HIROSHIMA MATH. J.
24 (1994), 565–571

On a vector-valued interpolation theoretical proof of the generalized Clarkson inequalities

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(Received April 30, 1993)

Introduction

In [4] Kato gave the generalized Clarkson inequalities by using the Littlewood matrices. Later, Tonge gave in his interesting paper [11] their second proof based on an algebraic structure of these matrices, where the generalized Hausdorff-Young inequality by Williams and Wells [12] is used. He proved them directly for L_p without dealing the scalar case. On the other hand, Maligranda and Persson [8] (see also [9]) recently discussed them in a more generalized form, where an interpolation theoretical treatment is found for the scalar case. (Such a treatment for scalar case is also found in Pietsch's work [10].)

The aim of this paper is, applying complex vector-valued interpolation, to give another direct proof of the generalized Clarkson inequalities. (Unfortunately, 'simple application' to L_p of the argument for the scalar case in Pietsch [10] or Maligranda and Persson [8] stated above does not work well.) Our proof reveals the 'structure' of these inequalities well and it seems to be easily applicable to obtaining these inequalities for some other Banach spaces (cf. the authors [6]). In a special case, our proof may provide one of the most concise proofs of classical Clarkson's inequalities.

ACKNOWLEDGEMENTS. The authors are indebted to the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture (04640172, 1992). They would like to thank Professor L. Maligranda for sending them the recent paper [9]. The second author would like to express his hearty thanks to Professor W. Schachermayer for his hospitality during his stay at the University of Vienna by the KIT Research Fellowship Program.

1. Clarkson's and generalized Clarkson's inequalities

In this section, we recall Clarkson's and generalized Clarkson's inequalities, and prepare our tool concerning the complex method of vector-valued interpolation.

Let $L_p = L_p(\Omega, \Sigma, \mu)$, $1 < p < \infty$, be the usual L_p-space on an arbitrary but fixed measure space (Ω, Σ, μ). Let $l_r^n(L_p)$, $1 \leqq r \leqq \infty$, be the space of

L_p-valued sequences $\{f_j\}$ of length n with the norm

$$\|\{f_j\}\|_{r(p)} = \begin{cases} \left(\sum_{j=1}^{n} \|f_j\|_p^r\right)^{1/r} & \text{if } 1 \leqq r < \infty, \\ \max_{1 \leqq j \leqq n} \|f_j\|_p & \text{if } r = \infty. \end{cases}$$

Let A_n be the Littlewood matrices, that is,

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad A_{n+1} = \begin{pmatrix} A_n & A_n \\ A_n & -A_n \end{pmatrix} \qquad (n = 1, 2, \dots).$$

We denote by ε_{ij} the entries of A_n.

In the followings, let p', r', s', ... be the conjugate numbers of p, r, s, ... respectively, i.e., $1/p + 1/p' = 1/r + 1/r' = 1/s + 1/s' = \cdots = 1$.

CLARKSON'S INEQUALITIES (Clarkson [3]). *For all f and g in L_p,*

$$(1) \qquad (\|f+g\|_p^{p'} + \|f-g\|_p^{p'})^{1/p'} \leqq 2^{1/p'}(\|f\|_p^p + \|g\|_p^p)^{1/p} \qquad \text{if } 1 < p \leqq 2,$$

$$(2) \qquad (\|f+g\|_p^{p} + \|f-g\|_p^{p})^{1/p} \leqq 2^{1/p}(\|f\|_p^{p'} + \|g\|_p^{p'})^{1/p'} \qquad \text{if } 2 < p < \infty.$$

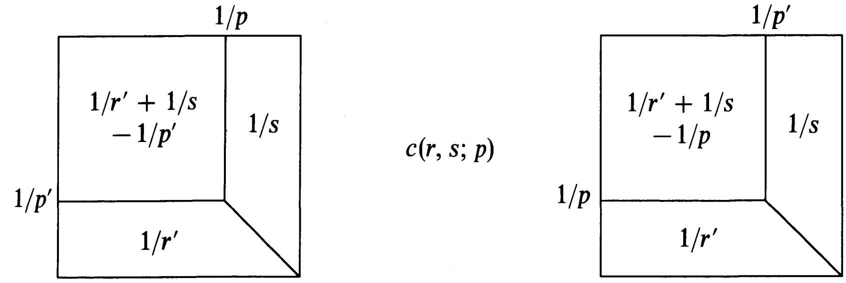
GENERALIZED CLARKSON'S INEQUALITIES (Kato [4]). *Let $1 < p < \infty$ and $1 \leqq r, s \leqq \infty$. Then, for an arbitrary positive integer n and for all $f_1, f_2, \dots, f_{2^n} \in L_p$,*

