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Simple setting for white noise calculus using Bargmann space and Gauss transform

Yoshitaka YOKOI

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0. Introduction

Let E_0 be a real separable infinite-dimensional Hilbert space with an inner product $(\cdot, \cdot)_0$ and suppose that we are given a densely defined selfadjoint operator D of E_0 such that D^{-1} is of Hilbert-Schmidt type and $D > 1$. Let $E \subset E_0 \subset E^*$ be a real Gel'fand triplet rigged by the system of norms $\{\|D^p \cdot\|_0; p \in \mathbf{R}\}$ and $H \subset H_0 \subset H^*$ be its complexification. The canonical bilinear forms defined by the pairs of elements $(x, \xi) \in E^* \times E$ and $(z, \eta) \in H^* \times H$ are denoted by $\langle x, \xi \rangle$ and $\langle z, \eta \rangle$, respectively. The functional $C(\xi) = \exp[-\frac{1}{2}\|\xi\|_0^2]$, which is continuous and positive definite in $\xi \in E$, determines a unique probability measure μ on E^* such that

$$\int_{E^*} \exp[\sqrt{-1}\langle x, \xi \rangle]\, d\mu(x) = \exp\left[-\frac{1}{2}\|\xi\|_0^2\right].$$

If $H^* = E^* + \sqrt{-1}\,E^*$ is identified with the product space $E^* \times E^*$, it is possible to define the product measure $\nu = \mu \times \mu$ on H^*. Let $\mathscr{P}(E^*)$ be the space of all polynomials in $\{\langle x, \xi \rangle; \xi \in E\}$ with complex coefficients and $\mathscr{P}(H^*)$ be the space of all polynomials in $\{\langle z, \xi \rangle; \xi \in H\}$, where $x \in E^*$ and $z \in H^*$. Then $\mathscr{P}(E^*)$ is dense in $(L^2) \equiv L^2(E^*, \mu)$. The L^2-closure of $\mathscr{P}(H^*)$ is a proper subspace of $L^2(H^*, \nu)$. This subspace is denoted by $(\mathfrak{F}_0)$. It is called a Bargmann space ([4]).

For $\varphi(x) \in \mathscr{P}(E^*)$, $\varphi(x)$ has a natural analytic continuation $\varphi(w) \in \mathscr{P}(H^*)$ and its restriction to E^* is trivially the original $\varphi(x)$. Thus we can define a map $G: \mathscr{P}(E^*) \to \mathscr{P}(H^*)$ by

$$G\varphi(w) \equiv \int_{E^*} \varphi(x + w/\sqrt{2})\, d\mu(x), \tag{0.1}$$

(ref. Kondrat'ev [17], Hida [10]). This map is called Gauss transform because of its similarity with Gauss transform $\mathscr{G}_t[F]$ of a function $F(v)$ of one real variable v:

$$\mathscr{G}_t[F](u) = \int_{-\infty}^{\infty} F(v + u)(2\pi t)^{-1/2} \exp[-v^2/(2t)]\, dv.$$

Further, the inverse of the map $G: \mathscr{P}(E^*) \to \mathscr{P}(H^*)$ is given by

$$G^{-1}f(x) = \int_{E^*} f(\sqrt{2}(x + \sqrt{-1}\,y))\,d\mu(y). \tag{0.2}$$

It can be shown that G preserves the L^2-norm. On the other hand, the conditions imposed on D make the operators $\{D^p; p \in \mathbf{R}\}$ act continuously on H^*. So it is natural to introduce the system of operators $\{\Lambda(D^p); p \in \mathbf{R}\}$ whose actions in $\mathscr{P}(H^*)$ are described as

$$\Lambda(D^p)f(z) = f(D^p z) \qquad \text{for } f \in \mathscr{P}(H^*),\ (z \in H^*,\ p \in \mathbf{R}).$$

The action of $\Lambda(D^p)$ is intelligible and convenient to treat: the eigensystem of $\Lambda(D^p)$ is derived almost directly from the eigensystem of D^p; the introduction of norms

$$\|f\|^2_{(\mathfrak{F}_p)} \equiv \|\Lambda(D^p)f\|^2_{(\mathfrak{F}_0)} = \int_{H^*} |f(D^p z)|^2\,d\nu(z) \qquad (p \in \mathbf{R})$$

into $\mathscr{P}(H^*)$ is also very direct; the completions of $\mathscr{P}(H^*)$ by these norms afford us a Gel'fand triplet of Bargmann space $(\mathfrak{F}_0)$ quite naturally. We will denote the completions by $(\mathfrak{F}_p)$ $(p \in \mathbf{R})$ and the Gel'fand triplet by $(\mathfrak{F}) \subset (\mathfrak{F}_0) \subset (\mathfrak{F}')$. Now let us combine G and $\Lambda(D^p)$, $p \in \mathbf{R}$. Then the operators $\{\Gamma(D^p) \equiv G^{-1}\Lambda(D^p)G; p \in \mathbf{R}\}$ act on $\mathscr{P}(E^*)$ as $\{\Lambda(D^p); p \in \mathbf{R}\}$ do on $\mathscr{P}(H^*)$. Therefore we can bring over the structure of Gel'fand triplet of $(\mathfrak{F}_0)$ to (L^2). Thus we get a Gel'fand triplet of (L^2). We will denote this Gel'fand triplet by $(\mathscr{S}) \subset (L^2) \subset (\mathscr{S}')$. We propose this construction of the triplet of (L^2) as a simple setting of white noise calculus. This setting can clarify the problem of white noise analysis and simplify the related matters considerably, which will be seen in §6 and §7.

As we saw above, if $\varphi \in \mathscr{P}(E^*)$, φ can be analytically continued to H^* naturally. Furthermore $G\varphi(w)$ is an analytic continuation of $S\varphi(\xi/\sqrt{2})$, the composition of S-transform and $(1/\sqrt{2})$-multiplication to its variable. It is a well-known fact that $S\varphi(\xi)$ can be defined for functionals $\varphi \in (L^2)$ and for the variable $\xi \in E$ (see [23]). This observation suggests that even if φ belongs to a much wider class than polynomials', φ and $G\varphi$ could have an analytic continuation $\tilde{\varphi}(w)$ and an analytic version $\tilde{f}(w)$, respectively on H^*. Hence $S\tilde{\varphi}(w/\sqrt{2})$ could be defined for the variable $w \in H$ and $\tilde{f}(w) = S\tilde{\varphi}(w/\sqrt{2})$ could hold. We will show that this is possible.

The organization of this paper is as follows: §1 is for the notations. In §2 we will extend the Gauss transform $G: \mathscr{P}(E^*) \to \mathscr{P}(H^*)$ to an isometric isomorphism from (L^2) onto $(\mathfrak{F}_0)$. In §3 we introduce the system of operators $\{\Lambda(D^p); p \in \mathbf{R}\}$ into $\mathscr{P}(H^*)$ and construct a nuclear rigging $(\mathfrak{F}) \subset (\mathfrak{F}_0) \subset (\mathfrak{F}_{-p}) \subset$

$(\mathfrak{F}')$ by using this operator system. The analyticity of functionals of $(\mathfrak{F}_p)$ is stated in this section. The operator $\Lambda(D^p)$ $(p \in \mathbf{R})$ will turn out to act on $(\mathfrak{F}')$. In §4 we will derive a Gel'fand triplet $(\mathscr{S}) \subset (L^2) \subset (\mathscr{S}')$ of white noise analysis from $(\mathfrak{F}) \subset (\mathfrak{F}_0) \subset (\mathfrak{F}')$ with the help of $\{G^{-1}\Lambda(D^p)G; p \in \mathbf{R}\}$. It will be noted in some concrete case that this construction of the triplet is the same as the usual one. In §5 we will show the following things: the existence of analytic continuation $\tilde{\varphi}$ of $\varphi \in (\mathscr{S}_p)$ for $p > p_0$ and the continuous version $\tilde{\varphi}$ for $p > s_0$ (p_0 and s_0 are given in §1); some estimate inequalities of these functionals $\tilde{\varphi}$; the integral representation of G and G^{-1}; and the rigorous meaning of $S\tilde{\varphi}(w/\sqrt{2})$ $(w \in H^*)$. In the estimate inequalities mentioned above, the functional $\exp[\frac{1}{2}\|x\|_{-p}^2]$ appears as follows:

$$|\tilde{\varphi}(x)| \leq \alpha_p \|\varphi\|_{(\mathscr{S}_p)} \exp\left[\frac{1}{2}\|x\|_{-p}^2\right]. \tag{0.3}$$

In §6 we will prove that this functional belongs to $(\mathscr{S}_q)$ for p and q such that $0 \leq q < p - p_0$. That $(\mathfrak{F})$ and $(\mathscr{S})$ are algebras is also mentioned in this section. In §7 using (0.3) and the results of §6, we will refine the theorem about positive functionals which we got before (Theorem 5.1 in [39]) and obtain a somewhat delicate characterization about positive functionals and their associated measures.

Some results similar to the ones in §7 of this paper are already obtained in Lee's paper [29]. But, because of the use of the different method from [29], in particular the use of Bargmann space and the operators $\{\Lambda(D^p); p \in \mathbf{R}\}$, we can clarify the roblem and can obtain the refined results, including new ones, more simply. The point of [29] is to construct the space $\mathscr{A}_\infty$ of analytic *versions* of test white noise functionals. The method is complicated. But if our method is applied, the construction of the space which would correspond to $\mathscr{A}_\infty$ is quite natural and simple. It is just the inverse image of $(\mathfrak{F})$ by Gauss transform G, i.e., $\{G^{-1}f(w); f \in (\mathfrak{F})\}$ $(w \in H^*)$. And this space coincides with the space $\{\tilde{\varphi}(w); \varphi \in (\mathscr{S})\}$ $(w \in H^*)$, where $\tilde{\varphi}(w)$ is the analytic continuation in H^* of $\varphi \in (\mathscr{S})$. These three spaces $(\mathfrak{F})$, $\{G^{-1}f(w); f \in (\mathfrak{F})\}$ $(w \in H^*)$, and $(\mathscr{S})$, at least their roles, should rigorously be distinguished.

