HIROSHIMA MATH. J.
25 (1995), 595–621

Taylor expansion of Riesz potentials

Dedicated to Professor Fumi-Yuki Maeda on the occasion of his sixtieth birthday

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(Received June 3, 1994)

Abstract: This paper deals with Riesz potentials $U_\alpha f(x) = \int |x-y|^{\alpha-n} f(y)dy$ of functions f satisfying Orlicz condition with weight ω in the form:

$$\int \Phi_p(|f(y)|)\omega(|y|)dy < \infty.$$

We are mainly concerned with the case when $\Phi_p(r)/r^p$, $p > 1$, is nondecreasing and $\omega(r)$ is of the form r^β, $-n < \beta \leq \alpha p - n$. Letting ℓ be the integer such that $\ell \leq \alpha - (n+\beta)/p < \ell + 1$, we examine when

$$\lim_{x\to 0, x\in R^n - E} [\kappa(|x|)]^{-1}[U_\alpha f(x) - P(x)] = 0$$

holds for an exceptional set E, a weight function κ and a polynomial P of degree at most ℓ.

1. Introduction

For $0 < \alpha < n$ and a nonnegative measurable function f on R^n, we define $U_\alpha f$ by

$$U_\alpha f(x) = \int_{R^n} |x-y|^{\alpha-n} f(y)dy.$$

Here it is natural to assume that $U_\alpha f \not\equiv \infty$, which is equivalent to

$$\int_{R^n} (1+|y|)^{\alpha-n} f(y)dy < \infty. \tag{1.1}$$

To obtain general results, we treat functions f satisfying a condition of the form:

$$\int_{R^n} \Phi_p(f(y))\omega(|y|)dy < \infty. \tag{1.2}$$

Here $\Phi_p(r)$ and $\omega(r)$ are positive monotone functions on the interval $(0, \infty)$ with the following properties:

($\varphi 1$) $\Phi_p(r)$ is of the form $r^p \varphi(r)$, where $1 \le p < \infty$ and φ is a positive nondecreasing function on the interval $(0, \infty)$; set $\varphi(0) = \lim_{r \to 0} \varphi(r)$.

($\varphi 2$) φ is of logarithmic type, that is, there exists $A_1 > 0$ such that

$$A_1^{-1} \varphi(r) \le \varphi(r^2) \le A_1 \varphi(r) \qquad \text{whenever } r > 0.$$

($\omega 1$) ω satisfies the doubling condition; that is, there exists $A_2 > 0$ such that

$$A_2^{-1} \omega(r) \le \omega(2r) \le A_2 \omega(r) \qquad \text{whenever } r > 0.$$

It is known (see [7]) that if $p > 1$ and

$$\int_0^1 [r^{n-\alpha p} \varphi(r^{-1})]^{-1/(p-1)} r^{-1} dr < \infty, \tag{1.3}$$

then $U_\alpha f$ is continuous everywhere on R^n possibly except at the origin; in case $\alpha p > n$, (1.3) holds by condition ($\varphi 2$) and the continuity also follows from Sobolev's theorem. More precisely, we shall show (Theorem 4.2) that if $p = n/\alpha > 1$, $\omega(r) \equiv 1$ and (1.3) holds, then

$$U_\alpha f(x) - U_\alpha f(0) = o(\varphi^*(|x|)) \tag{1.4}$$

as $x \to 0$, where

$$\varphi^*(r) = \left(\int_0^r [\varphi(t^{-1})]^{-1/(p-1)} t^{-1} dt \right)^{1-1/p}.$$

This gives an extension of Sobolev's theorem as far as we restrict ourselves to the limiting case $\alpha p = n$; for this, see also Maz'ya [2, Theorem 5.4]. Typical examples of φ satisfying (1.3) in case $\alpha p = n$ are

$$[\log(1+r)]^\delta,\ [\log(1+r)]^{p-1} [\log(1+\log(1+r))]^\delta, \cdots$$

for $\delta > p - 1$.