$$(3) \qquad \left\{\sum_{i=1}^{2^n} \left\|\sum_{j=1}^{2^n} \varepsilon_{ij} f_j\right\|_p^s\right\}^{1/s} \leqq 2^{nc(r,s;p)} \left\{\sum_{j=1}^{2^n} \|f_j\|_p^r\right\}^{1/r},$$

where

$$c(r, s; p) = \begin{cases} \dfrac{1}{r'} + \dfrac{1}{s} - \min\left(\dfrac{1}{p}, \dfrac{1}{p'}\right) & \text{if (i)} \quad \min(p, p') \leqq r \leqq \infty, \\ & \qquad\quad 1 \leqq s \leqq \max(p, p'), \\ \dfrac{1}{s} & \text{if (ii)} \quad 1 \leqq r \leqq \min(p, p'), \\ & \qquad\quad 1 \leqq s \leqq r', \\ \dfrac{1}{r'} & \text{if (iii)} \quad s' \leqq r \leqq \infty, \\ & \qquad\quad \max(p, p') \leqq s \leqq \infty. \end{cases}$$

The constant $c(r, s; p)$ is best possible in (3) ([4], Theorem 1) and it is represented in the following unit squares with axes $1/r$ (horizontal) and $1/s$ (vertical):



$1 < p \leqq 2$ $\qquad$ $2 < p < \infty$

NOTE. The inequalities (3) include the generalizations of (1) and (2) by Boas ([2]; especially, Theorem 1) and Koskela ([7]; especially, Theorem 2).

As special cases of (3) we have the following high dimensional versions of classical Clarkson's inequalities (1) and (2). As we shall see later, they are the heart of the generalized Clarkson inequalities.

CLARKSON'S INEQUALITIES OF 2^n-DIMENSION (Kato [4]). *For an arbitrary positive integer n and for all $f_1, f_2, \ldots, f_{2^n} \in L_p$,*

$$\left\{\sum_{i=1}^{2^n}\left\|\sum_{j=1}^{2^n}\varepsilon_{ij}f_j\right\|_p^{p'}\right\}^{1/p'} \leqq 2^{n/p'}\left\{\sum_{j=1}^{2^n}\|f_j\|_p^p\right\}^{1/p} \quad \textit{if } 1 < p \leqq 2, \tag{4}$$

$$\left\{\sum_{i=1}^{2^n}\left\|\sum_{j=1}^{2^n}\varepsilon_{ij}f_j\right\|_p^{p}\right\}^{1/p} \leqq 2^{n/p}\left\{\sum_{j=1}^{2^n}\|f_j\|_p^{p'}\right\}^{1/p'} \quad \textit{if } 2 < p < \infty. \tag{5}$$

For later use we note that these inequalities (4) and (5) are interpreted by means of operator norms of A_n as

$$\|A_n : l_p^{2^n}(L_p) \to l_{p'}^{2^n}(L_p)\| \leqq 2^{n/p'} \quad \textit{if } 1 < p \leqq 2 \tag{6}$$

and

$$\|A_n : l_{p'}^{2^n}(L_p) \to l_p^{2^n}(L_p)\| \leqq 2^{n/p} \quad \textit{if } 2 < p < \infty \tag{7}$$

respectively.