1. Notations

Let E_0 be a real separable Hilbert space with $\dim E_0 = \infty$ and $(\cdot, \cdot)_0$ be its inner product. Let D be a densely defined and selfadjoint operator of E_0 such that $D > 1$ and D^{-1} is of Hilbert-Schmidt type. Further we assume that the eigensystem of D^{-1},

$$\{(\lambda_j, \zeta_j)\}_{j=0}^{\infty} \quad \text{with} \quad D^{-1}\zeta_j = \lambda_j\zeta_j \ (j = 0, 1, 2, \cdots),$$

satisfies

$$1 > \lambda_j \geq \lambda_{j+1} \quad (j = 0, 1, 2, \cdots)$$

and that $\{\zeta_j; j = 0, 1, 2, \cdots\}$ is an orthonormal basis of E_0. The following constants t_0, s_0, and p_0 will appear frequently:

$$t_0 = -\log 2/2 \log \lambda_0, \text{ i.e., } \lambda_0^{2t_0} = 1/2,$$
$$s_0 = \inf\{s; \textstyle\sum_{j=0}^{\infty} \lambda_j^{2s} < \infty\},$$
$$p_0 = \max(t_0, s_0).$$

Since $\|D^{-1}\|_{\mathrm{HS}}^2 = \sum_{j=0}^{\infty} \lambda_j^2$ is finite, s_0 is in [0, 1].

For any real number $p > 0$ write E_p = the domain of D^p and define the inner product $(x, y)_p$ for $x, y \in E_p$ by

$$(x, y)_p = (D^p x, D^p y)_0.$$

Then $(E_p, (\cdot, \cdot)_p)$ is a Hilbert space. If $0 \leq q < p$, then $E_p \subset E_q$. Every E_p contains ζ_j's, and so $E \equiv \bigcap_{p>0} E_p$ is not empty. Set $\|\xi\|_p = \sqrt{(\xi, \xi)_p}$ for $\xi \in E$. The system of norms $\{\|\xi\|_p; p \geq 0\}$ is compatible. Since D^{-1} is of Hilbert-Schmidt type, the space E equipped with the projective limit topology of $\{(E_p, \|\cdot\|_p); p > 0\}$ is a nuclear space. We can easily see that $D^p(E_p) = E_0$ for $p > 0$. For $p > 0$, let E_{-p} be the completion of E_0 with respect to the norm $\|\cdot\|_{-p} \equiv \|D^{-p}\cdot\|_0$. Clearly, if $0 \leq q < p$, then $E_0 \subset E_{-q} \subset E_{-p}$. Let $E^* = \bigcup_{p>0} E_{-p}$ and let it be equipped with the inductive limit topology of $\{(E_{-p}, \|\cdot\|_{-p}); p > 0\}$. We have $E \subset E_0 \subset E^*$. Once the increasing family $\{E_p; p \in \mathbf{R}\}$ of Hilbert spaces is set, the operator $D^q (q \in \mathbf{R})$ acts naturally and isometrically as

$$D^q \colon E_p \longrightarrow E_{p-q} \quad \text{(surjective)} \ (p \in \mathbf{R}),$$

and so it acts continuously on E^* with respect to the inductive limit topology. We can naturally identify the dual space of E_p with $E_{-p} (p \in \mathbf{R})$ and the dual space of E with E^*.

Let H_p be the complexification of E_p, i.e., $H_p = E_p + \sqrt{-1}\, E_p$. Then D^q extends to an isometry from H_p onto H_{p-q} naturally by setting

$$D^q(x + \sqrt{-1}\, y) = D^q x + \sqrt{-1}\, D^q y \qquad \text{for } x, y \in E_p \ (p, q \in \mathbf{R}).$$

According to this way the real spaces E and E^* also have their complexifications H and H^*, respectively. The letters w and z are often used for elements in H^* or H_{-p} and letters x and y for ones in E^* or E_{-p}, where $p \geq 0$. Like in the real case, the operator D^q acts on H (also on H^*) continuously. Obviously,

$$\langle D^q w, \zeta\rangle = \langle w, D^q\zeta\rangle$$

holds for any $w \in H^*$ and any $\zeta \in H$.

Suppose that X is a locally convex topological vector space and X^* is the dual of X. Then $x^*(x)$, the canonical bilinear form, defined by each pair $(x, x^*) \in X \times X^*$ is always denoted by $\langle x^*, x\rangle$. Here, to be bilinear does not mean to be sesquilinear. Denote by $\mathscr{P}(X^*)$ the space of all polynomials in $\{\langle x^*, x\rangle; x \in X\}$ with complex coefficients; that is,

$$\mathscr{P}(X^*) = \{\text{finite sums of } c\textstyle\prod_j \langle x^*, x_j\rangle;\ x_j \in X,\ c \in \mathbf{C}\}.$$

If X is a nuclear space or a Hilbert space over $\mathbf{R}$ or $\mathbf{C}$, then the n-fold symmetric tensor product of X is denoted by $X^{\hat{\otimes} n}$. If $x_1, x_2, \cdots, x_n \in X$, then $\hat{\bigotimes}_{j=1}^n x_j$ is the symmetrization of $x_1 \otimes x_2 \cdots \otimes x_n$. In particular the n-fold tensor product of x is denoted by $x^{\hat{\otimes} n}$.

The following notation on infinite-dimensional indices of non-negative integers will be used:

$$\mathscr{N} = \{\text{all sequences of non-negative integers}\},$$

$$\mathscr{N}_0 = \{\mathbf{n} = (n_0, n_1, n_2, \cdots);\ \mathbf{n} \in \mathscr{N},\ n_j = 0 \text{ for almost all } j\}.$$

Let $\mathbf{n}, \mathbf{k} \in \mathscr{N}_0$. Write $\mathbf{n} \geq \mathbf{k}$ if and only if $n_j \geq k_j (j \geq 0)$. Let p be a non-negative integer. Define

$$p\mathbf{n} = (pn_0, pn_1, pn_2, \cdots),\quad |\mathbf{n}| = n_0 + n_1 + n_2 + \cdots,$$

$$\mathbf{n} \wedge \mathbf{k} = (n_0 \wedge k_0, n_1 \wedge k_1, n_2 \wedge k_2, \cdots),$$

$$\mathbf{n}! = \textstyle\prod_j n_j! \quad \text{and} \quad \binom{\mathbf{n}}{\mathbf{k}} = \prod_j \binom{n_j}{k_j}.$$

For $r \in \mathbf{R}$ and $\mathbf{n} \in \mathscr{N}_0$ with $|\mathbf{n}| = n$, the symbols $\lambda^{r\mathbf{n}}$, $\zeta^{\hat{\otimes}\mathbf{n}}$, $h_\mathbf{n}$ and $\mathbf{z}^\mathbf{n}$ are defined as follows:

$$\lambda^{r\mathbf{n}} = \textstyle\prod_j \lambda_j^{rn_j},$$

$$\zeta^{\hat{\otimes}\mathbf{n}} = \hat{\bigotimes}_{n_j \neq 0} \zeta_j^{\hat{\otimes} n_j} = \text{the symmetrization of } \bigotimes_{n_j \neq 0} \zeta_j^{\hat{\otimes} n_j},$$

$$\mathbf{z}^\mathbf{n} = \mathbf{z}^\mathbf{n}(z) = (2^n \mathbf{n}!)^{-1/2} \langle z^{\hat{\otimes} n}, \xi^{\hat{\otimes}\mathbf{n}}\rangle \qquad \text{for } z \in H^*, \tag{1.1}$$

$$h_\mathbf{n} = h_\mathbf{n}(x) = (2^n \mathbf{n}!)^{-1/2} \textstyle\prod_j H_{n_j}(\langle x, \zeta_j\rangle/\sqrt{2}) \qquad \text{for } x \in E^*, \tag{1.2}$$

where $\{(\lambda_j, \zeta_j)\}_{j=0}^\infty$ is the eigensystem of D^{-1} and $H_n(u)$ is the Hermite polynomial of n degrees defined by

$$H_n(u) = (-1)^n \exp[u^2] (d/du)^n \exp[-u^2].$$

$\mathscr{B}$ is the smallest σ-algebra containing all cylindrical sets of E^*. Here, cylindrical sets of E^* are subsets of E^* of the form

$$\{x \in E^*; (\langle x, \xi_1\rangle, \cdots, \langle x, \xi_n\rangle) \in B_n\}$$

where n is any integer ≥ 1, B_n is any n-dimensional Borel set, and $\xi_1, \cdots, \xi_n$ are any elements of E.

2. The space of white noise functionals (L^2), the Bargmann space $(\mathfrak{F}_0)$ over a nuclear space, and Gauss transform G

The functional $C(\xi) = \exp[-\frac{1}{2}\|\xi\|_0^2]$ of ξ is positive definite and continuous on the nuclear space E. Bochner-Minlos theorem assures us that this functional defines a unique Gaussian measure μ in the measurable space $(E^*, \mathscr{B})$ such that

$$C(\xi) = \int_{E^*} \exp[\sqrt{-1}\langle x, \xi\rangle]\, d\mu(x),$$

(Minlos [30]).
Since D^{-s} is of Hilbert-Schmidt type for $s > s_0$, $\mu(E_{-s}) = 1$ holds. Hence, if a functional is defined in E_{-s} for $s > s_0$, then we may consider that it is given μ-a.e. in E^*.

The space $L^2(E^*, \mathscr{B}, \mu)$ is called the space of white noise functionals and denoted by (L^2) (Hida [9], [10]). Then $\mathscr{P}(E^*)$, the space of all polynomials in $\{\langle x, \xi\rangle; \xi \in E\}$ with complex coefficients, is dense in (L^2). It is readily seen that the system $\{h_{\mathbf{n}}; \mathbf{n} \in \mathscr{N}_0\}$ of (1.2) is contained in $\mathscr{P}(E^*)$ and is a complete orthonormal system of (L^2). From now on let CONS stand for complete orthonormal system.

Let us consider the product measure $\nu = \mu \times \mu$ in the space $H^* = E^* + \sqrt{-1}\, E^*$. Then the system $\{\mathbf{z}^{\mathbf{n}}; \mathbf{n} \in \mathscr{N}_0\}$ of (1.1) is orthonormal in the space $L^2(H^*, \nu)$. A *Bargmann space* $(\mathfrak{F}_0)$ is the closure of $\mathscr{P}(H^*)$ in $L^2(H^*, \nu)$, where $\mathscr{P}(H^*)$ is the space of all polynomials in $\{\langle z, \xi\rangle; \xi \in H\}$ with complex coefficients. It is evident that the system $\{\mathbf{z}^{\mathbf{n}}; \mathbf{n} \in \mathscr{N}_0\}$ is contained in $\mathscr{P}(H^*)$ and forms a CONS of $(\mathfrak{F}_0)$. It is well-known that the space of all entire functions, $\mathfrak{F}^n(\mathbf{C})$, which are defined on $\mathbf{C}^n$ and square integrable with respect to

$$dg(z) = (2\pi)^{-n} \exp[-(z\bar{z})/2](\sqrt{-1}/2)^n dz d\bar{z}$$

is closed in $L^2(\mathbf{C}^n, dg(z))$ (see Bargmann [1]). The space $(\mathfrak{F}_0)$ is an analogue of $\mathfrak{F}(\mathbf{C}^n)$ in passing from $\mathbf{C}^n$ to the infinite dimensional space H^*. $(\mathfrak{F}_0)$ has similar properties with $\mathfrak{F}(\mathbf{C}^n)$. For instance, every element of $(\mathfrak{F}_0)$ is associated with an analytic functional in H_0 whose Taylor series converges to the original functional in $(\mathfrak{F}_0)$ (see Kondrat'ev [17]). But it is not an analytic version in the sense of ν-a.e. because $\nu(H_0) = 0$. Throughout this paper, "a version" means an element of ν-a.e. (or μ-a.e.) equivalence class. If we introduce a nuclear rigging $(\mathfrak{F}) \subset (\mathfrak{F}_p) \subset (\mathfrak{F}_0) \subset (\mathfrak{F}_{-p}) \subset (\mathfrak{F}')$, we can see this more clearly. The construction of the nuclear rigging and the problem of version will be discussed in detail in §3, (cf. Berezansky and Kondrat'ev [4] [17]).