If (1.3) does not hold, then the potential may not be continuous anywhere, and the second author ([8]) studied the fine limits of $U_\alpha f$, that is,

$$\lim_{x \to 0, x \in R^n - E} U_\alpha f(x) = U_\alpha f(0)$$

with an exceptional set E which is thin at 0 in a certain sense (see also Adams-Meyers [1] and Meyers [5]). In this paper, we extend this result and in fact show that

$$\lim_{x \to 0, x \in R^n - E} [\kappa(|x|)]^{-1} [U_\alpha f(x) - P(x)] = 0$$

with an exceptional set E, a weight function κ and a polynomial P; we are concerned mainly with the case $\kappa(0) = 0$.

For this purpose, let $R_\alpha(x) = |x|^{\alpha - n}$ and consider the remainder term of Taylor's expansion:

$$R_{\alpha,\ell}(x, y) = R_\alpha(x - y) - \sum_{|\mu| \le \ell} \frac{x^\mu}{\mu!} [(D^\mu R_\alpha)(-y)].$$

Then our aim is to investigate the behavior at the origin of the function:

$$U_{\alpha,\ell} f(x) = \int_{R^n} R_{\alpha,\ell}(x, y) f(y) dy.$$

Here it is natural to assume that

$$\int_{B(0,1)} |y|^{\alpha - n - \ell} f(y) dy < \infty \tag{1.5}$$

and

$$\int_{R^n - B(0,1)} |y|^{\alpha - n - \ell - 1} f(y) dy < \infty, \tag{1.6}$$

instead of (1.1), where $B(0, 1)$ denotes the unit ball.

For simplicity, consider the case $\omega(r) = r^\beta$, where $-n < \beta \le \alpha p - n$, and let ℓ be the nonnegative integer such that

$$\ell \le \alpha - (n + \beta)/p < \ell + 1.$$

We shall show (in Corollary 5.1 given later) that if f satisfies (1.1) and (1.2) with $p > 1$, then there exist a set $E \in R^n$ and a polynomial P_ℓ such that

$$\lim_{x \to 0, x \in R^n - E} [\kappa(|x|)]^{-1} [U_\alpha f(x) - P_\ell(x)] = 0 \tag{1.7}$$

and

$$\sum_{j=1}^{\infty} 2^{j(n - \alpha p)} [\varphi(2^j)]^{-1} C_{\alpha, \Phi_p}(E_j; B_j) < \infty, \tag{1.8}$$

where $E_j = \{x \in E : 2^{-j} \le |x| < 2^{-j+1}\}$, $B_j = \{x : 2^{-j-1} < |x| < 2^{-j+2}\}$ and

$$\kappa(r) = r^\ell \left(\int_0^r [t^{n - \alpha p + \beta + \ell p} \varphi(t^{-1})]^{-1/(p-1)} t^{-1} dt \right)^{1 - 1/p};$$

see Section 5 for the definition of C_{α,Φ_p}. Note here that

$$C_{\alpha,\Phi_p}(A_j; B_j) \sim 2^{-j(n-\alpha p)}\varphi(2^j), \qquad A_j = B(0, 2^{-j+1}) - B(0, 2^{-j})$$

(cf. [8, Lemma 7.3]), and our definition of thinness differs from that of Adams-Meyers [1]. If in addition (1.3) holds, then the above fine limit is seen to be replaced by the usual limit similar to (1.4); moreover, (1.7) implies that $U_\alpha f$ is ℓ times differentiable at the origin.

To derive the radial limit result, we modify this as follows (see Corollary 6.1): there exist a set $E \subset R^n$ and a polynomial P_ℓ such that

$$\lim_{x \to 0, x \in R^n - E} |x|^{(n-\alpha p+\beta)/p}[U_\alpha f(x) - P_\ell(x)] = 0 \tag{1.9}$$

and

$$\sum_{j=1}^{\infty} C_{\alpha,\Phi_p}(2^j E_j; B_0) < \infty; \tag{1.10}$$

note here that $r^{(n-\alpha p+\beta)/p} \leq M[\kappa(r)]^{-1}$, and hence (1.9) is weaker than (1.7). It will be seen that (1.10) is more convenient than (1.8) to our aim of deriving the radial limit result.

2. Preliminary lemmas

Throughout this paper, let $M, M_1, M_2, \ldots,$ denote various constants independent of the variables in question.

First we collect properties which follow from conditions ($\varphi 1$) and ($\varphi 2$) (cf. [8, Preliminary lemmas]).

