved that a semisimple Lie algebra $L$, not isomorphic to $\mathrm{sl}(2, F)$, whose proper subalgebras are either abelian, almost-abelian or 3-dimensional non-split simple is simple ad-semisimple (that is, $\mathrm{ad}\, x$ is semisimple for every $x \in L$). From this it follows that $L$ contains no almost-abelian subalgebras (see Proposition 1.2 of [8]). Next, we give a proof of this last result which works for any field of characteristic neither 2 nor 3. More generally, we obtain the following

COROLLARY 2.4. Let $L$ be a semisimple Lie algebra over an arbitrary field $F$ with char $(F) \neq 2, 3$. Assume that every proper subalgebra of $L$ is either abelian, almostabelian or 3-dimensional simple. Then one of the following holds:

1) $L \cong \mathrm{sl}(2, F)$.

2) $L \cong W_{F,1}(\lambda)$ for some $\lambda \in F - F^p$.

3) $L$ is simple having no nonzero ad-nilpotent elements and every proper subalgebra of $L$ is either abelian or 3-dimensional non-split simple. If $F$ is perfect, then $L$ is ad-semisimple.

PROOF. If $L' \neq L$, then by our hypothesis $L'$ is 3-dimensional simple. Thus every derivation of $L'$ is inner. By [14, p. 11], $L = L' \oplus K$ for some ideal $K$. But since $K \cong L/L'$, we have that $K$ is abelian which contradicts the semisimplicity of $L$. Therefore $L' = L$. Then, by Theorem 2.2, we get that $L$ is simple. Now suppose that $L$ contains no nonzero ad-nilpotent elements. Then, for every $a \in L - (0)$ the Engel subalgebra $E_L(a)$ is abelian. This yields that $L$ contains no almost-abelian subalgebras. Since sl (2, $F$) does contain almost-abelian subalgebras, it follows that every proper subalgebra of $L$ is either abelian or 3-dimensional non-split simple. So, $L$ is an in (3). The last assertion in (3) follows from Lemma 1.11.

Next suppose that $L$ has a nonzero ad-nilpotent element $x$. Then $E_L(x) = L$ and $C_L(x) = Fx$ by Theorem 2.2. If dim $L = 3$, then we have $L \cong \mathrm{sl}(2, F)$. Suppose $\dim L > 3$. Let $S$ denote the kernel of the derivation $(\mathrm{ad}\, x)^p$. We have that $S$ is a simple subalgebra of $L$ of dimension $p$, by Theorem 2.8 of [4]. This yields $S = L$ and so $(\mathrm{ad}\, x)^p = 0$. Thus $L$ is a form of the Witt algebra $W_{\Omega,1}(0)$. Let $y \in L$ such that ad $y$ is not nilpotent. Since $E_L(y)$ cannot be either almost-abelian or 3-dimensional simple (Proposition 1.9), it follows that $E_L(y)$ is abelian. Thus $E_L(y)$ is a Cartan subalgebra of $L$. This yields, $\dim E_L(y) = 1$ and so $E_L(y) = Fy$. Let $L_\Omega = \Omega y \oplus (L_\Omega)_\alpha \oplus \cdots \oplus (L_\Omega)_{(p-1)\alpha}$ be the decomposition of $L_\Omega$ into its root spaces relative to the Cartan subalgebra $\Omega y$. Since every root space has dimension one, it follows that the minimal polynomial of ad $y$ has the form $X^p - \beta X$ where $\beta \in F - (0)$. So, the derivation $(\mathrm{ad}\, y)^p$ is inner. We conclude that $L$ is restricted. Therefore, $L$ is isomorphic to $W_{F,1}(\lambda)$ for some $\lambda \in F$ (see [1]). If $\lambda \in F^p$, then $L$ is isomorphic to the Witt algebra $W_{F,1}(0)$. But then, $L$ contains a subalgebra of codimension 1 which is neither abelian, almost-abelian nor simple. Therefore, $\lambda \notin F^p$. The proof is complete.

For perfect fields of characteristic $p \neq 2, 3$, we shall give in section 4 more information on the structure of a Lie algebra satisfying conditions in Corollary 2.4.

## 3. Modular and relatively complemented Lie algebras

A Lie algebra $L$ is called modular, upper semi-modular or relatively complemented if its lattice of all subalgebras has the corresponding property. The study of these classes of Lie algebras was begun by Kolman in [15]. Of course, every modular Lie algebra is upper semimodular; and by Maeda [17], every upper semi-modular Lie algebra is relatively complemented. For perfect fields of characterisric not 2 or 3, these three concepts are equivalent (Gein [12]).

For arbitrary fields, Gein [9] has proved that an upper semi-modular Lie algebra is either abelian, almost-abelian, or a Lie algebra in which any two linearly independent elements generate a simple subalgebra. Here, we obtain this same result for relatively complemented Lie algebras over arbitrary fields of characteristic $\neq 2, 3$.