In the rest of this section, we observe the isometric maps G and G^{-1} between $(\mathscr{P}(E^*), \|\cdot\|_{L^2(\mu)})$ and $(\mathscr{P}(H^*), \|\cdot\|_{L^2(\mu\times\mu)})$ and their extensions to isometric maps between (L^2) and $(\mathfrak{F}_0)$. (cf. Berezansky and Kondrat'ev [4]). Each polynomial $\varphi(x) \in \mathscr{P}(E^*)$ can be naturally and analytically extended to $\varphi(z) \in \mathscr{P}(H^*)$ replacing $\langle x, \xi \rangle$ by $\langle z, \xi \rangle$. So we can define a map G on $\mathscr{P}(E^*)$ by

$$G\varphi(w) = \int_{E^*} \varphi(x + w/\sqrt{2})\, d\mu(x) \qquad \text{for } \varphi \in \mathscr{P}(E^*). \tag{2.1}$$

Then obviously, $G\varphi$ belongs to $\mathscr{P}(H^*)$. The inverse map G^{-1} is given by

$$G^{-1}f(x) = \int_{E^*} f(\sqrt{2}(x + \sqrt{-1}\, y))\, d\mu(y) \qquad \text{for } f \in \mathscr{P}(H^*). \tag{2.2}$$

Actually, we see that

$$Gh_{\mathbf{n}} = \mathbf{z}^{\mathbf{n}} \quad \text{and} \quad G^{-1}\mathbf{z}^{\mathbf{n}} = h_{\mathbf{n}}. \tag{2.3}$$

Since $\{h_{\mathbf{n}}; \mathbf{n} \in \mathscr{N}_0\}$ and $\{\mathbf{z}_{\mathbf{n}}; \mathbf{n} \in \mathscr{N}_0\}$ are CONS' in (L^2) and $(\mathfrak{F}_0)$ respectively, the map G extends to an isometry from (L^2) onto $(\mathfrak{F}_0)$:

$$\| G\varphi \|_{(\mathfrak{F}_0)} = \| \varphi \|_{(L^2)} \qquad \text{for } \varphi \in (L^2). \tag{2.4}$$

The map given by the form such as (2.1) is often called Gauss transform ([4], [10], [17]), so we also call this isometric map G and its extension $G: (L^2) \to (\mathfrak{F}_0)$ *Gauss transform.* The integral representation (2.1) of G (resp. (2.2) of G^{-1}) is not valid on (L^2) (resp. on $(\mathfrak{F}_0)$). But in §5 we will show that these representations can extend to the ones between much wider spaces than $\mathscr{P}(E^*)$ and $\mathscr{P}(H^*)$. Furthermore, the expression (2.1) shows us that if φ is a good functional, then $G\varphi(w)$ is an analytic continuation of $S\varphi(\xi/\sqrt{2})(\xi \in E)$ to $H^* = E^* + \sqrt{-1}\, E^*$, where S is the S-transform in Kubo-Takenaka [23]. It will also be shown in §5 that this is possible.

3. The Gel'fand triplet $(\mathfrak{F}) \subset (\mathfrak{F}_0) \subset (\mathfrak{F}')$ rigged by the operator $\Lambda(D^p)$

Let D be the selfadjoint operator of H_0 introduced in §1. Since $D^p (p \in \mathbf{R})$ acts on H^* naturally and continuously, we can define an operator $\Lambda(D^p)$ on $\mathscr{P}(H^*)$ for $p \in \mathbf{R}$ by

$$\Lambda(D^p)f(z) = f(D^p z),\ f \in \mathscr{P}(H^*). \tag{3.1}$$

Let $f(z) = \prod_{j=1}^n \langle z, \xi_j \rangle \in \mathscr{P}(H^*)$. Then, by the relation

$$\Lambda(D^p)f(z) = \prod_{j=1}^n \langle D^p z, \xi_j \rangle = \prod_{j=1}^n \langle z, D^p \xi_j \rangle$$

we see that $\{(\lambda^{-p\mathbf{n}}, \mathbf{z}^{\mathbf{n}});\ \mathbf{n} \in \mathscr{N}_0\}$ is an eigensystem of $\Lambda(D^p)$:

$$\Lambda(D^p)\mathbf{z}^{\mathbf{n}}(z) = \left(\prod_j \lambda_j^{-pn_j}\right)\mathbf{z}^{\mathbf{n}}(z) = \lambda^{-p\mathbf{n}}\mathbf{z}^{\mathbf{n}}(z). \tag{3.2}$$

As is easily seen, $\mathscr{P}(H^*)$ is a pre-Hilbert space with the inner product

$$(\Lambda(D^p)f,\ \Lambda(D^p)g)_{(\mathfrak{F}_0)} = \int_{H^*} (\Lambda(D^p)f(z))\,\overline{\Lambda(D^p)g(z)}\,d\nu(z). \tag{3.3}$$

We will denote its completion by $(\mathfrak{F}_p)$ and the inner product by $(f, g)_{(\mathfrak{F}_p)}$. As well as in the case of D^q, we can see that the operator $\Lambda(D^q)$ is an isometry from the Hilbert space $(\mathfrak{F}_p)$ onto the Hilbert space $(\mathfrak{F}_{p-q})$. We can easily see the following.

PROPOSITION 3.1. *For any* $p \in \mathbf{R}$, $\{\lambda^{p\mathbf{n}}\mathbf{z}^{\mathbf{n}};\ \mathbf{n} \in \mathscr{N}_0\}$ *is a* CONS *of* $(\mathfrak{F}_p)$. *And hence any* $f \in (\mathfrak{F}_p)$ *can be expressed in the form*

$$f = \sum_{\mathbf{n} \in \mathscr{N}_0} c_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} \tag{3.4}$$

with coefficients $\{c_{\mathbf{n}};\ \mathbf{n} \in \mathscr{N}_0\}$ *satisfying*

$$\| f \|_{(\mathfrak{F}_p)}^2 = \sum_{\mathbf{n} \in \mathscr{N}_0} \lambda^{-2p\mathbf{n}} |c_{\mathbf{n}}|^2 < \infty. \tag{3.5}$$

Furthermore, for $f \in (\mathfrak{F}_p)$ *of the form* (3.4) *we have the following*:

$$\Lambda(D^q)f = \sum_{\mathbf{n} \in \mathscr{N}_0} \lambda^{-q\mathbf{n}} c_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} \in (\mathfrak{F}_{p-q}). \tag{3.6}$$

By the proposition, we can identify $(\mathfrak{F}_{-p})$ with the dual space of $(\mathfrak{F}_p)$ and get the inclusion relation for $p > q > 0$

$$(\mathfrak{F}_p) \subset (\mathfrak{F}_q) \subset (\mathfrak{F}_0) \subset (\mathfrak{F}_{-q}) \subset (\mathfrak{F}_{-p}).$$

Actually the canonical bilinear form $\langle F, f \rangle$ for $F \in (\mathfrak{F}_{-p})$ and $f \in (\mathfrak{F}_p)$ is realized by

$$\langle F, f \rangle = \int_{H^*} (\Lambda(D^{-p})F(z))\Lambda(D^p)f(z)\,d\nu(z).$$

Since D^{-1} is of Hilbert-Schmidt type, it follows that for any $p \in \mathbf{R}$ and for any $s > s_0$

$$\sum_{\mathbf{n} \in \mathscr{N}_0} \| \lambda^{(p+s)\mathbf{n}} \mathbf{z}^{\mathbf{n}} \|_{(\mathfrak{F}_p)}^2 = \prod_j (1 - \lambda_j^{2s})^{-1} < \infty. \tag{3.7}$$

This shows that the canonical injection from $(\mathfrak{F}_{p+s})$ into $(\mathfrak{F}_p)$ is also of

Hilbert-Schmidt type. Thus, if we write

$$(\mathfrak{F}) = \bigcap_{p=0}^{\infty}(\mathfrak{F}_p) \quad \text{and} \quad (\mathfrak{F}') = \bigcup_{p=0}^{\infty}(\mathfrak{F}_{-p}), \tag{3.8}$$

then the dual space of $(\mathfrak{F})$ is $(\mathfrak{F}')$. About this triplet several interesting properties have been obtained, e.g., this triplet is isometrically isomorphic to a triplet of "holomorphic functionals" of at most order 2 (ref. [4] and [17]). For the later use we paraphrase this within our setting as follows:

PROPOSITION 3.2. *For any* $p \in \mathbf{R}$ *and any* $f \in (\mathfrak{F}_p)$ *with the expression* (3.4), *the series*

$$\sum_{\mathbf{n} \in \mathcal{N}_0} c_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}(z) \tag{3.9}$$

converges absolutely and uniformly to a functional $\tilde{f}(z)$ *on any bounded set of* H_{-p}. *The limit functional* $\tilde{f}(z)$ *satisfies*

$$|\tilde{f}(z)| \leq \exp\left[\frac{1}{4}\|z\|_{-p}^2\right]\|f\|_{(\mathfrak{F}_p)} \qquad \textit{for any } z \in H_{-p}. \tag{3.10}$$

Further $\tilde{f}(z)$ *is not only continuous but analytic in* H_{-p} *in the sense of* [14] (E. Hille & R. S. Phillips).