LEMMA 2.1. *φ satisfies the doubling condition, that is, there exists $A > 1$ such that*

$$\varphi(r) \leq \varphi(2r) \leq A\varphi(r) \qquad \textit{whenever } r > 0.$$

LEMMA 2.2. *For any $\gamma > 0$, there exists $A(\gamma) > 1$ such that*

$$A(\gamma)^{-1}\varphi(r) \leq \varphi(r^\gamma) \leq A(\gamma)\varphi(r) \qquad \textit{whenever } r > 0.$$

LEMMA 2.3. *If $\gamma > 0$, then*

$$s^\gamma \varphi(s^{-1}) \leq Mt^\gamma \varphi(t^{-1}) \qquad \textit{whenever } 0 < s < t.$$

PROOF. We know ([8, ($\varphi 5$)])

$$s^\gamma \varphi(s^{-1}) \leq A_1 t^\gamma \varphi(t^{-1}) \qquad \text{whenever } 0 < s < t \leq A_1^{-1/\gamma},$$

so that

$$s^{\gamma}\varphi(s^{-1}) \le Mt^{\gamma}\varphi(t^{-1}) \quad \text{whenever } 0 < s < t \le 1. \tag{2.1}$$

If we apply (2.1) with $\psi(r) = [\varphi(r^{-1})]^{-1}$, then

$$\frac{s^{\gamma}}{\varphi(s)} \le M\frac{t^{\gamma}}{\varphi(t)} \quad \text{whenever } 0 < s < t \le 1. \tag{2.2}$$

In particular,

$$M^{-1}\varphi(1) \le s^{-\gamma}\varphi(s) \quad \text{whenever } 0 < s \le 1.$$

Hence, in case $0 < s < 1 \le t$, we have by (2.1) and the last inequality

$$s^{\gamma}\varphi(s^{-1}) \le M\varphi(1) \le M't^{\gamma}\varphi(t^{-1}).$$

In case $1 < s < t$, we have by (2.2)

$$\frac{t^{-\gamma}}{\varphi(t^{-1})} \le M\frac{s^{-\gamma}}{\varphi(s^{-1})}.$$

Thus Lemma 2.3 is proved.

Lemma 2.4. *If $a > 0$ and $b > 0$, then for $0 < r < 1$,*

$$\int_r^1 t^{-a}[\varphi(t^{-1})]^{-b}t^{-1}\,dt \le Mr^{-a}[\varphi(r^{-1})]^{-b}.$$

Remark 2.1. The converse inequality also holds for $0 < r < 1/2$. In fact, by the doubling condition on φ,

$$\int_r^1 t^{-a}[\varphi(t^{-1})]^{-b}t^{-1}\,dt \ge \int_r^{2r} t^{-a}[\varphi(t^{-1})]^{-b}t^{-1}\,dt \ge Mr^{-a}[\varphi(r^{-1})]^{-b}.$$

Proof of Lemma 2.4. Letting $0 < \gamma < a/b$, we have by Lemma 2.3,

$$\begin{aligned}\int_r^1 t^{-a}[\varphi(t^{-1})]^{-b}t^{-1}\,dt &\le Mr^{-\gamma b}[\varphi(r^{-1})]^{-b}\int_r^1 t^{-a+\gamma b-1}\,dt \\ &\le Mr^{-a}[\varphi(r^{-1})]^{-b}.\end{aligned}$$

Lemma 2.5. *If $a > 0$ and b is a real number, then for $r > 0$,*

$$\int_0^r t^{a}[\varphi(t^{-1})]^{b}t^{-1}\,dt \le Mr^{a}[\varphi(r^{-1})]^{b}.$$

In fact, if $b \le 0$, then the required inequality follows since $[\varphi(r^{-1})]^{-1}$ is nondecreasing. The case $b > 0$ can be obtained by applying Lemma 2.3 and the proof of Lemma 2.4.

3. The estimates of $U_{\alpha,\ell} f$

For an integer ℓ, we consider the potential

$$U_{\alpha,\ell} f(x) = \int_{R^n} R_{\alpha,\ell}(x, y) f(y) dy;$$

in case $\ell \leq -1$, $U_{\alpha,\ell} f(x)$ is nothing but $U_\alpha f(x)$, so that, in this paper, we assume that $\ell \geq 0$.

