HIROSHIMA MATH. J.
24 (1994), 647–655

Remark on the characterization of weighted Besov spaces via temperatures

Dedicated to Professor Makoto Ohtsuka on the occasion of his 70*th* birthday

Huy-Qui BUI

(Received April 30, 1993)

1. Introduction and statements of the results

The aim of this note is to complete the characterization of the weighted homogeneous Besov spaces by means of solutions of the heat equation on $\boldsymbol{R}_+^{n+1}$, which was initiated in [2]. We retain all the notations and terminologies in [1, 2] to which the reader is referred also for background and references to related works in the literature.

In [2, Theorem 1′] we proved that

$$\|f\|_{\dot{B}(s,w,p,q)} \le C\left(\int_0^\infty (t^{k-s/2}\|(\partial/\partial t)^k(W_t * f)\|_{p,w})^q \frac{dt}{t}\right)^{1/q} \tag{1}$$

for any $f \in \mathscr{S}'$, where $w \in A_\infty$, $0 < p < \infty$, $0 < q \le \infty$, $s \in \boldsymbol{R}$, k is a non-negative integer greater than $s/2$, $W_t(x) = (4t)^{-n/2}e^{-|x|^2/4t}$ is the Gauss-Weierstrass kernel on $\boldsymbol{R}_+^{n+1}$, and as usual, we use C, c, ... to denote positive constants whose values might change from one occurrence to the next one; they might depend on the parameters s, p, q, k, ..., but not on the distribution f. If a constant depends also on f, we use subscript to denote this dependence, e.g., C_f.

For the opposite direction to (1), in the same quoted theorem, we had to assume that w is furthermore in $\dot{\mathscr{M}}_d$. In this note we shall remove this restriction. Namely, we shall prove the following result.

THEOREM. *If $f \in \dot{B}_{p,q}^{s;w}$ and k is a non-negative integer greater than $s/2 + \max(0, 1/p - 1, 1/q - 1)$, then there is a polynomial P such that*

$$\left(\int_0^\infty (t^{k-s/2}\|(\partial/\partial t)^k(W_t * (f-P))\|_{H(p,w)})^q \frac{dt}{t}\right)^{1/q} \le C\|f\|_{\dot{B}(s,w,p,q)}. \tag{2}$$

REMARK. Though we have removed the restriction $w \in \dot{\mathscr{M}}_d$, we have introduced a new restriction on the range of k in the case either $p < 1$ or $q < 1$. This new restriction is, however, more satisfactory than the other, as it does not depend on the weight function w, and the result in the Theorem is sufficient for our study of the integrability of the Fourier transform on weighted spaces in [3].

REMARK. In the paper [9], H. Triebel has studied various equivalent quasi-norms on the unweighted Besov and Triebel-Lizorkin spaces. His method was based on a rather sharp multiplier's criterion for spaces of analytic functions of exponential type. It would be an interesting problem to find out how much of his results can be extended to the present weighted case.

2. Lemmata

In the proof of the Theorem we shall need a number of Lemmata.

LEMMA 1. *Let $s < 0$, and assume that u is a temperature on $\boldsymbol{R}_+^{n+1}$ for which*

$$\|u\|_{s,w,p,q} = \left(\int_0^\infty (t^{-s/2}\|u(\cdot, t)\|_{p,w})^q \frac{dt}{t}\right)^{1/q} < \infty .$$

Then there exist $\rho > 0$ and $f \in \mathscr{S}'$ such that $u = W_t * f$, and

$$|u(x, t)| \leq C\|u\|_{s,w,p,q} t^{s/2-n/2\rho}(1 + t)^{n/2\rho}(1 + |x|)^{n/\rho} \tag{3}$$

for all $(x, t) \in \boldsymbol{R}_+^{n+1}$. Consequently, f is in $\dot{B}_{p,q}^{s,w}$,

$$\|f\|_{\dot{B}(s,w,p,q)} \leq C\|u\|_{s,w,p,q} , \tag{4}$$

and the semi-group formula holds for u, i.e., $u(\cdot, r + t) = W_t * u(\cdot, r)$ for all r, $t > 0$.