PROOF. By Schwarz' inequality and (1.1), we see that for any $z \in H_{-p}$

$$\begin{aligned}
\sum_{n \in \mathcal{N}_0} |c_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}(z)| &= \sum_{n=0}^{\infty} \sum_{|\mathbf{n}|=n} |c_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}(z)| \\
&= \sum_{n=0}^{\infty} \sum_{|\mathbf{n}|=n} |c_{\mathbf{n}} \langle z^{\hat{\otimes} n}, (2^n \mathbf{n}!)^{-1/2} \zeta^{\hat{\otimes} \mathbf{n}} \rangle| \\
&= \sum_{n=0}^{\infty} (2^n n!)^{-1/2} \sum_{|\mathbf{n}|=n} |c_{\mathbf{n}}| \lambda^{-p\mathbf{n}} \left(\frac{n!}{\mathbf{n}!}\right)^{1/2} |\langle z^{\hat{\otimes} n}, \lambda^{p\mathbf{n}} \zeta^{\hat{\otimes} \mathbf{n}} \rangle| \\
&\leq \|f\|_{(\mathfrak{F}_p)} \exp\left[\frac{1}{4}\|z\|_{-p}^2\right].
\end{aligned}$$

Therefore the series converges to a continuous functional $\tilde{f}$ in H_{-p} absolutely and uniformly on any bounded set of H_{-p} and hence $\tilde{f}$ satisfies (3.10). The finite sums of (3.9) are functionals analytic and locally uniformly bounded in H_{-p} in the sense of [14]. Applying Theorem 3.18.1 of [14], we can see the analyticity of $\tilde{f}$ in H_{-p}. □

DEFINITION 3.1. The functional $\tilde{f}$ given in Proposition 3.2 is called *the analytic functional associated with* $f \in (\mathfrak{F}_p)$ or *the associated functional of* f ($p \in \mathbf{R}$).

For $p < s_0$ and $f \in (\mathfrak{F}_p)$, the associated functional $\tilde{f}$ of f can not be a version in the sense of ν-a.e., because of $\nu(H_{-p}) = 0$. However, the functional

$\tilde{f}$ recovers f by means of Taylor coefficients (ref. [4], [17]). We will discuss this point in a forthcoming paper. For $p > s_0$, we have the following:

PROPOSITION 3.3. *Let $p > s_0$. If $f \in (\mathfrak{F}_p)$, then the analytic functional $\tilde{f}$ associated with f is a unique continuous version of f in H_{-p}; that is, $\tilde{f}(z) = f(z)$ holds for ν-a.e. $z \in H^*$. Moreover, if $p > q + s_0$, then $\tilde{f}(D^q z)$ coincides with the continuous version of $\Lambda(D^q)f(z)$ in H_{-p+q}.*

PROOF. Since $\nu(H_{-p}) = 1$ for $p > s_0$, f of (3.4) is equal to $\tilde{f}$ ν-a.e. in H^*. Since every non-void open set in H_{-p} has strictly positive ν-measure, the continuous version of f is uniquely given on H_{-p}. If $p > s_0 + q$ and $z \in H_{-p+q}$, then $D^q z \in H_{-p}$ and $p - q > s_0$. Therefore we see that

$$\tilde{f}(D^q z) = \sum_{\mathbf{n} \in \mathcal{N}_0} x_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}(D^q z) = \sum_{\mathbf{n} \in \mathcal{N}_0}^{\infty} \lambda^{-q\mathbf{n}} c_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}(z)$$

converges uniformly on any bounded set in H_{-p+q}. Therefore we have the last assertion. □

If $f \in (\mathfrak{F})$, then $\tilde{f}(z)$ can be defined on H_{-p} for any $p \in \mathbf{R}$ and so $\tilde{f}(z)$ is a functional defined on H^*. Moreover, if $p > q$, the continuity of $\tilde{f}(z)$ on H_{-p} implies the one on H_{-q}. From this it follows that $\tilde{f}(z)$ is continuous in $z \in H^* = \varinjlim H_{-p}$ with the inductive limit topology. But we omit the proof. Besides we can say that $\tilde{f}(z)$ is not merely entire of at most order 2 on any $H_{-p}(p \in \mathbf{R})$ but also of minimal type (ref. [4], [17]), as we can easily see in the following as a corollary of Proposition 3.2.

COROLLARY 3.1. *If $f \in (\mathfrak{F})$, then for any $p \in \mathbf{R}$, any $k > 0$, and for any $z \in H_{-p}$ we have*

$$|\tilde{f}(z)| \leq \|f\|_{(\mathfrak{F}_{p+k})} \exp\left[\frac{1}{4} \lambda_0^{2k} \|z\|_{-p}^2\right]. \tag{3.11}$$

PROOF. Let $z \in H_{-p}$. Then this is clear from (3.10) and

$$\|z\|_{-(p+k)}^2 \leq \lambda_0^{2k} \|z\|_{-p}^2. \quad \square$$

4. The triplet $(\mathscr{S}) \subset (L^2) \subset (\mathscr{S}')$ derived by Gauss transform from the triplet $(\mathfrak{F}) \subset (\mathfrak{F}_0) \subset (\mathfrak{F}')$

In § 2 we introduced Gauss transform G which is an isometric isomorphism from (L^2) onto $(\mathfrak{F}_0)$. But in this section, we begin by reconsidering G as a map from $\mathscr{P}(E^*)$ onto $\mathscr{P}(H^*)$. Next, we define the system of operators $\{\Gamma(D^p) \equiv G^{-1} \Lambda(D^p) G;\, p \in \mathbf{R}\}$ which acts on $\mathscr{P}(E^*)$ and by using this system we construct the nuclear rigging of white noise functionals:

$$(\mathscr{S}) \subset (\mathscr{S}_p) \subset (L^2) \subset (\mathscr{S}_{-p}) \subset (\mathscr{S}'). \tag{4.1}$$

It will turn out that the rigging (4.1) is obtained as the image of

$$(\mathfrak{F}) \subset (\mathfrak{F}_p) \subset (\mathfrak{F}_0) \subset (\mathfrak{F}_{-p}) \subset (\mathfrak{F}')$$

by the extended G^{-1}.

Let us define the operator $\Gamma(D^p)$ from $\mathscr{P}(E^*)$ onto itself. G is an isometry from $\mathscr{P}(E^*)$ onto $\mathscr{P}(H^*)$:

$$(\mathscr{P}(E^*), \|\cdot\|_{(L^2)}) \xrightarrow[\text{isometric}]{G} (\mathscr{P}(H^*), \|\cdot\|_{(\mathfrak{F}_0)}) \tag{4.2}$$

and $\Lambda(D^p)$ maps $\mathscr{P}(H^*)$ onto $\mathscr{P}(H^*)$. Therefore we can define $\Gamma(D^p)$ for each $p \in \mathbf{R}$ by setting

$$\Gamma(D^p)\varphi \equiv G^{-1}\Lambda(D^p)G\varphi \qquad \text{for } \varphi \in \mathscr{P}(E^*). \tag{4.3}$$

It is easy to see that $\mathscr{P}(E^*)$ is a pre-Hilbert space with the inner product

$$(\Gamma(D^p)\varphi, \Gamma(D^p)\psi)_{(L^2)} = \int_{E^*} (\Gamma(D^p)\varphi(x))\,\overline{\Gamma(D^p)\psi(x)}\, d\mu(x). \tag{4.4}$$

Let us denote its completion by $(\mathscr{S}_p)$ and the inner product by $(\varphi, \psi)_{(\mathscr{S}_p)}$. We evidently see that $(\mathscr{S}_0) = (L^2)$. Let us recall the relations (2.3) and (3.2), that is,

$$Gh_{\mathbf{n}} = \mathbf{z}^{\mathbf{n}},\ G^{-1}\mathbf{z}^{\mathbf{n}} = h_{\mathbf{n}}, \text{ and}$$
$$\Lambda(D^p)\mathbf{z}^{\mathbf{n}}(z) = (\textstyle\prod_j \lambda_j^{-pn_j})\,\mathbf{z}^{\mathbf{n}}(z) = \lambda^{-p\mathbf{n}}\mathbf{z}^{\mathbf{n}}(z).$$

Then, corresponding to the eigensystem of $\Lambda(D^p)$, $\Gamma(D^p)$ has the eigensystem:

$$\Gamma(D^p)h_{\mathbf{n}}(x) = (\textstyle\prod_j \lambda_j^{-pn_j})h_{\mathbf{n}}(x) = \lambda^{-p\mathbf{n}}h_{\mathbf{n}}(x). \tag{4.5}$$

The system $\{h_{\mathbf{n}};\ \mathbf{n} \in \mathscr{N}_0\}$ is a CONS of (L^2), so we can easily see the following.

PROPOSITION 4.1. *For any* $p \in \mathbf{R}$, $\{\lambda^{p\mathbf{n}}h_{\mathbf{n}};\ \mathbf{n} \in \mathscr{N}_0\}$ *is a* CONS *of* $(\mathscr{S}_p)$. *And hence any* $\varphi \in (\mathscr{S}_p)$ *can be expressed in the form*

$$\varphi = \textstyle\sum_{\mathbf{n} \in \mathscr{N}_0} c_{\mathbf{n}} h_{\mathbf{n}} \tag{4.6}$$

with coefficients $\{c_{\mathbf{n}};\ \mathbf{n} \in \mathscr{N}_0\}$ *satisfying*

$$\|\varphi\|^2_{(\mathscr{S}_p)} = \textstyle\sum_{\mathbf{n} \in \mathscr{N}_0} \lambda^{-2p\mathbf{n}} |c_{\mathbf{n}}|^2 < \infty. \tag{4.7}$$

Furthermore, for any p *and* $q \in \mathbf{R}$, $\Gamma(D^q)$ *can extend its domain to* $(\mathscr{S}_p)$ *as an isometry from* $(\mathscr{S}_0)$ *to* $(\mathscr{S}_{p-q})$ *such that, for* $\varphi \in (\mathscr{S}_p)$ *of the form* (4.6),

$$\Gamma(D^q)\varphi = \textstyle\sum_{\mathbf{n} \in \mathscr{N}_0} \lambda^{-q\mathbf{n}} c_{\mathbf{n}} h_{\mathbf{n}} \in (\mathscr{S}_{p-q}). \tag{4.8}$$

By the proposition above we can identify the dual space of $(\mathscr{S}_p)$ with $(\mathscr{S}_{-p})$ for $p \in \mathbf{R}$. In fact, the bilinear form, $\langle \psi, \varphi \rangle$, of $(\varphi, \psi) \in (\mathscr{S}_p) \times (\mathscr{S}_{-p})$ is given by

$$\langle \psi, \varphi \rangle = \int_{E^*} \Gamma(D^{-p})\psi(x)(\Gamma(D^p)\varphi(x))\, d\mu(x). \tag{4.9}$$

Let us write

$$(\mathscr{S}) = \bigcap_{p=0}^{\infty} (\mathscr{S}_p) \quad \text{and} \quad (\mathscr{S}') = \bigcup_{p=0}^{\infty} (\mathscr{S}_{-p}). \tag{4.10}$$

Corresponding to (3.7), for any $p \in \mathbf{R}$ and any $s > s_0$ we have

$$\sum_{\mathbf{n} \in \mathscr{N}_0} \| \lambda^{(p+s)\mathbf{n}} h_{\mathbf{n}} \|^2_{(\mathscr{S}_p)} = \prod_j (1 - \lambda_j^{2s})^{-1} < \infty.$$

Thus we obtain a nuclear rigging

$$(\mathscr{S}) \subset (\mathscr{S}_p) \subset (L^2) \subset (\mathscr{S}_{-p}) \subset (\mathscr{S}'), \qquad p > 0. \tag{4.11}$$

As well as $(\mathfrak{F})$ and $(\mathfrak{F}')$, $(\mathscr{S})$ is a nuclear space and $(\mathscr{S}')$ is the dual space of $(\mathscr{S})$. We call $(\mathscr{S})$ the space of test white noise functionals and $(\mathscr{S}')$ the space of generalized white noise functionals, as usual.