Write $U_{\alpha,\ell} f(x) = U_1(x) + U_2(x) + U_3(x)$ for $x \in R^n - \{0\}$, where

$$U_1(x) = \int_{R^n - B(0,2|x|)} R_{\alpha,\ell}(x, y) f(y) dy,$$

$$U_2(x) = \int_{B(0,|x|/2)} R_{\alpha,\ell}(x, y) f(y) dy,$$

$$U_3(x) = \int_{B(0,2|x|) - B(0,|x|/2)} R_{\alpha,\ell}(x, y) f(y) dy.$$

LEMMA 3.1. *If* $y \in B(0, |x|/2)$, *then*

$$|R_{\alpha,\ell}(x, y)| \leq M |x|^\ell |y|^{\alpha - n - \ell}.$$

PROOF. Since $|y| < |x|/2$, we have

$$\begin{aligned} |R_{\alpha,\ell}(x, y)| &\leq |R_\alpha(x - y)| + \sum_{|\mu| \leq \ell} \left| \frac{x^\mu}{\mu!} [(D^\mu R_\alpha)(-y)] \right| \\ &\leq (|x|/2)^{\alpha - n} + M \sum_{|\mu| \leq \ell} \frac{|x|^{|\mu|}}{\mu!} |y|^{\alpha - n - |\mu|} \\ &\leq M |x|^\ell |y|^{\alpha - n - \ell}. \end{aligned}$$

LEMMA 3.2. *If* $y \in B(0, 2|x|) - B(0, |x|/2)$, *then*

$$|R_{\alpha,\ell}(x, y)| \leq M |x - y|^{\alpha - n}.$$

PROOF. We have as above

$$\begin{aligned} |R_{\alpha,\ell}(x, y)| &\leq |R_\alpha(x - y)| + \sum_{|\mu| \leq \ell} \left| \frac{x^\mu}{\mu!} [(D^\mu R_\alpha)(-y)] \right| \\ &\leq |x - y|^{\alpha - n} + M |x|^\ell |y|^{\alpha - n - \ell} \\ &\leq M |x - y|^{\alpha - n}. \end{aligned}$$

LEMMA 3.3. *If* $|y| \geq 2|x|$, *then*

$$|R_{\alpha,\ell}(x, y)| \leq M |x|^{\ell + 1} |y|^{\alpha - n - \ell - 1}.$$

PROOF. By Taylor's theorem, we obtain

$$|R_{\alpha,\ell}(x, y)| \le M \sum_{|\mu|=\ell+1} \frac{|x|^{|\mu|}}{\mu!} |\theta x - y|^{\alpha-n-|\mu|} \qquad (0 < \theta < 1)$$

$$\le M\left(\sum_{|\mu|=\ell+1} \frac{1}{\mu!}\right)|x|^{\ell+1}\left(\frac{|y|}{2}\right)^{\alpha-n-\ell-1}$$

$$= M|x|^{\ell+1}|y|^{\alpha-n-\ell-1}.$$

LEMMA 3.4 (cf. [8, Lemma 2.1]). *Let $p > 1$ and f be a nonnegative measurable function on R^n. If $0 \le 2r < a < 1$ and $0 < \delta < \beta$, then*

$$\int_{R^n - B(0,r)} |y|^{\beta-n} f(y)dy \le \int_{R^n - B(0,a)} |y|^{\beta-n} f(y)dy + Ma^{\beta-\delta}$$

$$+ M\left(\int_r^a [t^{n-\beta p}\eta(t)]^{-p'/p} t^{-1} dt\right)^{1/p'} \left(\int_{B(0,a)} \Phi_p(f(y))\omega(|y|)dy\right)^{1/p},$$

and if $0 \le 2r < a < 1$ and $\delta > 0 \ge \beta$, then

$$\int_{R^n - B(0,r)} |y|^{\beta-n} f(y)dy \le \int_{R^n - B(0,a)} |y|^{\beta-n} f(y)dy + Mr^{\beta-\delta}$$

$$+ M\left(\int_r^a [t^{n-\beta p}\eta(t)]^{-p'/p} t^{-1} dt\right)^{1/p'} \left(\int_{B(0,a)} \Phi_p(f(y))\omega(|y|)dy\right)^{1/p},$$

where $\eta(r) = \varphi(r^{-1})\omega(r)$ and $1/p + 1/p' = 1$.