LEMMA 1 (cf. [1], Theorems 5.1.2, 4.1.2 and 4.2.1). (i) *Let $1 \leqq p_0$, $p_1 < \infty$, $1 \leqq r_0$, $r_1 \leqq \infty$ (not both $= \infty$), and let $0 < \theta < 1$. Let $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/r = (1-\theta)/r_0 + \theta/r_1$. Then,*

$$(l_{r_0}^n(L_{p_0}), l_{r_1}^n(L_{p_1}))_{[\theta]} = l_r^n(L_p) \quad \textit{with equal norms}.$$

(ii) *Let further* $1 \leqq q_0,\ q_1 < \infty$ *and* $1 \leqq s_0,\ s_1 \leqq \infty$ *(not both* $= \infty$*), and let* $1/q = (1-\theta)/q_0 + \theta/q_1,\ 1/s = (1-\theta)/s_0 + \theta/s_1$. *Let*

$$T: \begin{array}{l} l^n_{r_0}(L_{p_0}) \to l^n_{s_0}(L_{q_0}) \\ l^n_{r_1}(L_{p_1}) \to l^n_{s_1}(L_{q_1}) \end{array}$$

with the norms M_0 *and* M_1 *respectively. Then,*

$$T: l^n_r(L_p) \to l^n_s(L_q)$$

with the norm M *satisfying*

$$M \leqq M_0^{1-\theta} M_1^{\theta}.$$

2. Proof of 2^n-dimensional Clarkson's inequalities (4) and (5)

Let $1 < p < 2$. To prove the inequality (4) or equivalently (6), we need the following norms of A_n in two special cases $p = 1$ and 2, which are easily calculated:

$$\begin{aligned} M_1 &= \|A_n : l_1^{2^n}(L_1) \to l_\infty^{2^n}(L_1)\| \\ &= \sup\left\{ \max_{1 \leqq i \leqq 2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_1 \Big/ \sum_{j=1}^{2^n} \|f_j\|_1 : \sum_{j=1}^{2^n} \|f_j\|_1 \neq 0 \right\} \\ &= 1 \end{aligned} \tag{8}$$

and

$$M_2 = \|A_n : l_2^{2^n}(L_2) \to l_2^{2^n}(L_2)\| = 2^{n/2} \tag{9}$$

since $2^{-n/2}A_n$ is unitary, or

$$\left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_2^2 \right\}^{1/2} = 2^{n/2} \left\{ \sum_{j=1}^{2^n} \|f_j\|_2^2 \right\}^{1/2} \tag{10}$$

for all $\{f_j\}$ in $l_2^{2^n}(L_2)$ (cf. [5]). Put $\theta = 2/p'$ $(0 < \theta < 1)$. Then, since $(1-\theta)/1 + \theta/2 = 1/p$ and $(1-\theta)/\infty + \theta/2 = 1/p'$, we have by Lemma 1 (i)

$$(l_1^{2^n}(L_1), l_2^{2^n}(L_2))_{[\theta]} = l_p^{2^n}(L_p) \qquad \textit{with equal norms}$$

and

$$(l_\infty^{2^n}(L_1), l_2^{2^n}(L_2))_{[\theta]} = l_{p'}^{2^n}(L_p) \qquad \textit{with equal norms}.$$

By Lemma 1 (ii) with (8) and (9) we obtain

$$\|A_n : l_p^{2^n}(L_p) \to l_{p'}^{2^n}(L_p)\| \leqq M_1^{1-\theta} M_2^{\theta} = 2^{n/p'},$$

or (6), as is desired. (For $p = 2$, (4) (with equality) is none other than (10)).

Let $2 < p < \infty$. Since A_n is symmetric, we have by (6)

$$\|A_n : l_{p'}^{2^n}(L_p) \to l_p^{2^n}(L_p)\| = \|A_n : l_{p'}^{2^n}(L_{p'}) \to l_p^{2^n}(L_{p'})\| \leqq 2^{n/p},$$

or (7). This completes the proof.

3. Proof of the generalized Clarkson inequality (3)

At first, we derive from 2^n-dimensional Clarkson's inequality (4), or (6), the following inequality (11), which is a part of (3) and is just what Tonge [11] derived from the generalized Hausdorff-Young inequality by Williams and Wells [12]:

Lemma 2. *Let $1 < t < p \leqq 2$. Then, for all $f_1, f_2, \ldots, f_{2^n}$ in L_p,*

$$\left\{\sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_p^{t'}\right\}^{1/t'} \leqq 2^{n/t'} \left\{\sum_{j=1}^{2^n} \|f_j\|_p^t\right\}^{1/t}, \tag{11}$$

or equivalently

$$\|A_n : l_t^{2^n}(L_p) \to l_{t'}^{2^n}(L_p)\| \leqq 2^{n/t'}. \tag{12}$$

Proof. In the same way as (8) we have

$$M_3 = \|A_n : l_1^{2^n}(L_p) \to l_\infty^{2^n}(L_p)\| = 1.$$

On the other hand, by (6) we have

$$M_4 = \|A_n : l_p^{2^n}(L_p) \to l_{p'}^{2^n}(L_p)\| \leqq 2^{n/p'}.$$

Put $\theta = p'/t'$ $(0 < \theta < 1)$. Then, since $(1-\theta)/1 + \theta/p = 1/t$ and $(1-\theta)/\infty + \theta/p' = 1/t'$, we obtain by Lemma 1

$$\begin{aligned}\|A_n : l_t^{2^n}(L_p) \to l_{t'}^{2^n}(L_p)\| &\leqq M_3^{1-\theta} M_4^{\theta} \\ &\leqq 2^{n\theta/p'} = 2^{n/t'},\end{aligned}$$

or (12).

Now, the rest of our proof for $1 < p \leqq 2$ is the same as Tonge's, and it may be regarded as a vector-valued version of a part of Pietsch's argument in [10]. For the case $2 < p < \infty$, we use duality. For convenience of the reader, we state it in full with operator theoretical treatment.

Let us proceed in the proof of (3) according to the cases indicated in the representation of $c(r, s; p)$.

The case $1 < p \leqq 2$: (i) Let $p \leqq r \leqq \infty$ and $1 \leqq s \leqq p'$. Then, by (6) we have

$$\begin{aligned}
&\|A_n : l_r^{2^n}(L_p) \to l_s^{2^n}(L_p)\| \\
&\quad \leqq \|I : l_r^{2^n}(L_p) \to l_p^{2^n}(L_p)\| \, \|A_n : l_p^{2^n}(L_p) \to l_{p'}^{2^n}(L_p)\| \, \|I : l_{p'}^{2^n}(L_p) \to l_s^{2^n}(L_p)\| \\
&\quad \leqq 2^{n(1/p-1/r)} 2^{n/p'} 2^{n(1/s-1/p')} \\
&\quad = 2^{n(1/r'+1/s-1/p')},
\end{aligned}$$

which implies (3).

(ii) Let $1 \leqq r \leqq p$ and $1 \leqq s \leqq r'$. Then, by Lemma 2 with $t = r$ we have

$$\begin{aligned}
&\|A_n : l_r^{2^n}(L_p) \to l_s^{2^n}(L_p)\| \\
&\quad \leqq \|A_n : l_r^{2^n}(L_p) \to l_{r'}^{2^n}(L_p)\| \, \|I : l_{r'}^{2^n}(L_p) \to l_s^{2^n}(L_p)\| \\
&\quad \leqq 2^{n/r'} 2^{n(1/s-1/r')} \\
&\quad = 2^{n/s}.
\end{aligned}$$

(iii) Let $s' \leqq r \leqq \infty$ and $p' \leqq s \leqq \infty$. Then, by Lemma 2 with $t = s'$,

$$\begin{aligned}
&\|A_n : l_r^{2^n}(L_p) \to l_s^{2^n}(L_p)\| \\
&\quad \leqq \|I : l_r^{2^n}(L_p) \to l_{s'}^{2^n}(L_p)\| \, \|A_n : l_{s'}^{2^n}(L_p) \to l_s^{2^n}(L_p)\| \\
&\quad \leqq 2^{n(1/s'-1/r)} 2^{n/s} \\
&\quad = 2^{n/r'}.
\end{aligned}$$

The case $2 < p < \infty$: By duality, we have

$$\begin{aligned}
\|A_n : l_r^{2^n}(L_p) \to l_s^{2^n}(L_p)\| &= \|A_n : l_{s'}^{2^n}(L_{p'}) \to l_{r'}^{2^n}(L_{p'})\| \\
&\leqq 2^{nc(s',r';p')}.
\end{aligned}$$

Observe here $c(s', r'; p') = c(r, s; p)$ (we have only to note that the points $(1/r, 1/s)$ and $(1/s', 1/r')$ are symmetric with respect to the segment $1/s = 1 - 1/r$). Then, we have the desired inequality (3). This completes the proof.

ADDED NOTE. A unified consideration on some relations between inequalities (including Clarkson's) and interpolation is given in the recent paper [9] of Maligranda and Persson.

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