Let $p \in \mathbf{R}$. It follows from (4.2) that for any $f \in \mathscr{P}(H^*)$

$$\| G^{-1} f \|_{(\mathscr{S}_p)} = \| \Gamma(D^p) G^{-1} f \|_{(L^2)} = \| \Lambda(D^p) f \|_{(\mathfrak{F}_0)} = \| f \|_{(\mathfrak{F}_p)}.$$

Therefore G^{-1} can extend uniquely to the isometric map G_p^{-1} from $(\mathfrak{F}_p)$ onto $(\mathscr{S}_p)$. These extensions $\{G_p^{-1}; p \in \mathbf{R}\}$ are consistent. That is, if $p < q$, then G_p^{-1} coincides with G_q^{-1} on $(\mathfrak{F}_q)$. So we have a unique continuous extension from $(\mathfrak{F}')$ onto $(\mathscr{S}')$, which we denote by the same symbol G^{-1}. It satisfies the property that for any $f, g \in (\mathfrak{F}_p)$ and any $p \in \mathbf{R}$

$$(G^{-1} f, G^{-1} g)_{(\mathscr{S}_p)} = (f, g)_{(\mathfrak{F}_p)}. \tag{4.12}$$

Moreover, we can easily see that for $F \in (\mathfrak{F}_{-p})$ and $f \in (\mathfrak{F}_p)$

$$\langle G^{-1} F, G^{-1} f \rangle = \langle F, f \rangle. \tag{4.13}$$

We note that the above construction of the nuclear rigging of white noise calculus is the same as the ordinary ones. Actually, we can see that the triplet $(\mathscr{S}) \subset (L^2) \subset (\mathscr{S}')$ is coincident with the one in [20] or [39]. Here, let us see this in the following concrete case: Let $L^2(\mathbf{R}^n; \mathbf{R})$ (resp. $L^2(\mathbf{R}^n; \mathbf{C})$) be the Hilbert space constructed with all $\mathbf{R}$-valued (resp. $\mathbf{C}$-valued) square integrable functions defined on $\mathbf{R}^n$. Let E_0 be $L^2(\mathbf{R}; \mathbf{R})$ and H_0 be its complexification. Then we can easily see that

$$H_0 = L^2(\mathbf{R}; \mathbf{C}), \; E_0^{\hat{\otimes} n} = \hat{L}^2(\mathbf{R}^n; \mathbf{R}), \quad \text{and} \quad H_0^{\hat{\otimes} n} = \hat{L}^2(\mathbf{R}^n; \mathbf{C}),$$

where $\hat{L}^2$ denotes the space of symmetric L^2-functions ($n \geq 2$). Next, let $D = 1 + u^2 - (d/du)^2$. Then D has an eigensystem:

$$D\zeta_j(u) = 2(j+1)\zeta_j(u) \qquad (j = 0, 1, 2, \cdots),$$

where

$$\zeta_j(u) = (2^j j! \sqrt{\pi})^{-1/2} H_j(u) \exp[-u^2/2].$$

Hence $(D^p)^{\otimes n}$ has an eigensystem:

$$(D^p)^{\otimes n} \zeta^{\hat{\otimes} \mathbf{n}} = \hat{\bigotimes}_j (D^p \zeta_j)^{\hat{\otimes} n_j} = \{\textstyle\prod_j (2(j+1))^{pn_j}\} \zeta^{\hat{\otimes} \mathbf{n}}. \tag{4.14}$$

Further the equalities on spaces

$$H_p = \left\{ f; f \in H_0, \int_{\mathbf{R}} |D^p f(u)|^2 \, du < \infty \right\} \quad \text{and}$$

$$H_p^{\hat{\otimes} n} = \left\{ f; f \in H_0^{\hat{\otimes} n}, \int_{\mathbf{R}^n} |(D^p)^{\otimes n} f(u_1, \cdots, u_n)|^2 \, du_1 \cdots du_n < \infty \right\}$$

hold for $p > 0$. All the conditions of our setting are fulfilled; accordingly we have the $(\mathscr{S}) \subset (L^2) \subset (\mathscr{S}')$. But the usual construction of Gel'fand triplet of white noise calculus is apparently different from this (see Kubo & Takenaka [22–24]). In this case the usual $(\mathscr{S}_p)$ for $p > 0$, which we denote by $(\mathscr{S}_p^{\mathrm{u}})$, is constructed by means of S-transform, a reproducing kernel, and the multiple Wiener integral, etc.; as a result $(\mathscr{S}_p^{\mathrm{u}})$ is the totality of functionals

$$\varphi = \textstyle\sum_{n=0}^{\infty} I_n(f_n) \quad \text{with} \quad \sum_{n=0}^{\infty} n! (\| f_n \|_{H_p^{\hat{\otimes} n}})^2 < \infty \tag{4.15}$$

where $I_n(f_n)$ is the multiple Wiener integral of $f_n \in H_p^{\hat{\otimes} n}$. The inner product for $\varphi, \psi \in (\mathscr{S}_p^{\mathrm{u}})$ is given by

$$(\varphi, \psi)_{(\mathscr{S}_p^{\mathrm{u}})} = \textstyle\sum_{n=0}^{\infty} n! (f_n, g_n)_{H_p^{\hat{\otimes} n}} \tag{4.16}$$

if $\varphi = \sum_{n=0}^{\infty} I_n(f_n)$ and $\psi = \sum_{n=0}^{\infty} I_n(g_n)$. Especially, by Theorem 3.1 in Itô [15], we can see the important relation

$$I_n(\zeta^{\hat{\otimes} \mathbf{n}} / \sqrt{\mathbf{n}!}) = h_{\mathbf{n}} = (2^n \mathbf{n}!)^{-1/2} \textstyle\prod_j H_{n_j}(I_1(\zeta_j)/\sqrt{2}). \tag{4.17}$$

From (4.14), (4.15), and (4.17) it follows that $(\mathscr{S}_p^{\mathrm{u}})$ is the totality of elements of the form

$$\textstyle\sum_{\mathbf{n} \in \mathscr{N}_0} c_{\mathbf{n}} h_{\mathbf{n}} \quad \text{with} \quad \sum_{\mathbf{n} \in \mathscr{N}_0} (\prod_j \{2(j+1)\}^{pn_j})^2 |c_{\mathbf{n}}|^2 < \infty.$$

But λ_j corresponds to $(2(j+1))^{-1}$ in this concrete case. Hence (4.6) and (4.7) in Proposition 4.1 show that the space $(\mathscr{S}_p)$ coincides completely with the space $(\mathscr{S}_p^{\mathrm{u}})$ including their norms.

5. Integral representation of Gauss transform and analytic continuation of $S\varphi(\xi/\sqrt{2})$

At first the maps G and G^{-1} were introduced as isometric isomorphisms between $\mathscr{P}(E^*)$ and $\mathscr{P}(H^*)$ with L^2-norm (in § 2). And in the preceding section we have extended them to continuous maps between $(\mathscr{S}')$ and $(\mathfrak{F}')$. In this section we will show that if they are restricted to the spaces $(\mathscr{S}_p)$ and $(\mathfrak{F}_p)$ for $p > p_0$, then they have integral representations as well as between $\mathscr{P}(E^*)$ and $\mathscr{P}(H^*)$. To see this, we prepare two lemmas.

LEMMA 5.1. *If* $p > p_0$, *then* $\exp[\frac{1}{2}\|x\|_{-p}^2]$ *belongs to* (L^2) *and the square of its* (L^2)*-norm is*

$$\gamma_p \equiv \int_{E_{-p}} \exp[\|x\|_{-p}^2]\, d\mu(x) = \prod_j (1 - 2\lambda_j^{2p})^{-1/2}.$$

If $p > s_0$, *then* $\exp[\frac{1}{2}\|x\|_{-p}^2]$ *belongs to* (L^1) *and its* (L^1)*-norm is*

$$\alpha_p \equiv \int_{E_{-p}} \exp\left[\frac{1}{2}\|x\|_{-p}^2\right] d\mu(x) = \prod_j (1 - \lambda_j^{2p})^{-1/2}.$$

PROOF. Recall the definition of constants s_0, t_0, and p_0: $s_0 = \inf\{s; \sum_{j=0}^{\infty} \lambda_j^{2s} < \infty\}$, $1/2 = \lambda_0^{2t_0}$, and $p_0 = \max(s_0, t_0)$. Then by direct computation we have, if $2c\lambda_0^{2p} < 1$, then

$$\begin{aligned}
&\int_{E^*} \exp[c\|x\|_{-p}^2]\, d\mu(x) \\
&= \prod_{j=0}^{\infty} \int_{-\infty}^{\infty} (2\pi)^{-1/2} \exp\left[-\left(\frac{1}{2} - c\lambda_j^{2p}\right)u^2\right] du \\
&= \prod_{j=0}^{\infty} (1 - 2c\lambda_j^{2p})^{-1/2}.
\end{aligned}$$

This is equal to γ_p if $c = 1$ and $p > p_0$, and to α_p if $c = 1/2$ and $p > s_0$. □

LEMMA 5.2. *Let* $p \in \mathbf{R}$ *and* $f \in (\mathfrak{F}_p)$. *Then for* $x, y \in E_{-p}$ *and* $w \in H_{-p}$, *the analytic functional* $\tilde{f}$ *associated with* f *satisfies the following inequalities,*

$$|\tilde{f}(\sqrt{2}(x + w + \sqrt{-1}))| \le \|f\|_{(\mathfrak{F}_p)} \exp[\|x\|_{-p}^2] \exp[\|w\|_{-p}^2] \exp[\|y\|_{-p}^2]$$

and especially for $w = 0$

$$\tilde{f}(\sqrt{2}(x + \sqrt{-1}y))| \le \|f\|_{(\mathfrak{F}_p)} \exp\left[\frac{1}{2}\|x\|_{-p}^2\right] \exp\left[\frac{1}{2}\|y\|_{-p}^2\right].$$

PROOF. Trivial by Proposition 3.2. □

In the next theorem it should be noticed that $p_0 \geq s_0$.