PROOF. Let $0 < a < 1$. We write

$$\int_{B(0,a)-B(0,r)} |y|^{\beta-n} f(y)dy = \int_{\{y\in B(0,a)-B(0,r): f(y) > |y|^{-\delta}\}} |y|^{\beta-n} f(y)dy$$

$$+ \int_{\{y\in B(0,a)-B(0,r): 0 < f(y) \le |y|^{-\delta}\}} |y|^{\beta-n} f(y)dy$$

$$= U_{11} + U_{12}.$$

From Hölder's inequality, we obtain

$$U_{11} \le \left(\int_{\{y\in B(0,a)-B(0,r): f(y) > |y|^{-\delta}\}} f(y)^p \varphi(f(y))\omega(|y|)dy\right)^{1/p}$$

$$\times \left(\int_{\{y\in B(0,a)-B(0,r): f(y) > |y|^{-\delta}\}} |y|^{(\beta-n)p'} [\varphi(f(y))\omega(|y|)]^{-p'/p} dy\right)^{1/p'}.$$

In view of Lemma 2.2, we see that if $f(y) > |y|^{-\delta}$, then

$$\varphi(f(y)) \geq \varphi(|y|^{-\delta}) \geq M\varphi(|y|^{-1}).$$

Hence it follows that

$$U_{11} \leq M\left(\int_r^a [t^{n-\beta p}\eta(t)]^{-p'/p}t^{-1}dt\right)^{1/p'}\left(\int_{B(0,a)} \Phi_p(f(y))\omega(|y|)dy\right)^{1/p}.$$

On the other hand, we have

$$\begin{aligned} U_{12} &\leq \int_{B(0,a)-B(0,r)} |y|^{\beta-\delta-n}dy \\ &\leq M\begin{cases} a^{\beta-\delta}, & \text{in case } \beta-\delta>0, \\ r^{\beta-\delta}, & \text{in case } \beta-\delta<0, \end{cases} \end{aligned}$$

and thus Lemma 3.4 is proved.

Setting $\eta(r)=\varphi(r^{-1})\omega(r)$ as above, we define

$$\kappa_1(r) = \begin{cases} \left(\int_r^1 [t^{n-\alpha p+(\ell+1)p}\eta(t)]^{-p'/p}t^{-1}dt\right)^{1/p'}, & \text{in case } p>1, \\ \sup_{r\leq t<1} t^{\alpha-\ell-1-n}[\eta(t)]^{-1}, & \text{in case } p=1, \end{cases}$$

for $0<r\leq 1/2$; further, set $\kappa_1(r)=\kappa_1(1/2)$ when $r>1/2$.

REMARK 3.1. In view of the doubling conditions on φ and ω, we see that

$$\kappa_1(r) \geq M[r^{n-\alpha p+(\ell+1)p}\eta(r)]^{-1/p} \qquad \text{whenever } 0<r\leq 1/2.$$

LEMMA 3.5. *Let f be a nonnegative measurable function on R^n. If $0<2|x|<a<1$ and $0<\delta<\alpha-\ell-1$, then*

$$\begin{aligned} |U_1(x)| &\leq M|x|^{\ell+1}\left\{\int_{R^n-B(0,a)} |y|^{\alpha-\ell-1-n}f(y)dy + Ma^{\alpha-\ell-1-\delta}\right\} \\ &\quad + M|x|^{\ell+1}\kappa_1(|x|)\left(\int_{B(0,a)} \Phi_p(f(y))\omega(|y|)dy\right)^{1/p}, \end{aligned}$$

and if $0<2|x|<a<1$ and $\delta>0\geq\alpha-\ell-1$, then

$$\begin{aligned} |U_1(x)| &\leq M|x|^{\ell+1}\int_{R^n-B(0,a)} |y|^{\alpha-\ell-1-n}f(y)dy + M|x|^{\alpha-\delta} \\ &\quad + M|x|^{\ell+1}\kappa_1(|x|)\left(\int_{B(0,a)} \Phi_p(f(y))\omega(|y|)dy\right)^{1/p}, \end{aligned}$$

where M is a positive constant independent of x and a.