PROOF. The proof of (3) and the existence of f is similar to that of the sufficient part of [2, Proposition 7(i)] (cf. (29) of [2] for (3)), while (4) follows from (1). The semi-group formula holds for u because $u = W_t * f$.

LEMMA 2. *Let $s < 0$, and let $\dot{T}_{p,q}^{s,w}$ denote the space of all temperatures u on $\boldsymbol{R}_+^{n+1}$ for which*

$$\|u\|_{\dot{T}(s,w,p,q)} = \left(\int_0^\infty (t^{-s/2}\|u(\cdot, t)\|_{H(p,w)})^q \frac{dt}{t}\right)^{1/q} < \infty .$$

Then $\dot{T}_{p,q}^{s,w}$ is a quasi-Banach space.

PROOF. First we note that, if $u \in \dot{T}_{p,q}^{s,w}$, then, as

$$\|u(\cdot, t)\|_{p,w} \leq \|u(\cdot, t)\|_{H(p,w)}$$

for all $t > 0$ (by Lebesgue differentiation theorem), we see that

$$\|u\|_{s,w,p,q} \leq \|u\|_{\dot{T}(s,w,p,q)} . \tag{5}$$

Next, observe that an equivalent quasi-norm on H_w^p is given by

$$\|g\|_{H(p,w)} = \|g^+\|_{p,w}$$

for $g \in \mathscr{S}'$, where $g^+(x) = \sup_{r>0} |W_r * g(x)|$, $x \in \boldsymbol{R}^n$. This is the quasi-norm on H_w^p which we shall use in the rest of this note. By (5) and Lemma 1, the semi-group formula holds for u, so that

$$\|u(\cdot, t)\|_{H(p,w)} \leq \|u(\cdot, \rho)\|_{H(p,w)}, \qquad t \geq \rho > 0.$$

This implies that

$$\|u\|_{\dot{T}(s,w,p,\infty)} = \sup_{t>0} t^{-s/2} \|u(\cdot, t)\|_{H(p,w)} \leq C \|u\|_{\dot{T}(s,w,p,q)} \tag{6}$$

for $0 < q \leq \infty$. (From which it follows that $\dot{T}_{p,q}^{s,w} \subseteq \dot{T}_{p,r}^{s,w}$, $0 < q \leq r \leq \infty$, but we do not need this embedding in our note.)

It is obvious that $\|\cdot\|_{\dot{T}(s,w,p,q)}$ is a quasi-norm. We next prove that $\dot{T}_{p,q}^{s,w}$ is complete. Let (u_j) be a Cauchy sequence in $\dot{T}_{p,q}^{s,w}$. Then (5) and (3) imply that (u_j) converges uniformly on each compact subset of $\boldsymbol{R}_+^{n+1}$, so that the limit function u is a temperature on $\boldsymbol{R}_+^{n+1}$. Furthermore, $u_j(\cdot, t)$ converges to $u(\cdot, t)$ in $\mathscr{S}'$ for each $t > 0$. On the other hand, from (6) we deduce that for each t, $(u_j(\cdot, t))$ is a Cauchy sequence in H_w^p and thus must converge to an element in H_w^p. Since $H_w^p \subset \mathscr{S}'$ (continuous embedding), we conclude that $u_j(\cdot, t)$ converges to $u(\cdot, t)$ in H_w^p for each $t > 0$. Fatou's lemma then implies that u_j converges to u in $\dot{T}_{p,q}^{s,w}$. The proof of the lemma is thus complete.

Lemma 3. *If P is a non-zero polynomial and $0 < p < \infty$, then $\|P\|_{p,w} = \infty$.*

Proof. As P is non-zero, there exist $\beta > 0$ and an unbounded cone $D \subseteq \boldsymbol{R}^n$ such that $|P(x)| \geq \beta$ for every x in D. In D we can choose a sequence of closed cubes (I_j) whose interiors are mutually disjoint such that there exists $M > 0$ for which $\bigcup_{j=1}^{\infty} I_j^* = \boldsymbol{R}^n$, where I_j^* is the cube I_j expanded M-times. Since w is doubling and positive almost everywhere, it follows that

$$\begin{aligned}
\|P\|_{p,w}^p &\geq \beta^p \int_D w(x)dx \\
&\geq \beta^p \sum_{j=1}^{\infty} w(I_j) \\
&\geq c \sum_{j=1}^{\infty} w(I_j^*) \\
&\geq c w(\boldsymbol{R}^n) = \infty .
\end{aligned}$$

The well-known fact that $w(\boldsymbol{R}^n) = \infty$ can be seen from the inequality in line 2, page 142 of [6] which is also an excellent source of references for properties of weight functions.