THEOREM 5.1. *Suppose* $p > p_0$ *and* $\varphi \in (\varphi_p)$. *Let* $f = G\varphi$ *and* $\tilde{f}$ *be the continuous version in* H_{-p} *of* f *in* Proposition 3.3, *which is analytic in* H_{-p}. *Define the functional* $\tilde{\varphi}(w)$ *in* H_{-p} *by*

$$\tilde{\varphi}(w) \equiv \int_{E_{-p}} \tilde{f}(\sqrt{2}(w + \sqrt{-1}\,y))\,d\mu(y). \tag{5.1}$$

Then $\tilde{\varphi}(w)$ *is analytic in* H_{-p} *and its restriction to* E_{-p} *satisfies* $\tilde{\varphi}(x) = \varphi(x)$ μ-a.e. $x \in E^*$, *and*

$$|\tilde{\varphi}(x + w)| \leq \gamma_p \|\varphi\|_{(\mathscr{S}_p)} \exp[\|x\|_{-p}^2] \exp[\|w\|_{-p}^2] \tag{5.2}$$

holds for $x \in E_{-p}$ *and* $w \in H_{-p}$. *Furthermore, under the weaker condition* $p > s_0$, *the variable* w *in* (5.1) *can be replaced with* $x \in E_{-p}$ *and then* $\tilde{\varphi}(x)$ *is continuous in* $x \in E_{-p}$, $\tilde{\varphi}(x)$ *satisfies* $\tilde{\varphi}(x) = \varphi(x)$ *for* μ-a.e. $x \in E^*$, *and*

$$|\tilde{\varphi}(x)| \leq \alpha_p \|\varphi\|_{(\mathscr{S}_p)} \exp\left[\frac{1}{2}\|x\|_{-p}^2\right] \tag{5.3}$$

holds for any $x \in E_{-p}$. PROOF. Let the Fourier expansion of $f \in (\mathfrak{F}_p)$ with respect to the CONS $\{\mathbf{z}^{\mathbf{n}}; \mathbf{n} \in \mathscr{N}_0\}$ of $(\mathfrak{F}_0)$ be

$$f = \sum_{\mathbf{n} \in \mathscr{N}_0} c_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} \quad \text{with} \quad \|f\|_{(\mathfrak{F}_p)}^2 = \sum_{\mathbf{n} \in \mathscr{N}_0} |c_{\mathbf{n}}|^2 \lambda^{-2p\mathbf{n}} < \infty.$$

Let $z, w \in H_{-p}$ and J be any finite subset of $\mathscr{N}_0$. Let us set

$$f_{\mathrm{J}}(z) \equiv \sum_{\mathbf{n} \in \mathrm{J}} c_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}(z) \quad \text{and} \quad \varphi_{\mathrm{J}}(w) \equiv \sum_{\mathbf{n} \in \mathrm{J}} c_{\mathbf{n}} h_{\mathbf{n}}(w).$$

If $x, y \in E_{-p}$, it follows from Proposition 3.2 that

$$\lim_{\mathrm{J} \to \mathscr{N}_0} f_{\mathrm{J}}(\sqrt{2}(x + w + \sqrt{-1}\,y)) = \tilde{f}(\sqrt{2}(x + w + \sqrt{-1}\,y)). \tag{5.4}$$

By the property of Gauss transform between polynomials we have

$$\varphi_{\mathrm{J}}(x + w) = \int_{E_{-p}} f_{\mathrm{J}}(\sqrt{2}(x + w + \sqrt{-1}\,y))\,d\mu(y), \tag{5.5}$$

while it follows from the first inequality of Lemma 5.2 that

$$\begin{aligned} |f_{\mathrm{J}}(\sqrt{2}(x + w + \sqrt{-1}\,y))| &\leq \sum_{\mathbf{n} \in \mathscr{N}_0} |c_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}(\sqrt{2}(x + w + \sqrt{-1}\,y)| \\ &\leq \|f\|_{(\mathfrak{F}_p)} \exp[\|x\|_{-p}^2] \exp[\|w\|_{-p}^2] \exp[\|y\|_{-p}^2]. \end{aligned} \tag{5.6}$$

In addition, if $p > p_0$, $\exp[\|y\|_{-p}^2]$ is μ-integrable. So for $p > p_0$, it follows from Lebesgue's dominated convergence theorem that

$$\lim_{J\to\mathcal{N}_0}\varphi_J(x+w)$$

$$=\lim_{J\to\mathcal{N}_0}\int_{E_{-p}} f_J(\sqrt{2}(x+w+\sqrt{-1}\,y))\,d\mu(y)$$

$$=\int_{E_{-p}}\tilde{f}(\sqrt{2}(x+w+\sqrt{-1}\,y))\,d\mu(y)=\tilde{\varphi}(x+w). \tag{5.7}$$

Therefore $\sum_{\mathbf{n}\in\mathcal{N}_0}c_{\mathbf{n}}h_{\mathbf{n}}(x+w)$ converges absolutely to $\tilde{\varphi}(x+w)$. Besides, (5.5), (5.6), and the isometric property of G^{-1}, i.e.,

$$\|\varphi\|_{(\mathcal{S}_p)}=\|G^{-1}f\|_{(\mathcal{S}_p)}=\|f\|_{(\mathfrak{F}_p)},$$

imply that

$$|\varphi_J(x+w)|$$

$$\leq\|f\|_{(\mathfrak{F}_p)}\exp[\|x\|_{-p}^2]\exp[\|w\|_{-p}^2]\int_{E_{-p}}\exp[\|y\|_{-p}^2]\,d\mu(y)$$

$$=\gamma_p\|\varphi\|_{(\mathcal{S}_p)}\exp[\|x\|_{-p}^2]\exp[\|w\|_{-p}^2].$$

By (5.7) we have

$$|\tilde{\varphi}(x+w)|\leq\gamma_p\|\varphi\|_{(\mathcal{S}_p)}\exp[\|x\|_{-p}^2]\exp[\|w\|_{-p}^2].$$

Thus putting $x=0$, we can see that $\tilde{\varphi}(w)$ is analytic in H_{-p} as the limit of $\{\varphi_J(w); J\,(\text{finite})\subset\mathcal{N}_0\}$ which consists of analytic and locally uniformly bounded functionals in H_{-p} (see [14]).

Furthermore, putting $w=0$ in the above but under the condition $p>s_0$, from the second inequality of Lemma 5.2 it follows that

$$|f_J(\sqrt{2}(x+\sqrt{-1}\,y))|\leq\|f\|_{(\mathfrak{F}_p)}\exp\left[\frac{1}{2}\|x\|_{-p}^2\right]\exp\left[\frac{1}{2}\|y\|_{-p}^2\right].$$

Since $p>s_0$, $\exp\left[\frac{1}{2}\|y\|_{-p}^2\right]$ is μ-integrable by Lemma 5.1 and so

$$|\varphi_J(x)|\leq\alpha_p\|\varphi\|_{(\mathfrak{F}_p)}\exp\left[\frac{1}{2}\|x\|_{-p}^2\right] \tag{5.8}$$

holds. Therefore the series

$$\sum_{\mathbf{n}\in\mathcal{N}_0}c_{\mathbf{n}}h_{\mathbf{n}}(x)=\lim_{J\to\mathcal{N}_0}\varphi_J(x)$$

converges to $\tilde{\varphi}(x)$ absolutely and uniformly on any bounded set of

E_{-p}. Consequently $\tilde{\varphi}(x)$ is continuous on E_{-p} and $\tilde{\varphi} = \varphi$ μ-a.e. Letting $J \to \mathcal{N}_0$ in (5.8), we obtain

$$|\tilde{\varphi}(x)| \leq \alpha_p \|\varphi\|_{(\mathcal{S}_p)} \exp\left[\frac{1}{2}\|x\|_{-p}^2\right]. \quad \square$$

DEFINITION 5.1. For $p > p_0$ and $\varphi \in (\mathcal{S}_p)$, the analytic functional $\tilde{\varphi}(w)$ in H_{-p} given by Theorem 5.1 is called *an analytic continuation of* φ *from* E_{-p} *to* H_{-p}. For $p > s_0$ and $\varphi \in (\mathcal{S}_p)$, the continuous functional $\tilde{\varphi}(x)$ in E_{-p} given by Theorem 5.1 is called *a continuous version of* φ *in* E_{-p}.

REMARK. Let $p > p_0$. For $\varphi \in (\mathcal{S}_p)$, let $\tilde{\varphi}(w)$ be the analytic continuation of φ. It is clear that the restriction of $\tilde{\varphi}(w)$ ($w = x + \sqrt{-1}\,y$) to E_{-p} is equal to the continuous version $\tilde{\varphi}(x)$ of φ in E_{-p}.

Now it is easy to show that if $p > p_0$, the Gauss transform G from $(\mathcal{S}_p)$ onto $(\mathfrak{F}_p)$ has an integral representation. That is, for $\varphi \in (\mathcal{S}_p)$, φ and $f = G\varphi$ have the analytic continuation $\tilde{\varphi}$ and the continuous version $\tilde{f}$, respectively; and then $\tilde{f}$ is expressed as

$$\tilde{f}(w) = \int_{E_{-p}} \tilde{\varphi}(x + w/\sqrt{2})\, d\mu(x) \qquad \text{for any } w \in H_{-p}.$$

Moreover this representation can be considered as a modified S-transform of φ, i.e., $\tilde{f}(w) = S\tilde{\varphi}(w/\sqrt{2})$ for $w \in H_{-p}$. It is well-known that since the measure $\mu_\xi(\cdot) \equiv \mu(\cdot - \xi)$ for $\xi \in E$ in absolutely continuous with respect to $\mu(\cdot)$, S-transform can be defined as follows: for $\varphi(L^2)$,

$$S\varphi(\xi) = \int_{E^*} \varphi(x + \xi)\, d\mu(x), \qquad \xi \in E.$$

Because of modifying S and restricting the domain of S, we can enlarge the domain of variable of the transformed functional.

THEOREM 5.2. *Let* $p > p_0$ *and* $\varphi \in (\mathcal{S}_p)$. *Suppose that* $f = G\varphi$. *Let* $\tilde{\varphi}$ *be the analytic continuation of* φ *from* E_{-p} *to* H_{-p} *in* Definition 5.1 *and* $\tilde{f}$ *be the continuous version of* f *in* H_{-p}. *Then*

$$\tilde{f}(w) = S\tilde{\varphi}(w/\sqrt{2}) = \int_{E_{-p}} \tilde{\varphi}(x + w/\sqrt{2})\, d\mu(x) \tag{5.9}$$

holds for any $w \in H_{-p}$.