PROOF. By Lemma 3.3, we have

$$|U_1(x)| \le M|x|^{\ell+1} \int_{R^n - B(0,2|x|)} |y|^{\alpha-\ell-1-n} f(y)dy.$$

The case $p > 1$ follows readily from Lemma 3.4 with $r = |x|$, and the case $p = 1$ is trivial.

In view of Lemma 3.5, we have the following results.

COROLLARY 3.1. *Let f be a nonnegative measurable function on R^n satisfying* (1.2) *and* (1.6). *If $\alpha - \ell - 1 > 0$ and $\kappa_1(0) = \infty$, then*

$$\lim_{x \to 0} [|x|^{\ell+1}\kappa_1(|x|)]^{-1} U_1(x) = 0.$$

PROOF. By Lemma 3.5, we have

$$\limsup_{x \to 0} [|x|^{\ell+1}\kappa_1(|x|)]^{-1} U_1(x) \le M\left(\int_{B(0,a)} \Phi_p(f(y))\omega(|y|)dy\right)^{1/p}$$

for any $a > 0$, which implies that the left hand side is equal to zero.

COROLLARY 3.2. *Let f be a nonnegative measurable function on R^n satisfying conditions* (1.2) *and* (1.6). *If $\alpha - \ell - 1 \le 0$ and*

$$\lim_{r \to 0} r^{\alpha-\delta}[r^{\ell+1}\kappa_1(r)]^{-1} = 0 \qquad \textit{for some } \delta > 0,$$

then

$$\lim_{x \to 0} [|x|^{\ell+1}\kappa_1(|x|)]^{-1} U_1(x) = 0.$$

This can be proved in a way similar to the proof of Corollary 3.1.

In view of Lemmas 3.1 and 3.4, we can establish the following result.

LEMMA 3.6. *If $0 < \delta < \alpha - \ell$, then there exists a positive constant M such that*

$$|U_2(x)| \le M|x|^{\ell}\kappa_2(|x|)\left(\int_{B(0,|x|/2)} \Phi_p(f(y))\omega(|y|)dy\right)^{1/p} + M|x|^{\alpha-\delta}$$

for any $x \in B(0, 1/2) - \{0\}$, where

$$\kappa_2(r) = \begin{cases} \left(\int_0^r [t^{n-\alpha p+\ell p}\eta(t)]^{-p'/p} t^{-1} dt \right)^{1/p'}, & \textit{in case } p > 1, \\ \sup_{0<t\le r} t^{\alpha-\ell-n}[\eta(t)]^{-1}, & \textit{in case } p = 1. \end{cases}$$

REMARK 3.2. As in Remark 3.1, we see that

$$\kappa_2(r) \ge M[r^{n-\alpha p+\ell p}\eta(r)]^{-1/p}.$$

With the aid of Lemma 3.6, we have the following result.

COROLLARY 3.3. *Let f be a nonnegative measurable function on R^n satisfying* (1.2). *If $0 < \delta < \alpha - \ell$, $\kappa_2(1) < \infty$ and*

$$\lim_{r\to 0} r^{\alpha-\delta}[r^{\ell}\kappa_2(r)]^{-1} = 0,$$

then

$$\lim_{x\to 0} [|x|^{\ell}\kappa_2(|x|)]^{-1} U_2(x) = 0.$$

REMARK 3.3. Let $\omega(r) = r^{\beta}$. If $\alpha - (n+\beta)/p < \ell + 1$, then Lemma 2.4 implies that

$$\kappa_1(r) \sim [r^{n-\alpha p+(\ell+1)p+\beta}\varphi(r^{-1})]^{-1/p} \qquad \text{as } r \to 0$$

and thus

$$\kappa_1(0) = \infty.$$

If in addition $n + \beta > 0$, then we see by Lemma 2.3 that

$$\limsup_{r\to 0} r^{\alpha-\delta}[r^{\ell+1}\kappa_1(r)]^{-1} \le M \limsup_{r\to 0} r^{(n+\beta)/p-\delta}[\varphi(r^{-1})]^{1/p} = 0$$

for $0 < \delta < (n+\beta)/p$.