First we note that the Lie algebras in which any two linearly independent elements generate a simple subalgebra are precisely those in which every subalgebra of dimension greater than one is simple (called supersimple Lie algebras). Particular cases of supersimple Lie algebras are the Lie algebras of dimension $> 2$ whose non-trivial subalgebras are 1-dimensional (in [12], these algebras are called $\mu$-algebras).

In [19] it is proved that for a Lie algebra $L$ over an arbitrary field $F$, the following are equivalent: (i) $L$ is supersimple, (ii) $\dim L > 2$ and $L$ contains no 2-dimensional subalgebras, (iii) $E_L(a) = Fa$ for every $a \in L - (0)$ and $\dim L > 1$. By Lemma 1.11, it follows that a supersimple Lie algebra is ad-semisimple. For perfect fields of characteristic $\neq 2, 3$, a supersimple Lie algebra must be 3-dimensional non-split simple (Lemma 1.1 of [21]). An example of a supersimple Lie algebra of dimension 7 over a perfect field of characteristic 3 is given in [12, Example 2], in this algebra any two linearly independent elements generate a 3-dimensional non-split subalgebra.

The existence of supersimple Lie algebras over a field $F$ of characteristic $> 3$, other than the 3-dimensional non-split simple Lie algebras, is an interesting open problem. We prove that such Lie algebras cannot contain 3-dimensional non-split simple subalgebras. Therefore, there exists such a Lie algebra if and only if there is a $\mu$-algebra over $F$ of dimension greater than three.

PROPOSITION 3.1. Let $L$ be supersimple of dimension greater than 3 over a field $F$ of characteristic $p > 3$. Then $L$ contains no 3-dimensional non-split simple subalgebras.

PROOF. Assume on the contrary that $L$ contains a 3-dimensional non-split simple subalgebra $S$. Pick $a \in S - (0)$. Let $K$ be a splitting field of the minimum polynomial of $\operatorname{ad} a$ over $F$. Let $L_K := L \otimes_F K$. By [19] $Ka$ is a Cartan subalgebra of $L_K$, and by [6, Corollary 3.8] each root space $V_\alpha$ of $L_K$ relative to $Ka$ is one-dimensional. Since $S_K \cong \mathrm{sl}(2, K)$, we have $S_K = Ka \oplus V_\alpha \oplus V_{-\alpha}$ for some root $\alpha$. Let $G$ denote the Galois group of $K$ over $F$. Let $\sigma \in G$ and $\sigma'$ be the Lie automorphism of $L$ defined by $\sigma$. Say $V_\alpha = Ke_\alpha$. Since every element of $L$ is fixed by $\sigma'$, we have

$$[\sigma'(e_\alpha), a] = \sigma'[e_\alpha, a] = \sigma(\alpha)\sigma'(e_\alpha).$$

Moreover, $\sigma'(e_\alpha) \in S_K$. We deduce that $\sigma(\alpha) = \pm\alpha$. Let $(L_K)^{(\alpha)}$ be the 1-section of $L_K$ corresponding to $\alpha$, that is

$$(L_K)^{(\alpha)} = Ka \oplus V_\alpha \oplus V_{2\alpha} \oplus \cdots \oplus V_{(p-1)\alpha}.$$

We have that $(L_K)^{(\alpha)}$ is a subalgebra of $L_K$ invariant under the action of $\sigma'$ for every $\sigma \in G$. As $K$ is a Galois extension of $F$, it follows that $(L_K)^{(\alpha)} = U_K$ for some subalgebra $U$ of $L$ (see [7, p. 54]). We deduce that $U$ is a form of the Witt algebra $W_1(\Omega)$. On the other hand, we have that $U$ is ad-semisimple since so is $L$. However, the forms of $W_1(\Omega)$ are known not to be ad-semisimple (c.f. [1]). This contradiction shows that $L$ contains no 3-dimensional non-split simple subalgebra, as desired.

THEOREM 3.2. Let $L$ be a relatively complemented Lie algebra over an arbitrary field $F$ of characteristc $\neq 2, 3$. Then $L$ is either abelian, almost-abelian or supersimple.

PROOF. By Theorem 9 of Gein and Muhin [10], it follows that $L$ is either abelian, almost-abelian or semisimple. Note that every subalgebra of $L$ is also a relatively complemented Lie algebra, so $L$ is an $X$-algebra.