PROOF. By replacing w with $w/\sqrt{2}$ in Theorem 5.1 and its proof, we have

$$\lim_{J\to\mathcal{N}_0} \varphi_J(x + w/\sqrt{2}) = \tilde{\varphi}(x + w/\sqrt{2}),$$

while we have

$$|\varphi_J(x + w/\sqrt{2})| \le \gamma_p \|f\|_{(\mathfrak{F}_p)} \exp[\|x\|_{-p}^2] \exp\left[\frac{1}{2}\|w\|_{-p}^2\right].$$

Therefore, from Lebesgue's dominated convergence theorem and the evident equation

$$f_J(w) = \int_{E^*} \varphi_J(x + w/\sqrt{2})\, d\mu(x),$$

it follows that

$$\begin{aligned}\tilde{f}(w) &= \lim_{J\to\mathcal{N}_0} f_J(w) = \lim_{J\to\mathcal{N}_0} \int_{E_{-p}} \varphi_J(x + w/\sqrt{2})\, d\mu(x) \\ &= \int_{E_{-p}} \tilde{\varphi}(x + w/\sqrt{2})\, d\mu(x). \quad \square\end{aligned}$$

6. Other properties of two triplets

THEOREM 6.1. *Let* $0 \le p < q - p_0$. *Then the functional* $\exp\left[\frac{1}{2}\|x\|_{-q}^2\right]$ *defined in* E_{-q} *belongs to* $(\mathscr{S}_p)$. *Actually, the* $(\mathscr{S}_p)$*-norm is evaluated as*

$$\left\|\exp\left[\frac{1}{2}\|\cdot\|_{-q}^2\right]\right\|_{(\mathscr{S}_p)} = \prod_j ((1 - \lambda_j^{2q})^2 - \lambda_j^{4(q-p)})^{-1/4}.$$

PROOF. If $q > p_0$, then the functional $\exp[\frac{1}{2}\|x\|_{-q}^2]$ belongs to $(L^2) = (\mathscr{S}_0)$ by Lemma 5.1. So it is expanded into a Fourier series. Let us compute the Fourier coefficients

$$c_{\mathbf{n}} = \int_{E^*} \exp\left[\frac{1}{2}\|x\|_{-q}^2\right] h_{\mathbf{n}}(x)\, d\mu(x) \tag{6.1}$$

with respect to the CONS $\{h_{\mathbf{n}}(x); \mathbf{n} \in \mathcal{N}_0\}$ of (L^2). To get the values $c_{\mathbf{n}}$, if we note the equality

$$\exp\left[\frac{1}{2}\|x\|_{-q}^2\right] = \prod_{j=0}^{\infty} \exp\left[\frac{1}{2}\lambda_j^{2q}\langle x, \zeta_j\rangle^2\right]$$

and independentness of $\langle x, \zeta_j\rangle$'s, we have only to calculate the integrals

$$\int_{E^*} \exp\left[\frac{1}{2}\lambda_j^{2q}\langle x, \zeta_j\rangle^2\right] H_n(\langle x, \zeta_j\rangle/\sqrt{2})\, d\mu(x).$$

But if n is odd, then the integral is equal to zero and if n is even, say $n = 2k$, then it is equal to

$$(1 - \lambda_j^{2q})^{-1/2}\frac{n!}{k!}(\lambda_j^{2q}/(1 - \lambda_j^{2q}))^k.$$

So we have for $\mathbf{n} = 2\mathbf{k} = (2k_0, 2k_1, 2k_2, \cdots)$

$$c_{\mathbf{n}} = \alpha_q(2^{\mathbf{n}}\mathbf{n}!)^{-1/2}\frac{\mathbf{n}!}{\mathbf{k}!}\prod_j(\lambda_j^{2q}/(1 - \lambda_j^{2q}))^{k_j}$$

else $c_{\mathbf{n}} = 0$, where α_q is the constant in Lemma 5.1, i.e.,

$$\alpha_q = \prod_{j=0}^{\infty}(1 - \lambda_j^{2q})^{-1/2}.$$

Therefore

$$\left\|\exp\left[\frac{1}{2}\|\cdot\|_{-q}^2\right]\right\|_{(\mathscr{S}_p)}^2 = \sum_{\mathbf{n}\in\mathscr{N}_0}\lambda^{-2p\mathbf{n}}|c_{\mathbf{n}}|^2$$

$$= \alpha_q^2\sum_{\mathbf{k}\in\mathscr{N}_0}2^{-2\mathbf{k}}\binom{2\mathbf{k}}{\mathbf{k}}\prod_j(\lambda_j^{4(q-p)}/(1 - \lambda_j^{2q})^2)^{k_j}. \tag{6.2}$$

If we recall the definition of the constant p_0 and the formula

$$2^{-2k}\binom{2k}{k} = (-1)^k\binom{-1/2}{k},$$

(6.2) is followed by

$$\alpha_q^2\sum_{\mathbf{k}\in\mathscr{N}_0}\prod_j\binom{-1/2}{k_j}(-\lambda_j^{2(q-p)}/(1 - \lambda_j^{2q}))^{2k_j}.$$

But $0 \le p < q - p_0$ implies that $\lambda_j^{2(q-p)}/(1 - \lambda_j^{2q}) < 1$ and so this infinite sum of the finite product is equal to

$$\alpha_q^2\prod_j\sum_{k=0}^{\infty}\binom{-1/2}{k}(-\lambda_j^{2(q-p)}/(1 - \lambda_j^{2q}))^{2k}$$

$$= \alpha_q^2\prod_j(1 - \lambda_j^{4(q-p)}/(1 - \lambda_j^{2q})^2)^{-1/2}$$

$$= \prod_j((1 - \lambda_j^{2q})^2 - \lambda_j^{4(q-p)})^{-1/2}. \quad \square$$

THEOREM 6.2. *Let $s_0 < s$ and $p_0 < p$. Then we have*

$$(\mathfrak{F}_{s+p})\cdot(\mathfrak{F}_{s+p}) \subset (\mathfrak{F}_s)$$

and for $f, g \in (\mathfrak{F}_{s+p})$

$$\|f \cdot g\|_{(\mathfrak{F}_s)} \le \gamma_p^2 \|f\|_{(\mathfrak{F}_{s+p})} \|g\|_{(\mathfrak{F}_{s+p})} \tag{6.3}$$

where γ_p *is given in* Lemma 5.1. *Hence* $(\mathfrak{F})$ *is an algebra.*

PROOF. First we note that for $\mathbf{m}, \mathbf{n} \in \mathcal{N}_0$

$$\binom{\mathbf{m}+\mathbf{n}}{\mathbf{n}} \le 2^{|\mathbf{m}|+|\mathbf{n}|} \quad \text{and} \quad \mathbf{z}^{\mathbf{m}}(z) = \binom{\mathbf{m}+\mathbf{n}}{\mathbf{n}}^{1/2} \mathbf{z}^{\mathbf{m}+\mathbf{n}}(z).$$

Let $c_{\mathbf{n}} = (f, \mathbf{z}^{\mathbf{n}})_{(\mathfrak{F}_0)}$ and $d_{\mathbf{n}} = (g, \mathbf{z}^{\mathbf{n}})_{(\mathfrak{F}_0)}$. Then we have

$$\tilde{f}(z) = \sum_{\mathbf{n} \in \mathcal{N}_0} c_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}(z) \quad \text{and} \quad \tilde{g}(z) = \sum_{\mathbf{n} \in \mathcal{N}_0} d_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}(z).$$

By Proposition 3.1 these two series are absolutely convergent on H_{-s-p}. Therefore we have

$$\tilde{f}(z) \cdot \tilde{g}(z) = \sum_{\mathbf{m},\mathbf{n} \in \mathcal{N}_0} c_{\mathbf{m}} d_{\mathbf{n}} \binom{\mathbf{m}+\mathbf{n}}{\mathbf{n}}^{1/2} \mathbf{z}^{\mathbf{m}+\mathbf{n}}(z)$$

and so, using Schwarz' inequality,

$$\begin{aligned}
\|f \cdot g\|_{(\mathfrak{F}_s)} &\le \sum_{\mathbf{m},\mathbf{n} \in \mathcal{N}_0} |c_{\mathbf{m}}| |d_{\mathbf{n}}| 2^{(|\mathbf{m}|+|\mathbf{n}|)/2} \lambda^{-s(\mathbf{m}+\mathbf{n})} \\
&\le \left(\sum_{\mathbf{m} \in \mathcal{N}_0} |c_{\mathbf{m}}| \lambda^{-(s+p)\mathbf{m}} 2^{|\mathbf{m}|/2} \lambda^{p\mathbf{m}}\right)\left(\sum_{\mathbf{n} \in \mathcal{N}_0} |d_{\mathbf{n}}| \lambda^{-(s+p)\mathbf{n}} 2^{|\mathbf{n}|/2} \lambda^{p\mathbf{n}}\right) \\
&\le \|f\|_{(\mathfrak{F}_{s+p})} \|g\|_{(\mathfrak{F}_{s+p})} \prod_j \sum_{n=0}^{\infty} (2\lambda_j^{2p})^n \\
&= \|f\|_{(\mathfrak{F}_{s+p})} \|g\|_{(\mathfrak{F}_{s+p})} \prod_j (1 - 2\lambda_j^{2p})^{-1}. \quad \square
\end{aligned}$$

For the use of the next section let us mention the fact that $(\mathscr{S})$ is an algebra. How to conclude this result was shown in [23]. But our setting described above makes some computations a little bit simple. A rewritten form about this fact within our framework is:

PROPOSITION 6.1. *Let* $s_0 < s$ *and* $2p_0 < p$. *If the functionals* φ *and* ψ *are in* $(\mathscr{S}_{s+p})$, *then* $\varphi \cdot \psi$ *belongs to* $(\mathscr{S}_s)$ *and*

$$\|\varphi \cdot \psi\|_{(\mathscr{S}_s)} \le \beta_s \kappa_p \|\varphi\|_{(\mathscr{S}_{s+p})} \|\varphi\|_{(\mathscr{S}_{s+p})}$$

where

$$\beta_s = \prod_j (1 - \lambda_j^{4s}/4)^{-1/2} \quad \textit{and} \quad \kappa_p = \prod_j (1 - 4\lambda_j^{2p})^{-1}.$$

PROOF. Let $\varphi, \psi \in (\mathscr{S}_{s+p})$. Suppose that φ and ψ have the expansions as elements of $(\mathscr{S}_s)$:

$$\varphi = \sum_{\mathbf{n} \in \mathcal{N}_0} c_{\mathbf{n}} h_{\mathbf{n}} \quad \text{with} \quad \sum_{\mathbf{n} \in \mathcal{N}_0} \lambda^{-2s\mathbf{n}} |c_{\mathbf{n}}|^2 < \infty$$

and

$$\psi = \sum_{\mathbf{n}\in\mathcal{N}_0} d_{\mathbf{n}} h_{\mathbf{n}} \quad \text{with} \quad \sum_{\mathbf{n}\in\mathcal{N}_0} \lambda^{-2s\mathbf{n}} |d_{\mathbf{n}}|^2 < \infty.$$