REMARK 3.4. Let $\omega(r) = r^{\beta}$. If $\ell < \alpha - (n+\beta)/p$, then Lemma 2.5 implies that

$$\kappa_2(r) \sim [r^{n-\alpha p+\ell p+\beta}\varphi(r^{-1})]^{-1/p} \qquad \text{as } r \to 0.$$

If in addition $n + \beta > 0$, then we see by Lemma 2.3 that

$$\limsup_{r\to 0} r^{\alpha-\delta}[r^{\ell}\kappa_2(r)]^{-1} \le M \limsup_{r\to 0} r^{(n+\beta)/p-\delta}[\varphi(r^{-1})]^{1/p} = 0$$

for $0 < \delta < (n+\beta)/p$. If $p > 1$ and $\ell = \alpha - (n+\beta)/p$, then $\kappa_2(1) < \infty$ is equivalent to

$$\int_0^1 [\varphi(r^{-1})]^{-p'/p} r^{-1} dr < \infty.$$

4. Taylor expansion

Throughout this section, let $p > 1$. Set

$$\varphi^*(r) = \left(\int_0^r [t^{n-\alpha p}\varphi(t^{-1})]^{-p'/p} t^{-1} dt \right)^{1/p'}$$

and

$$\kappa_3(r) = [\omega(r)]^{-1/p} \varphi^*(r).$$

If $\varphi^*(1) < \infty$, then $U_\alpha f$ is continuous everywhere on R^n possibly except at the origin when f satisfies (1.1) and (1.2) (see [7, Theorem 1]).

LEMMA 4.1. *If $0 < \delta < \alpha$, then there exists a positive constant M such that*

$$|U_3(x)| \le M\kappa_3(|x|)\left(\int_{B(0,2|x|)-B(0,|x|/2)} \Phi_p(f(y))\omega(|y|)dy \right)^{1/p} + M|x|^{\alpha-\delta}$$

for any $x \in B(0, 1/2) - \{0\}$.

PROOF. Let $0 < \delta < \alpha$, and consider the function

$$\tilde{f}(y) = \begin{cases} f(y), & \text{for } y \in B(0, 2|x|) - B(0, |x|/2), \\ 0, & \text{otherwise.} \end{cases}$$

Note by Lemma 3.2 that

$$\begin{aligned} |U_3(x)| &\le M \int_{B(0,2|x|)-B(0,|x|/2)} |x-y|^{\alpha-n} f(y)dy \\ &= M \int_{B(0,3|x|)} |z|^{\alpha-n} \tilde{f}(x+z)dz. \end{aligned}$$

Hence it follows from Lemma 3.4 that

$$\begin{aligned} &|U_3(x)| \\ &\le M\left(\int_0^{3|x|} [r^{n-\alpha p}\varphi(r^{-1})]^{-p'/p} r^{-1} dr \right)^{1/p'} \left(\int \Phi_p(\tilde{f}(x+z))dz \right)^{1/p} + M|x|^{\alpha-\delta} \\ &\le M\varphi^*(|x|)\left(\int_{B(0,2|x|)-B(0,|x|/2)} \Phi_p(f(y))dy \right)^{1/p} + M|x|^{\alpha-\delta} \end{aligned}$$

$$\leq M\kappa_3(|x|)\left(\int_{B(0,2|x|)-B(0,|x|/2)} \Phi_p(f(y))\omega(|y|)dy\right)^{1/p} + M|x|^{\alpha-\delta},$$

as required.

We consider the function

$$K(r) = r^{\ell+1}\kappa_1(r) + r^{\ell}\kappa_2(r) + \kappa_3(r).$$

Here note that

$$K(r) \geq M[r^{n-\alpha p}\eta(r)]^{-1/p} \tag{4.1}$$

for $r > 0$.