3. Proof of the Theorem

Let $f \in \dot{B}^{s,w}_{p,q}$. We retain all the notations in the proof of Theorem 1′ in [2, pp. 60–62]. The proof there relied on the representation

$$f - P = \lim_{m\to\infty} \sum_{j=-m}^{\infty} (\psi_j * f + P_m) \qquad \text{in } \mathscr{S}',$$

where $\deg(P_m) \le N_f$ for all m, and in that proof we had used the equation

$$(\partial/\partial t)^k (W_t * P_m) = W_t * (-\Delta)^k P_m = 0 .$$

In the present case, we do not have control of N_f, but if we assume that $k > N_f/2$, then the above equation holds. A careful examination of the proof in [2] quoted above shows that the inequality

$$\left(\int_0^\infty (t^{k-s/2} \|(\partial/\partial t)^k (W_t * (f-P))\|_{H(p,w)})^q \frac{dt}{t}\right)^{1/q} \le C_f \|f\|_{\dot{B}(s,w,p,q)} \tag{7}$$

holds for all $k > \max(s/2, N_f/2)$. The dependence of C_f on f comes from the fact that various constants appearing in the proof depend on k (and hence implicitly on f).

Fix a non-negative integer k for which (7) holds. Let l be a non-negative integer greater than $s/2 + \max(0, 1/p - 1, 1/q - 1)$. We shall prove that (7) holds with k replaced by l (with possibly a different polynomial P). If $l = k$, then the result is obvious. Assume next that $l > k$, and let $m = l - k$. Put $K_t(x) = (\partial/\partial t)^m W_t(x)$. Then, as

$$D^\kappa K_t(x) = t^{-m-|\kappa|/2} W_t(x) R(x_1/2\sqrt{t}, \dots, x_n/2\sqrt{t})$$

for every $t > 0$, $x \in \boldsymbol{R}^n$ and every multi-index $\kappa = (\kappa_1, \dots, \kappa_n)$, where $|\kappa| = \kappa_1 + \cdots + \kappa_n$, and R is a polynomial of degree $|\kappa| + 2m$ (cf. [5, Lemma 4 (iii)], it is easy to verify that

$$|D^\kappa K_t(x)| \le C_\kappa t^{-m} |x|^{-|\kappa|-n} .$$

Hence it follows from [1, Lemma 4.6] that

$$\|(\partial/\partial t)^m (W_t * g)\|_{H(p,w)} = \|K_t * g\|_{H(p,w)} \le C t^{-m} \|g\|_{H(p,w)} \tag{8}$$

for every $g \in H^p_w$. The result for l in this case then follows from (7) (for k) and (8).

The result for the case $l < k$ will follow if we can prove that (7) implies the result for $k-1$; for then we can use similar proofs to show that (7) holds for $k-2, \dots, l$. Note that the assumptions on l imply that

$$k - 1 > k - 2 > \cdots = l > s/2 + \max(0, 1/p - 1, 1/q - 1) .$$

To prove that (7) holds for $k-1$, let $v=(\partial/\partial t)^k(W_t*(f-P))$ and $u=(\partial/\partial t)^{k-1}(W_t*(f-P))$. As $v=(\partial/\partial t)u$, $s-2/k<0$, and

$$\|v\|_{s-2k,w,p,q}\leq\|v\|_{\dot{T}(s-2k,w,p,q)}<\infty$$

by (5) and (7), we see that, for $1<t_1<t_2$ and $x\in\boldsymbol{R}^n$,

$$\begin{aligned}|u(x,t_1)-u(x,t_2)|&\leq\int_{t_1}^{t_2}|v(x,r)|dr\\&\leq C_r\|v\|_{s-2k,w,p,q}\int_{t_1}^{t_2}r^{-(k-s/2)}dr\to 0\end{aligned}$$