Let us first suppose that $L$ is simple. Let $a \in L - (0)$. Then by Theorem 2.2 either $E_L(a)$ is abelian or $C_L(a) = Fa$ and $\dim E_L(a) > 3$. In the latter case, from the Jordan canonical form for the transformation $\operatorname{ad} a$, it follows that $E_L(a)$ has a basis $y_{-1}, y_0, \ldots, y_m$ with $[y_i, a] = y_{i-1}$ and $y_{-1} = a$. We see that $Fy_0 + Fy_{-1}$ is the only two-dimensional subspace of $E_L(a)$ which is invariant under $\operatorname{ad} y_{-1}$. Now we take a complement $K$ of the subalgebra $Fy_0 + Fy_{-1}$ in the interval $[Fy_{-1} : E_L(a)]$; so that

$$K \vee (Fy_0 + Fy_{-1}) = E_L(a) \quad \text{and} \quad K \cap (Fy_0 + Fy_{-1}) = Fy_{-1}.$$

Since $K$ contains $y_{-1}$, $K$ is invariant under $\operatorname{ad} y_{-1}$. Nilpotency of $\operatorname{ad} y_{-1}$ implies that $K$ contains a two-dimensional subspace which is invariant under $\operatorname{ad} y_{-1}$. This yields $y_0 \in K$, which is a contradiction. Therefore, $E_L(a)$ is abelian for every $a \in L - (0)$. From this it follows that every subalgebra of $L$ which contains nonzero ad-nilpotent elements is abelian. So that, $L$ contains no almost-abelian subalgebras. Hence, every subalgebra of $L$ is either abelian or semisimple. Moreover, the field $F$ must be infinite since otherwise $L$ itself would contain a nonzero ad-nilpotent element (see [20]). Next we claim that every semisimple subalgebra of $L$ is simple. To prove this assume that $S$ is a nonsimple, semisimple subalgebra of $L$ of minimal dimension. Then by Theorem 2.2 it follows that $\dim S/S' = 1$ and $S'$ is simple. Pick $b \in 0$. We have $E_S(b) \leq E_L(b)$, so $E_S(b)$ is abelian. Since every proper subalgebra of $S$ containing the Engel subalgebra $E_S(b)$ is self-normalizing (see [3]), we have $E_S(b) \nleq S'$. So, $E_S(b) = E_S(b) \cap S' + Fc$ *for some nonzero element* $c$. Take a complement $C$ of $E_S(b)$ in $[Fc\colon S]$. We have $0 \neq C \cap S'$, so that $C$ is not simple. Assume that $C$ is abelian. Then, since $S = E_S(b) \vee C$ we have $Fc \lhd S$ which is a contradiction. It follows that $C$ is nonsimple and semisimple, which contradicts the minimality of $S$. The claim is proved. Now we prove that $L$ contains no abelian subalgebras of dimension greater than one. Assume on the contrary that $A$ is a maximal abelian subalgebra of $L$ of dimension $> 1$. If $N_L(A) \neq A$, then as $N_L(A)$ is not simple, $N_L(A)$ is abelian. Therefore, $A = N_L(A)$ by the maximality of $A$. Thus, $A$ is a Cartan subalgebra of $L$. Let $a \in A - (0)$. Pick a complement $U$ of $E_L(a)$ *in* $[Fa\colon L]$. Since $L = E_L(a) \vee U$ we have that $U$ is simple; otherwise we would have $Fa \lhd L$, which is a contradiction. Since $E_U(a) = E_L(a) \cap U = Fa$, by Lemma 1.1 it follows that $\operatorname{ad} a$ is diagonalizable on $U_\Omega$. Let $V$ be the sum of the eigenspaces of $L_\Omega$ relative to $\operatorname{ad} a$. We have $U_\Omega + (E_L(a))_\Omega \subseteq V$ and since $V$ is a subalgebra of $L_\Omega$, $V = L_\Omega$. We deduce that $\operatorname{ad} a$ is diagonalizable over $L_\Omega$ for every $a \in A$. Let $\Phi$ be the root system of $L_\Omega$ relative to the Cartan subalgebra $A_\Omega$. Every root in $\Phi$ is linear since $A_\Omega$ is abelian. Then, as $F$ is infinite, there exists an element $a_0$ in $A$ such that $\alpha(a_0) \neq 0$ and $\alpha(a_0) \neq \beta(a_0)$ for every $\alpha, \beta \in \Phi$, $\alpha \neq \beta$. Take a complement $U$ of $E_L(a_0)$ in $[Fa_0\colon L]$. We have that $U$ is simple. Let $u \in U_\Omega$ be an eigenvector relative to $\operatorname{ad} a_0$ corresponding to a nonzero eigenvalue $\lambda$. By our choice of $a_0$, there exists a unique root $\alpha \in \Phi$ such that $\alpha(a_0) = \lambda$. It follows that $u$ lies in the corresponding root space $(L_\Omega)_\alpha$ of $L_\Omega$. As every element of $A$ is diagonalizable on $L_\Omega$, we deduce that $[A, u] \subseteq \Omega u$. Therefore, $[A, U] \leq U$. We get that $A + U$ is a subalgebra of $L$ and that $U$ is an ideal of $A + U$, which is a contradiction. We conclude that every subalgebra of $L$ is either 1-dimensional or simple, which means that $L$ is supersimple.