The absolute convergence for $x \in E_{-s}$ of the series

$$\tilde{\varphi}(x) = \sum_{\mathbf{n}\in\mathcal{N}_0} c_{\mathbf{n}} h_{\mathbf{n}}(x) \quad \text{and} \quad \tilde{\psi}(x) = \sum_{\mathbf{n}\in\mathcal{N}_0} d_{\mathbf{n}} h_{\mathbf{n}}(x)$$

implies the absolute convergence of

$$\tilde{\varphi}(x) \cdot \tilde{\psi}(x) = \sum_{\mathbf{m},\mathbf{n}\in\mathcal{N}_0} c_{\mathbf{m}} d_{\mathbf{n}} h_{\mathbf{m}}(x) h_{\mathbf{n}}(x).$$

Therefore we have

$$\begin{aligned} \|\varphi\psi\|_{(\mathcal{S}_s)} &\le \sum_{\mathbf{m},\mathbf{n}\in\mathcal{N}_0} |c_{\mathbf{m}} d_{\mathbf{n}}| \, \|h_{\mathbf{m}} h_{\mathbf{n}}\|_{(\mathcal{S}_s)} \\ &\le \sum_{\mathbf{m},\mathbf{n}\in\mathcal{N}_0} \lambda^{-(s+p)(\mathbf{m}+\mathbf{n})} |c_{\mathbf{m}} d_{\mathbf{n}}| \, \|\lambda^{s\mathbf{m}} h_m \lambda^{s\mathbf{n}} h_{\mathbf{n}}\|_{(\mathcal{S}_s)} \lambda^{p(\mathbf{m}+\mathbf{n})}. \end{aligned}$$

But if we apply the formula

$$H_m(u) H_n(u) = \sum_{k=0}^{m\wedge n} 2^k k! \binom{n}{k} \binom{n}{k} H_{m+n-2k}(u),$$

the fact that $\{(\lambda^{-s\mathbf{n}}, h_{\mathbf{n}}); \mathbf{n} \in \mathcal{N}_0\}$ is an eigensystem of $\Gamma(D^s)$, and the inequality $\binom{m}{k} \le 2^m$ to the norm $\|\lambda^{s\mathbf{m}} h_{\mathbf{m}} \cdot \lambda^{s\mathbf{n}} h_{\mathbf{n}}\|_{(\mathcal{S}_s)}$, we have

$$\|\lambda^{s\mathbf{m}} h_{\mathbf{m}} \lambda^{s\mathbf{n}} h_{\mathbf{n}}\|_{(\mathcal{S}_s)}^2 = \sum_{\mathbf{k}\le\mathbf{m}\wedge\mathbf{n}} \binom{\mathbf{m}}{\mathbf{k}} \binom{\mathbf{n}}{\mathbf{k}} \binom{\mathbf{m}+\mathbf{n}-2\mathbf{k}}{\mathbf{n}-\mathbf{k}} \lambda^{4s\mathbf{k}} \le \beta_s^2 4^{\mathbf{m}+\mathbf{n}}.$$

After all we obtain

$$\begin{aligned} \|\varphi\psi\|_{(\mathcal{S}_s)} &\le \beta_s \sum_{\mathbf{m},\mathbf{n}\in\mathcal{N}_0} \lambda^{-(s+p)(\mathbf{m}+\mathbf{n})} |c_{\mathbf{m}} d_{\mathbf{n}}| (2\lambda^p)^{(\mathbf{m}+\mathbf{n})} \\ &= \beta_s \|\varphi\|_{(\mathcal{S}_{s+p})} \|\psi\|_{(\mathcal{S}_{s+p})} \sum_{\mathbf{n}\in\mathcal{N}_0} (2\lambda^p)^{2\mathbf{n}} \\ &= \beta_s \kappa_p \|\varphi\|_{(\mathcal{S}_{s+p})} \|\psi\|_{(\mathcal{S}_{s+p})}. \quad \square \end{aligned}$$

From this proposition we can easily conclude that $(\mathcal{S})$ is an algebra (cf. also [37], [38], [39]).

7. Integrability of $\exp[\frac{1}{2}\|x\|_{-p}^2]$ by the measures associated with positive generalized white noise functionals

A generalized white noise functional $\Psi \in (\mathcal{S}')$ is called a positive functional if $\langle \Psi, \varphi \rangle \ge 0$ for any $\varphi \in (\mathcal{S})$ which is μ-a.e. non-negative. The following theorem on positive generalized white noise functionals is already known (ref. [16], [38], [39]):

To every positive generalized white noise functional $\Psi\in(\mathscr{S}')$ *there corresponds a unique finite measure* ν_Ψ *on* $(E^*, \mathscr{B})$ *such that for any* $\varphi\in(\mathscr{S})$

$$\langle \Psi, \varphi\rangle = \int_{E^*} \tilde{\varphi}(x)\, d\nu_\Psi(x) \tag{7.1}$$

where $\tilde{\varphi}$ *is the continuous version of* φ *on* E^* *with the inductive limit topology.*

In Theorem 7.1 below, we firstly refine the above theorem and, using this refined form, we obtain the estimate of Fernique type about the measure ν_Ψ. Finally, in Theorem 7.2 we show that every measure which has such an estimate defines a positive generalized white noise functional.

THEOREM 7.1. *Let* Ψ *be any positive generalized white noise functional. Then there exist a real number* $q_0\geq 0$ *and a unique finite measure* ν_Ψ *on* $(E^*, \mathscr{B})$ *such that*

$$\Psi\in(\mathscr{S}_{-q_0}), \tag{7.2}$$

$$\langle \Psi, e^{\sqrt{-1}\langle\cdot,\xi\rangle}\rangle = \int_{E^*} e^{\sqrt{-1}\langle x,\xi\rangle}\, d\nu_\Psi(x), \qquad \xi\in E, \tag{7.3}$$

$$\nu_\Psi(E^*\setminus E_{-q_0-s}) = 0 \qquad \text{for } s > s_0, \tag{7.4}$$

and that if $p > q_0 + s_0 + 2p_0$, *then for any* $\varphi\in(\mathscr{S}_p)$

$$\langle \Psi, \varphi\rangle = \int_{E_{-p}} \tilde{\varphi}(x)\, d\nu_\Psi(x), \tag{7.5}$$

where $\tilde{\varphi}(x)$ *is the continuous version of* φ *on* E_{-p} *defined in* Definition 5.1. *Moreover, if* $q_0 + s_0 + 3p_0 < q$ *and* $q_0 + s_0 + 2p_0 < p < q - p_0$, *then*

$$\int_{E_{-p}} \exp\left[\frac{1}{2}\|x\|_{-q}^2\right] d\nu_\Psi(x) < \infty. \tag{7.6}$$

PROOF. The existence of the number $q_0\geq 0$ such that (7.2) holds is clear. The existence of the measure ν_Ψ which satisfies (7.3) and the equality $\nu_\Psi(E^*\setminus E_{-q_0-s}) = 0$ for $s > s_0$ follow from Minlos' theorem. Compared with (7.1), the space to which the functional φ in (7.5) belongs is larger. To prove the equality (7.5), we need Proposition 6.1 with the condition $p > q_0 + s_0 + 2p_0$. But the proof is almost the same as the one of Theorem 5.1 in [39], so we omit the proof of this part. Let us prove (7.6) by using (7.5). If $q > q_0 + s_0 + 3p_0$, there exists a number p such that $q_0 + s_0 + 2p_0 < p < q - p_0$. It follows from Theorem 6.1 that

$$\exp\left[\frac{1}{2}\|\cdot\|_{-q}^2\right]\in(\mathscr{S}_p).$$

Clearly, $\exp[\frac{1}{2}\|x\|_{-q}^2]$ is continuous in $x \in E_{-p}$. Applying these to (7.5), we have

$$\int_{E_{-p}} \exp\left[\frac{1}{2}\|x\|_{-q}^2\right] d\nu_\Psi(x) = \left\langle \Psi, \exp\left[\frac{1}{2}\|\cdot\|_{-q}^2\right]\right\rangle < \infty. \quad \square$$

THEOREM 7.2. *Let γ be a positive measure on $(E^*, \mathscr{B})$ which satisfies the following property: there exists a number $q_0 \geq 0$ such that for $p > q_0$,*

$$\gamma(E^* \setminus E_{-p}) = 0 \quad \text{and} \quad \int_{E_{-p}} \exp\left[\frac{1}{2}\|x\|_{-p}^2\right] d\gamma(x) < \infty. \tag{7.7}$$

Then the functional Ψ_γ on $(\mathscr{S})$ defined by

$$\langle \Psi_\gamma, \varphi\rangle \equiv \int_{E^*} \tilde{\varphi}(x)\, d\gamma(x) \tag{7.8}$$

is positive; and for any $p > s_0 \vee q_0$ it follows that $\Psi_\gamma \in (\mathscr{S}_{-p})$ and

$$\|\Psi_\gamma\|_{(\mathscr{S}_{-p})} \leq \alpha_p \int_{E_{-p}} \exp\left[\frac{1}{2}\|x\|_{-p}^2\right] d\gamma(x), \tag{7.9}$$

where $\tilde{\varphi}$ is the continuous version on E^* of $\varphi \in (\mathscr{S})$.

PROOF. The linearity and the positivity of Ψ_γ is trivial. Let $\varphi \in (\mathscr{S})$. If $p > s_0 \vee q_0$, then the restriction $\tilde{\varphi} \mid E_{-p}$ of the continuous version $\tilde{\varphi}$ in E^* on φ is a continuous version in E_{-p}. If we apply the estimate (5.3) in Theorem 5.1 to $\tilde{\varphi} \mid E_{-p}$, we have

$$\left|\int_{E^*} \tilde{\varphi}(x)\, d\gamma(x)\right| = \left|\int_{E_{-p}} (\tilde{\varphi} \mid E_{-p})(x)\, d\gamma(x)\right|$$
$$\leq \alpha_p \|\varphi\|_{(\mathscr{S}_p)} \int_{E_{-p}} \exp\left[\frac{1}{2}\|x\|_{-p}^2\right] d\gamma(x) < \infty.$$

So Ψ_γ is a continuous functional on $(\mathscr{S})$ and belongs to $(\mathscr{S}_{-p})$. The norm of Ψ_γ is evaluated by

$$\|\Psi_\gamma\|_{(\mathscr{S}_{-p})} \leq \alpha_p \int_{E_{-p}} \exp\left[\frac{1}{2}\|x\|_{-p}^2\right] d\gamma(x). \quad \square$$

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Department of Mathematics
Faculty of General Education
Kumamoto University