as $t_1,\ t_2\to\infty$, by (3) and the condition $k-1>s/2$. It follows that

$$\lim_{t\to\infty}u(x,t)=\phi(x)$$

exists for every $x\in\boldsymbol{R}^n$. Now since

$$u(x,t)=-\int_0^\infty v(x,r+t)dr+\phi(x)\,, \tag{9}$$

and both u and the integral on the right-hand side of (9) are temperatures, ϕ is harmonic in $\boldsymbol{R}^n$. By [5, Theorem 17] and (3) we can find $M>0$ for which

$$|\phi(x)|\leq C(1+|x|)^M$$

for all $x\in\boldsymbol{R}^n$. This implies that ϕ must be a harmonic polynomial. By a result in [7, Proof of Proposition, p. 299] we have $W_t*\phi=\phi$ for all $t>0$, and by [8, Chapter IV, Proof of Theorem 2.1] we can find a polynomial Q such that $(-\Delta)^{k-1}Q=\phi$, so that

$$\phi=(\partial/\partial t)^{k-1}(W_t*Q)\,.$$

If we replace P by $P+Q$, then v is unchanged ($(\partial/\partial t)^k(W_t*Q)=W_t*(-\Delta)\phi=0$), and for this new P, (9) implies the relation

$$u(x,t)=-\int_0^\infty v(x,r+t)dr \tag{10}$$

for every $x\in\boldsymbol{R}^n$ and $t>0$. Let $u_t(x)=u(x,t)$ and $v_t(x)=v(x,t)$. Then the semi-group formula and (10) imply that

$$u_t^+(x)\leq tv_t^+(x)+\int_t^\infty v_r^+(x)dr\,. \tag{11}$$

We consider two cases.

Case 1: $1 \le p < \infty$.

Integrating (11) and using Minkowski's inequality we obtain

$$\|u_t\|_{H(p,w)} \le t\|v_t\|_{H(p,w)} + \int_t^\infty \|v_r\|_{H(p,w)} dr\,. \tag{12}$$

If $q \ge 1$, then Hardy's inequality implies that

$$\begin{aligned}\left(\int_0^\infty (t^{k-1-s/2}\|u_t\|_{H(p,w)})^q \frac{dt}{t}\right)^{1/q} &\le C\left(\int_0^\infty (t^{k-s/2}\|v_t\|_{H(p,w)})^q \frac{dt}{t}\right)^{1/q} \\ &\le C_f \|f\|_{\dot{B}(s,w,p,q)}\end{aligned}$$

by (7) for k. (See [8, Chapter V, Lemma 3.14 for Hardy's inequality].)

Assume next that $0 < q \le 1$. As the estimate for the first term in the right-hand side of (12) is obvious, we need only estimate the second term. Noting that

$$v_r^+(x) \le v_t^+(x)\,, \tag{13}$$

for $r \ge t > 0$ by the semi-group formula, and discretizing the integral in r, we obtain

$$\begin{aligned}&\int_0^\infty \left(t^{k-1-s/2}\int_t^\infty \|v_r\|_{H(p,w)} dr\right)^q \frac{dt}{t} \\ &\le \int_0^\infty t^{(k-1-s/2)q}\left(\sum_{j=1}^\infty t^q \|v_{jt}\|^q_{H(p,w)}\right)\frac{dt}{t} \\ &\le C\int_0^\infty t^{(k-s/2)q}\left(\sum_{j=1}^\infty \int_{jt/2}^{(j+1)t/2} \|v_r\|^q_{H(p,w)}\frac{dr}{t}\right)\frac{dt}{t} \\ &= C\int_0^\infty \left(\int_0^{2r} t^{(k-s/2)q}\frac{dt}{t^2}\right)\|v_r\|^q_{H(p,w)} dr \\ &= C'\int_0^\infty (r^{k-s/2}\|v_r\|_{H(p,w)})^q \frac{dr}{r}\,,\end{aligned}$$

where we have used Fubini's theorem and the condition $k-1 > s/2 + (1/q - 1)$. The proof in the case $1 \le p < \infty$ is thus complete.

Case 2: $0 < p < 1$.