What remains is to prove that every semisimple Lie algebra which is relatively complemented is simple. Let $L$ be a counterexample of minimal dimension. Then, by Theorem 2.2 we have $\dim L/L' = 1$ and $L'$ is simple. By the preceding paragraph, $L'$ is supersimple. Pick $a \in L' - (0)$. We have $E_{L'}(a) = Fa$. Since $E_L(a) \nleq L'$, it follows $\dim E_L(a) = 2$. Let $b \in E_L(a)$, $b \notin Fa$. Then $[ab] = ta$ for some $t \in F$. Assume $t \neq 0$ and let $L_1(a)$ denote the Fitting

1-component of ad $a$. Since ad $b$ stabilizes $L_1(a)$ (see [14, p. 38]) and $[\text{ad } a, \text{ad } b] = t(\text{ad } a)$, we have that the trace of $(\text{ad } a)|_{L_1(a)}$ is zero. This yields that $p \neq 0$ and $p$ divides $\dim L_1(a)$. As $L_1(a) \cong L'/Fa$, it follows that $p$ divides $\dim L'/Fa$. On the other hand, since $L'$ is supersimple, we have that $(L')_\Omega$ is either isomorphic to sl$(2, \Omega)$ or an Albert-Zassenhaus algebra. So, $\dim L' = 3$ or a power of $p$. This yields $p = 2$, which is a contradiction. Therefore $E_L(a)$ is abelian. Now take a complement $U$ of $E_L(a)$ in $[Fb: L]$. We see that $U$ cannot be abelian, since otherwise we would have $Fa \lhd L$ a contradiction. Since $0 \neq S' \cap U$, it folows that $U$ is not simple. Thus $U$ is not semisimple because of the minimality of $L$. Therefore $U$ is almost-abelian. Let $x \in L' \cap U$, $x \neq 0$. By above, we have that $E_L(x)$ is abelian. Moreover, we have $U \leq E_L(x)$ since $U$ is almost-abelian and $x \in U'$. This yields that $U$ is abelian, which is a contradiction. Now the proof is complete.

As a direct consequence of Theorem 3.2 and Lemma 1.1 of [21], we obtain the following result due to Gein [12].

COROLLARY 3.3 (Gein [12]). For a Lie algebra $L$ over a perfect field $F$ with char $(F) \neq 2, 3$, the following are equivalent: (i) $L$ is modular, (ii) $L$ is upper semi-modular, (iii) $L$ is relatively complemented, and (iv) $L$ is either abelian, almost-abelian or 3-dimensional non-split simple.

## 4. Minimal non-modular Lie algebras

A Lie algebra $L$ is called minimal non-modular if every proper subalgebra of $L$ is a modular Lie algebra but $L$ is not. Assume $F$ is perfect with char $(F) \neq 2, 3$. Then, a Lie algebra $L$ over $F$ is minimal non-modular if and only if every proper subalgebra of $L$ is either abelian, almost-abelian or 3-dimensional non-split simple but $L$ is not.

By Corollary 2.4, we have that if $L$ is semisimple and minimal non-modular then either $L \cong$ sl$(2, F)$ or $L$ is simple and ad-semisimple. This result was first proved by Gein in [12]. The following result gives us more information on the structure of a simple and minimal non-modular Lie algebra.

THEOREM 4.1. Let $L$ be a minimal non-modular Lie algebra over a perfect field $F$ with char $(F) \neq 2, 3$. Assume $L$ is simple and $L \ncong$ sl$(2, F)$. Then the following holds:

(1) $L$, regarded as a Lie algebra over its centroid, has only abelian subalgebras and is a form of a classical simple Lie algebra.

(2) If $L$ contains a nonabelian proper $F$-subalgebra, then $L$ is 3-dimensional non-split simple over its centroid.

PROOF. (1): Let us first suppose $L$ is central-simple. Since $\operatorname{ad} x$ is semisimple for every $x \in L$ (Corollary 2.4), it follows from [18, Corollary in p. 869] that $L$ is a form of a classical simple Lie algebra. Assume $L$ has a proper nonabelian subalgebra $S$. Then, by Corollary 2.4, $S$ is 3-dimensional non-split simple. Moreover, $L$ has rank $> 1$ since otherwise we would have $\dim L = 3$ a contradiction. We can take a basis $e_1, e_2, e_3$ for $S$ with product $[e_2, e_3] = e_1$, $[e_3, e_1] = \alpha e_2$, $[e_1, e_2] = \beta e_3$ where $\alpha, \beta \in F - (0)$, see [14, p. 13]. Let $x = e_1$. The characteristic polynomial of $\operatorname{ad} x$ on $S$ has the form $\lambda(\lambda^2 + \alpha\beta)$. Let $f(\lambda)$ denote the minimum polynomial of $\operatorname{ad} x$ on $L$. Since $\operatorname{ad} x$ is semisimple we can write $f(\lambda) = \pi_0(\lambda)\pi_1(\lambda)\cdots\pi_r(\lambda)$ where $\pi_0(\lambda) = \lambda$, $\pi_1(\lambda) = \lambda^2 + \alpha\beta$, $\pi_i(\lambda)$ is an irreducible polynomial on $F$ of degree $> 1$ for $i \geq 1$, and $\pi_i \neq \pi_j$ for $i \neq j$. Let $\mu \in \Omega$ be a root of $\pi_1(\lambda)$. Then there exists a maximal number $n$ such that $2^n\mu$ is an eigenvalue of $L_\Omega$ relative to $\operatorname{ad} x$. By Corollary 2.4, the subalgebra $E_L(x)$ is abelian and so it is a Cartan subalgebra of $L$. So, $(E_L(x))_\Omega$ is a Cartan subalgebra of $L_\Omega$. Let $\Phi$ be the root system of $L_\Omega$ relative to $(E_L(x))_\Omega$. We have that there exists $\sigma \in \Phi$ such that $\sigma(x) = 2^n\mu$. Since $L_\Omega$ is classical, $-\sigma$ is a root too. Thus, $-2^n\mu$ is also an eigenvalue of $L_\Omega$ relative to $\operatorname{ad} x$. This yields that the polynomial $\lambda^2 + 2^{2n}\alpha\beta$ divides $f(\lambda)$. So, $\pi_i(\lambda) = \lambda^2 + 2^{2n}\alpha\beta$ for some $1 \leq i \leq r$. Let $V$ denote the kernel of $\pi_i(\operatorname{ad} x)$. Since $\operatorname{ad} x$ *is diagonalizable on* $L_\Omega$ and $2^{n+1}\mu$ is not an eigenvalue of $\operatorname{ad} x$, we have $[V, V] \leq E_L(x)$. This yields that the subspace $E_L(x) + V$ is a nonabelian subalgebra of $L$. If $E_L(x) + V \neq L$, then $E_L(x) + V$ is 3-dimensional non-split simple. This yields, $E_L(x) = Fx$ and so $L$ is rank one which is a contradiction. Therefore, $E_L(x) + V = L$ and then $f(\lambda) = \lambda(\lambda^2 + \alpha\beta)$. We deduce that $\mu$ and $-\mu$ are the only nonzero eigenvalues of $\operatorname{ad} x$ on $L_\Omega$. Therefore, $\sigma(x) = \pm\mu$ for every $\sigma \in \Phi$. Since $L_\Omega$ is classical simple of rank greater than 1, there exist $\sigma_1, \sigma_2 \in \Phi$ such that $\sigma_1 + \sigma_2 \in \Phi$. Then, either $(\sigma_1 + \sigma_2)(x) = \sigma_1(x)$ or $(\sigma_1 + \sigma_2)(x) = -\sigma_1(x)$. In the former case we get $\sigma_2(x) = 0$. This yields, $(L_\Omega)_{\sigma_2} \leq C_L(x) = E_L(x)$ a contradiction. In the latter case, we get $\sigma_2(x) = -2\sigma_1(x)$. But since $\sigma_2(x) = \pm\sigma_1(x)$, we have $\sigma_1(x) = 0$ a contradiction again. We conclude that every proper subalgebra of $L$ is abelian.

Now suppose that $L$ is not central-simple and let $\Gamma$ denote the centroid of $L$. Let $S$ be a nonabelian proper $\Gamma$-subalgebra of $L$. Then, $S$ regarded as a $F$-subalgebra of $L$ is 3-dimensional non-split simple by Corollary 2.4. Then, since $\dim_F S = |\Gamma : F| \dim_\Gamma S$ and $\Gamma \neq F$, we get $\dim_\Gamma S = 1$. So, $S$ is abelian which is a contradiction. We conclude that $L$ regarded as a Lie algebra over $\Gamma$ has only abelian subalgebras. The proof of (1) is now complete.

(2): Assume $S$ is a nonabelian proper $F$-subalgebra of $L$. Then, by (1) the centroid $\Gamma$ of $L$ is a proper extension of $F$. Let $T$ be the $\Gamma$-subspace of $L$ generated by $S$. Clearly, $T$ is closed under the Lie bracket, so that $T$ is a subalgebra of $L$. Moreover, $T$ is nonabelian since $S$ is. Now, by (1) again we get $T = L$. On the other hand, we have that $S$ is 3-dimensional non-split simple over $F$ by Corollary 2.4. This yields, $\dim_\Gamma T \leq 3$. Since $L$ is simple over $\Gamma$ we get $\dim_\Gamma L = 3$. What remains is to show that $L$ is non-split over $\Gamma$. If not, then $L$ is isomorphic to $\mathrm{sl}(2, \Gamma)$. Thus the constants of multiplication of $L$, relative to a standard basis $B$ of $L$ over $\Gamma$, lie in $F$. But then, the $F$-span of $B$ is an $F$-subalgebra of $L$ isomorphic to $\mathrm{sl}(2, F)$, which contradicts Corollary 2.4. The proof is now complete.

## 5. Lie Algebras of length 3 and their Subalgebra Lattices

In this section we describe the Lie algebras $L$ with a subalgebra lattice $\mathscr{L}(L)$ of length 3 as well as their corresponding lattices of subalgebras. In particular, we show that the algebra $\mathrm{sl}(2, F)$ is determined by its subalgebra lattice.