Again we only estimate the second term in the right-hand side of (11). Using (13) to discretize the integral in r, we get

$$I(x) = \int_t^\infty v_r^+(x)dr \le \sum_{j=1}^\infty t v_{jt}^+(x)\,,$$

so that

$$\|I\|_{p,w} \le t\left(\sum_{j=1}^\infty \|v_{jt}\|_{H(p,w)}^p\right)^{1/p}.$$

If $0 < q \le p$, then, similar to the proof in the case $1 \le p < \infty$ and $0 < q \le 1$ given above, we have

$$\begin{aligned}&\left(\int_0^\infty t^{(k-1-s/2)q}\|I\|_{p,w}^q \frac{dt}{t}\right)^{1/q}\\ &\le C\left(\int_0^\infty \|v_r\|_{H(p,w)}^q\left(\int_0^{2r} t^{(k-s/2)q}\frac{dt}{t^2}\right)dr\right)^{1/q}\\ &= C'\left(\int_0^\infty (r^{k-s/2}\|v_r\|_{H(p,w)})^q\frac{dr}{r}\right)^{1/q}\end{aligned}$$

because $k-1 > s/2 + (1/q - 1)$ by our assumption.

On the other hand, if $p < q$, then by using again the discretization of the integral, we obtain

$$\begin{aligned}&\left(\int_0^\infty (t^{k-1-s/2}\|I\|_{p,w})^q \frac{dt}{t}\right)^{p/q}\\ &\le C\left(\int_0^\infty t^{(k-s/2)q}\left(\int_{t/2}^\infty \|v_r\|_{H(p,w)}^p\frac{dr}{t}\right)^{q/p}\frac{dt}{t}\right)^{p/q}\\ &= C'\left(\int_0^\infty t^{(k-s/2)q-q/p}\left(\int_t^\infty \|v_r\|_{H(p,w)}^p dr\right)^{q/p}\frac{dt}{t}\right)^{p/q}\\ &\le C''\left(\int_0^\infty (r^{k-s/2}\|v_r\|_{H(p,w)})^q\frac{dr}{r}\right)^{p/q}\end{aligned}$$

by Hardy's inequality as $k - 1 - s/2 > 1/p - 1$ by our assumption.

Since C_f in (7) depends on f, what we have proved is that, for any non-negative integer $k > s/2 + \max(0, 1/p - 1, 1/q - 1)$, there is a polynomial P such that

$$\left(\int_0^\infty (t^{k-s/2}\|(\partial/\partial t)^k(W_t * (f-P))\|_{H(p,w)})^q\frac{dt}{t}\right)^{1/q} < \infty\,. \tag{14}$$

To complete the proof of the Theorem, we shall verify the inequality in quasi-norm in (2). Assume first that $-s/2 > \max(0, 1/p - 1, 1/q - 1)$, and we shall prove (2) for $k = 0$; the inequality for $k > 0$ then follows from (8) as in the proof of (7) for $l > k$. In this case ($k = 0$) we claim that the polynomial P for which (14) holds is unique. For if Q is another such polynomial, then we deduce that

$$\| W_t * (P - Q) \|_{p,w} < \infty$$

for every t. Fixing a t and noticing that $W_t * (P - Q)$ is a polynomial, we derive from Lemma 3 that $W_t * (P - Q) = 0$, so that $P = Q$. Thus, $W_t * (f - P)$ in (14) depends only on the equivalent class of f in $\dot{B}^{s,w}_{p,q}$. It follows that

$$E : \dot{B}^{s,w}_{p,q} \to \dot{T}^{s,w}_{p,q}$$

given by $Ef = W_t * (f - P)$ is well-defined and is a linear map.