Throughout this section $F$ denotes a perfect field $F$ of characteristic $p \neq 2, 3$.

LEMMA 5.1. Assume $L$ has length 3. Then, every proper subalgebra of $L$ of dimension greater than 1 is either 2-dimensional or 3-dimensional non-split simple.

PROOF. Let $S$ be a proper subalgebra of $L$ of dimension greater than 1. The lattice $\mathscr{L}(S)$ has length 2, since it is isomorphic to the interval $[0: S]$ of $\mathscr{L}(L)$. So, the result follows from Proposition 1 of [12].

THEOREM 5.2. The lattice $\mathscr{L}(L)$ have length 3 if and only if one of the following holds:

1) $L$ is 3-dimensional solvable.

2) $L \cong \mathrm{sl}(2, F)$.

3) $L$ is a direct sum of a 3-dimensional non-split simple Lie algebra and a 1-dimensional Lie algebra.

4) $L$ is 3-dimensional non-split simple over a quadratic extension $\Gamma$ of $F$.

5) $L$ is central-simple having only abelian subalgebras and it is a form of a classical simple Lie algebra of type $A_2$, $B_2$ or $G_2$.

PROOF. First suppose that $\mathscr{L}(L)$ has length 3. If $L$ is solvable, then clearly $\dim L = 3$. Thus $L$ is as in (1). Then assume $L$ is not solvable. Suppose $Z(L) \neq 0$. Then $L/Z(L)$ must be nonsolvable with length 2 and $\dim Z(L) = 1$. By Proposition 1 of [12], we have $L/Z(L)$ is 3-dimensional non-split simple. Then, it is easy to see that $L = L' \oplus Z(L)$ (see the proof of Theorem 3.1 of [21]). Thus, $L$ is as in (3). Now suppose $Z(L) = 0$. Then by using Lemma 5.1, Theorem 2.2 and Corollary 2.4, we obtain that $L$ is simple. Let $\Gamma$ be the centroid of $L$. If $\Gamma = F$, then $L$ is as in (2) or (5) by Theorem 4.1. Then assume $\Gamma \neq F$. Let $x \in L - (0)$. Since $\Gamma x$ is an abelian $F$-subalgebra of $L$ and $\Gamma x \neq Fx$, we have $\dim_F \Gamma x = 2$. This yields, $|\Gamma : F| = 2$. If $L$ contains a nonabelian proper $F$-subalgebra, then we get that $L$ is as in (4) by Theorem 4.1. Suppose then that every proper $F$-subalgebra of $L$ is abelian. Let $S$ be a proper $\Gamma$-subalgebra of $L$. Then we have that $S$ is abelian and $\dim_F S = 2$. Since $\dim_F S = |\Gamma : F| \dim_\Gamma S$, it follows $\dim_\Gamma S = 1$. We conclude that $L$, regarded as a Lie algebra over $\Gamma$, has only 1-dimensional subalgebras. So, by Proposition 1 of [12] it follows that $L$ is as in (4). This completes the proof in one direction.

Clearly, Lie algebras as in (1), (2), (3) and (5) have length 3. Now suppose that $L$ is as in (4). Then we need to prove that every proper $F$-subalgebra of $L$ of dimension greater than 2 is 3-dimensional non-split simple. Let $x \in L - (0)$. Since $L$ has no proper $\Gamma$-subalgebras of dimension greater than 1, we have $C_L(x) = \Gamma x$. So, $\dim_F C_L(x) = |\Gamma : F| = 2$. Moreover, we have that if $[xy] = tx$ for some $y \in L - (0)$ and $t \in F$, then $[xy] = 0$ since $x$ and $y$ must be linearly dependent over $\Gamma$. We deduce that every 2-dimensional $F$-subalgebra of $L$ is abelian and that $L$ has no abelian $F$-subalgebras of dimension greater than 2. On the other hand, we have that the linear transformation $\operatorname{ad} x$ is semisimple over $F$ since so is over $\Gamma$. Then, according to Proposition 1.2 of [8], every solvable $F$-subalgebra of $L$ is abelian. We conclude that $L$ has no solvable $F$-subalgebras of dimension greater than 2. Now let $S$ be a proper $F$-subalgebra of $L$ of dimension $r > 2$. Let us suppose first $r = 3$. Then $S' = S$; otherwise we would have that $S$ is solvable which is a contradiction. It follows that $S$ is non-split simple. Therefore, we may assume $r > 3$. Since $\dim_F L = |\Gamma : F| \dim_\Gamma L = 6$, we have $r < 6$. Choose $x \in S - (0)$. If $x$ acts nilpotently on $S$, then $x \in Z(S)$ since $\operatorname{ad} x$ is semisimple. So, $S \leq C_L(x)$ which contradicts the fact that $\dim C_L(x) = 2$. Therefore, $S$ has no nonzero ad-nilpotent element. Next suppose $r = 4$. Let $N$ be a proper ideal of $S$. If $\dim N \leq 2$, then $N$ is abelian and hence every nonzero element of $N$ acts nilpotently on $S$ which is a contradiction. Therefore, $\dim N = 3$. Then, we have that $N$ is 3-dimensional non-split simple. Thus, every derivation of $N$ is inner. By [14, p. 11], it follows $S = N \oplus Fa$ for some $a \in S - (0)$. But then, we find $\operatorname{ad}_S a = 0$ a contradiction again. We deduce that $S$ is simple. Now, as $\operatorname{ad} x$ is semisimple for every $x \in S$, the corollary in page 869 of [18] applies and $S$ is a form of a classical simple Lie algebra, which contradicts the fact that $\dim S = 4$. Consequently, $L$ contains no $F$-subalgebras of dimension 4. Finally, suppose $r = 5$. Then, by Theorem 2.2 of [20] it follows that $L$ contains a nonzero ad-nilpotent element, which is a contradiction again. Now the proof is complete.