We next show that E has closed graph. Let f, f_j, $j = 1, 2, \ldots$, be in $\dot{B}^{s,w}_{p,q}$ such that $f_j \to f$ in $\dot{B}^{s,w}_{p,q}$ and $u_j = Ef_j = W_t * (f_j - P_j) \to u$ in $\dot{T}^{s,w}_{p,q}$. Lemma 2 implies that u is a temperature, and Lemma 1 and (5) imply that $u = W_t * g$ for some $g \in \dot{B}^{s,w}_{p,q}$. It then follows from (1) that

$$\begin{aligned} \| f_j - g \|_{\dot{B}(s,w,p,q)} &= \| f_j - P_j - g \|_{\dot{B}(s,w,p,q)} \\ &\leq C \| u_j - u \|_{\dot{T}(s,w,p,q)}, \end{aligned}$$

so that $f = g$ in $\dot{B}^{s,w}_{p,q}$, i.e., there exists a polynomial P such that $g = f - P$, or $u = W_t * (f - P) = Ef$. Thus the graph of E is closed. It follows from the closed graph theorem that E is continuous, and hence there is $C > 0$ for which

$$\| Ef \|_{\dot{T}(s,w,p,q)} \leq C \| f \|_{\dot{B}(s,w,p,q)}$$

for all $f \in \dot{B}^{s,}_{p,q}$, which is (2) in this case.

In the general case, let $f \in \dot{B}^{s,w}_{p,q}$ and $k > s/2 + \max(0, 1/p - 1, 1/q - 1)$. Then, as $(-\Delta)^k f$ is in $\dot{B}^{s-2k,w}_{p,q}$ and

$$-(s - 2k)/2 > \max(0, 1/p - 1, 1/q - 1),$$

the result in the previous case yields a polynomial Q for which

$$\begin{aligned} \| W_t * ((-\Delta)^k f - Q) \|_{\dot{T}(s-2k,w,p,q)} &\leq C \| (-\Delta)^k f \|_{\dot{B}(s-2k,w,p,q)} \\ &\leq C \| f \|_{\dot{B}(s,w,p,q)}. \end{aligned}$$

By [8, Chapter IV, Proof of Theorem 2.1], we can find a polynomial P such that $(-\Delta)^k P = Q$. This fact and the above inequality imply the desired result (2). The proof of the Theorem is thus complete.

The results in this paper were obtained during my stay at University of New South Wales and Flinders University from mid-January to mid-March 1993. The writing was done when I was at Washington University. I wish to thank my colleagues at these institutions, in particular, Tony Dooley, Garth Gaudry and Mitch Taibleson, for their hospitality and support.

Notes added in July 1993. We had recently obtained a simpler proof of the Theorem which also applies to the weighted Triebel-Lizorkin spaces by showing that we can take $\deg(P_m) \leq [s]$ in the representation of $f - P$ given at the beginning of Section 3, for every m, where $[s]$ is the greatest integer not exceeding s. More precisely, we proved that Theorem 1′ (i) and Theorem 4′(i) of [2] hold without the restriction $w \in \dot{\mathscr{M}}_d$ (see [3]). Consequently, the restriction in the range of k mentioned in the first remark after the Theorem is also removed. However, we feel that, though the proof in the present paper is more complicated, it introduces some new ideas which may be useful in other situations.

References

[1] H.-Q. Bui, Weighted Besov and Triebel spaces: Interpolation by the real method, Hiroshima Math. J., **12** (1982), 581–605.

[2] H.-Q. Bui, Characterizations of weighted Besov and Triebel-Lizorkin spaces via temperatures, J. Funct. Anal., **55** (1984), 39–62.

[3] H.-Q. Bui, Bernstein's theorem on weighted Besov spaces, in preparation.

[4] R. R. Coifman and C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, Studia Math., **51** (1974), 241–250.

[5] T. M. Flett, Temperatures, Bessel potentials and Lipschitz spaces, Proc. London Math. Soc., **22** (1971), 385–451.

[6] J. García-Cuerva and J. L. Rubio de Francia, Weighted norm inequalities and related topics, North Holland, Amsterdam, New York, 1985.

[7] R. Johnson, Temperatures, Riesz potentials, and the Lipschitz spaces of Herz, Proc. London Math. Soc., **27** (1973), 290–316.

[8] E. M. Stein and G. Weiss, Introduction to Fourier analysis on Euclidean spaces, Princeton University Press, Princeton, New Jersey, 1971.

[9] H. Triebel, Characterizations of Besov-Hardy-Sobolev spaces: A unified approach, J. Approx. Theory, **52** (1988), 162–203.

Department of Mathematics
University of Canterbury
Christchurch 1, *New Zealand*