COROLLARY 5.3. Let $F$ be a perfect field of type $(C_1)$ with char $(F) \neq 2, 3$. Then the lattice $\mathscr{L}(L)$ has length 3 if and only if $\dim L = 3$.

PROOF. Assume $\mathscr{L}(L)$ has length 3 and that $\dim L > 3$. Then we have that $L$ is as in (3), (4) or (5) in Theorem 5.2. In either case, $L'$ contains no nonzero ad-nilpotent element, which contradicts Corollary 1.3 of [20].

THEOREM 5.4. Let $\mathscr{L}$ be a lattice of length 3 and $L$ a Lie algebra over $F$ such that $\mathscr{L}(L) \cong \mathscr{L}$. Then,

1) $\mathscr{L}$ has no proper modular elements if and only if $L$ is as in (4) or (5) in Theorem 5.2.

2) $\mathscr{L}$ has just one modular atom which is complemented if and only if $L$ is as in (3) in Theorem 5.2.

3) $\mathscr{L}$ has just one modular atom which is not complemented if and only if $L$ has a basis $a, b, c$ with one of the following products:

(i) $[ab] = c$, $[ac] = [bc] = 0$.

(ii) $[ab] = 0$, $[ac] = a + b$, $[bc] = b$.

4) $\mathscr{L}$ has just two modular atoms if and only if $L$ has a basis $a, b, c$ with one of the following products:

(i) $[\mathrm{ab}] = a$, $[ac] = [bc] = 0$.

(ii) $[ab] = 0$, $[ac] = a$, $[bc] = \alpha b$ where $1 \neq \alpha \in F$

5) $\mathscr{L}$ is modular if and only if $L$ is either abelian or almost-abelian.

6) $L \cong \mathrm{sl}(2, F)$ if and only if $\mathscr{L}$ satisfies:

(i) for each atom $A$ there is another atom $B$ such that $A \vee B = L$,

(ii) there exists a modular co-atom, and

(iii) there exist $A, B \in \mathscr{L}$ such that $A \cap B \neq 0$.

7) $L$ has a basis $a, b, c$ with product $[ab] = 0$, $[ac] = b$, $[bc] = \beta a + \alpha b$ where $\alpha, \beta \in F$ and the polynomial $\lambda^2 - \alpha\lambda - \beta$ is irreducible on $F$ if and only if $\mathscr{L}$ satisfies:

(i) for each atom $A$ there is another atom $B$ such that $A \vee B = L$,

(ii) there exists a modular co-atom, and

(iii) $A \cap B = 0$ for every co-atoms $A, B$ with $A \neq B$.

PROOF. (1): Let us first suppose $L$ is as in (4) or (5) in Theorem 5.2. Let $x \in L - (0)$. We have $C_L(x) \neq Fx$, since otherwise we would have that $L$ is central simple of rank one which is a contradiction. By Lemma 1.5 of [2] it follows that $Fx$ is not a modular element in $\mathscr{L}(L)$. Now assume $S$ is a modular proper subalgebra of $L$ (that is, $S$ is a modular element of $\mathscr{L}(L)$ and $S \neq L$). We have $\dim S > 1$ and so $S$ is maximal since $L$ *has length* 3. Pick $x \in L - S$. Since $C_L(x) \neq Fx$, from the modularity of $S$ it follows $S \cap C_L(x) \neq 0$. If $S$ is abelian, then we have $S \cap C_L(x) \lhd L$ which contradicts the simplicity of $L$. Therefore, $S$ is 3-dimensional non-split simple and $L$ is as in (4) in Theorem

5.2. Take a basis $e_1, e_2, e_3$ for $S$ with product $[e_2, e_3] = e_1$, $[e_3, e_1] = \alpha e_2$, $[e_1, e_2] = \beta e_3$ where $\alpha, \beta \in F - (0)$. Let $\gamma \in \Gamma - F$. We find $S \cap C_L(e_1 + \gamma e_2) = 0$ a contradiction.

To prove the converse, suppose that $\mathscr{L}$ has no proper modular elements. Then $L$ must be simple. Since every 2-dimensional subalgebra of sl$(2, F)$ is modular, the result follows by Theorem 5.2.

The remaining statements can be easily proved by inspection of the 3-dimensional solvable Lie algebras and by using (1), Theorem 5.2 and the following facts:

i) the algebra sl$(2, F)$ has no modular atoms.

ii) every maximal subalgebra of sl$(2, F)$ of dimension greater than 1 is modular

iii) Lie algebras as in (3) in Theorem 5.2 have only one modular atom which is complemented.

DEFINITION 5.5. (Gein [11]). A lattice $\mathscr{L}$ is called an sl-lattice if it satisfies: (1) for each atom $A$ there is another atom $B$ such that $A \vee B = 1$, (2) there exists a modular co-atom, and (3) there exist $A, B \in \mathscr{L}$ such that $A \cap B \neq 0$.

COROLLARY 5.6. Let $L$ be a Lie algebra over a perfect field of characteristic not 2 or 3. Then, $L \cong$ sl$(2, F)$ if and only if $\mathscr{L}(L)$ is an sl-lattice.

## References

[1] H. Allen and M. Sweedler: A theory of linear descent based upon Hopf algebraic techniques, J. Algebra **12** (1969), 242–294.

[2] R. K. Amayo and J. Schwarz: Modularity in Lie algebras, Hiroshima Math. J. **10** (1980), 311–322.

[3] D. W. Barnes: On Cartan subalgebras of Lie algebras, Math. Z. **101** (1967), 350–355.

[4] G. M. Benkart, I. M. Isaacs and J. M. Osborn: Lie algebras with self-centralizing ad-nilpotent elements, J. Algebra **57** (1979), 279–309.

[5] G. M. Benkart, I. M. Isaacs and J. M. Osborn: Albert-Zassenhaus Lie algebras and isomorphisms, J. Algebra **57** (1979), 310–338.

[6] G. M. Benkart and J. Marshall Osborn: Rank one Lie algebras, Ann. of Math. (2) **119** (1984), 437–463.

[7] A. Borel: Linear Algebraic Groups, W.A. Benjamin, Inc. 1969

[8] R. Farnsteiner: Ad-semisimple Lie Algebras, J. Algebra **83** (1983), 510–519.

[9] A. G. Gein: Semimodular Lie algebras, Sibirsk. Mat. Z. **17** (1976) 243–248 (translated in Siberian Math. J. **17** (1976), 243–248).

[10] A. G. Gein and J. H. Muhin: Complements to subalgebras of Lie algebras, (Russian) Ural. Gos. Univ. Zap. **12** (1980), 2, 24–48.

[11] A. G. Gein: Modular subalgebras and projections of locally finite dimensional Lie algebras of characteristic zero, (Russian) Ural. Gos. Univ. Mat. Zap. **13** (1983), 39–51.

[12] A. G. Gein: Modular rule and relative complements in the lattice of subalgebras of a Lie algebra, Sov. Math. **31**, n°3, (1987) 22–32; Translation from Izv. Vyssh. Uchebn. Zaved., Mat. n°3, **83** (1987) 18-25

[13] A. G. Gein: Co-atomic lattices and Lie algebras. (Russian) Ural. Gos. Univ. Mat. Zap. **14** (1988), n°3 Algebr. Sistemy i ikh Mnogoobr., 66–74, ii.

[14] N. Jacobson: Lie Algebras, Wiley-Interscience, New York (1962).

[15] B. Kolman: Relatively Complemented Lie algebras, J. Sci. Hiroshima Univ. Ser. A-I **31** (1967), 1–11.

[16] A. Lashi: On Lie algebras with modular lattices of subalgebras, J. Algebra **99** (1986), 80–86.

[17] S. Maeda: On atomistic lattices with the covering property, J. Sci. Hiroshima Univ., Ser A-I **31** (1967), 105–121.

[18] A. A. Premet: Absolute zero divisors in Lie algebras over a perfect field, (Russian), Dokl. Akad. Nauk. BSSR **31** (10), (1987), 869–872.

[19] V. R. Varea: Lie algebras none of whose Engel subalgebras are in intermediate position, Comm. in Algebra, **15** (12) (1987), 2529–2543.

[20] V. R. Varea: On the existence of ad-nilpotent elements and simple Lie algebras with a subalgebra of codimension 1, Proc. Amer. Math. Soc. **104** (1988), 363–368.

[21] V. R. Varea: The subalgebra lattice of a supersolvable Lie algebra, Lie algebras, Madison 1987. Springer Lecture Notes in Math. **1373** (1989), 81–92.

[22] V. R. Varea: On modular subalgebras in Lie algebras of prime characteristic, Conference on Lie algebras and Related Topics, Madison 1988, Contemporary Mathematics **110** (1990), 289–307.

[23] D. J. Winter: Abstract Lie Algebras, MIT Press, Cambridge, Mass., 1972.

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