THEOREM 4.1. *Assume that* $\ell < \alpha$, $\lim_{r\to 0} K(r) = 0$ *and*

$\kappa_1(0) = \infty$	*in case* $\alpha - \ell - 1 > 0$,
$\lim_{r\to 0} r^{\alpha-\delta}[r^{\ell+1}\kappa_1(r)]^{-1} = 0$	*for some* $\delta > 0$ *in case* $\alpha - \ell - 1 \leq 0$,
$\lim_{r\to 0} r^{\alpha-\delta}[r^{\ell}\kappa_2(r)]^{-1} = 0$	*for some* δ *such that* $0 < \delta < \alpha - \ell$,
$\lim_{r\to 0} r^{\alpha-\delta}[\kappa_3(r)]^{-1} = 0$	*for some* $\delta > 0$.

If f is a nonnegative measurable function on R^n satisfying conditions (1.2) *and* (1.6), *then*

$$\lim_{x\to 0} [K(|x|)]^{-1} U_{\alpha,\ell}f(x) = 0.$$

PROOF. We may assume that $0 < \delta < \alpha$. Since $\lim_{r\to 0} r^{\alpha-\delta}[\kappa_3(r)]^{-1} = 0$, we see by Lemma 4.1 that

$$\lim_{x\to 0} [\kappa_3(|x|)]^{-1} U_3(x) = 0.$$

In view of Corollaries 3.1, 3.2 and 3.3, we have

$$\lim_{x\to 0} [K(|x|)]^{-1} \{U_1(x) + U_2(x)\} = 0,$$

and hence

$$\lim_{x\to 0} [K(|x|)]^{-1} U_{\alpha,\ell}f(x) = 0.$$

Thus we complete the proof of Theorem 4.1.

REMARK 4.1. Let $\omega(r) = r^{\beta}$. If $n + \beta > 0$, then we see by Lemma 2.3 that

$$\limsup_{r\to 0} r^{\alpha-\delta}[\kappa_3(r)]^{-1} = 0$$

for $0 < \delta < (n + \beta)/p$.

REMARK 4.2. Let $\omega(r) = r^{\beta}$, where $-n < \beta \le \alpha p - n$. Let ℓ be the integer such that

$$\ell \le \alpha - (n + \beta)/p < \ell + 1.$$

Then we see with the aid of Remarks 3.3, 3.4 and 4.1 that

$$K(r) \sim [r^{n-\alpha p+\beta}\varphi(r^{-1})]^{-1/p} \qquad \text{when } \ell < \alpha - (n+\beta)/p < \ell + 1,\ n - \alpha p < 0,$$

$$K(r) \sim r^{-\beta/p}\left(\int_0^r [\varphi(t^{-1})]^{-p'/p} t^{-1}\,dt\right)^{1/p'}$$
$$\text{when } \ell < \alpha - (n+\beta)/p < \ell + 1,\ n - \alpha p = 0,$$

$$K(r) \sim r^{\ell}\left(\int_0^r [\varphi(t^{-1})]^{-p'/p} t^{-1}\,dt\right)^{1/p'}$$
$$\text{when } \ell = \alpha - (n+\beta)/p.$$

In all cases, if $K(1) < \infty$, then

$$\lim_{r \to 0} K(r) = 0.$$

REMARK 4.3. Let $\omega(r) = r^{\beta}$, where $-n < \beta \le \alpha p - n$. If $\alpha - (n + \beta)/p < \ell + 1$ and f satisfies (1.2), then the proof of Lemma 3.4 shows that (1.6) is fulfilled.

COROLLARY 4.1. *Let $\omega(r) = r^{\beta}$ with $-n < \beta \le \alpha p - n$. Let f be a nonnegative measurable function on R^n satisfying conditions* (1.1) *and* (1.2). *If $\ell \le \alpha - (n + \beta)/p < \ell + 1$ and $K(1) < \infty$, then there exists a polynomial P_{ℓ} of degree at most ℓ such that*

$$\lim_{x \to 0} [K(|x|)]^{-1}[U_{\alpha}f(x) - P_{\ell}(x)] = 0$$

with K as in Remark 4.2.

In fact, since $\kappa_2(1) < \infty$, (1.5) holds, and further (1.6) holds by Remark 4.3. Hence

$$U_{\alpha,\ell}f(x) = U_{\alpha}f(x) - \sum_{|\mu| \le \ell} \frac{x^{\mu}}{\mu!} \int_{R^n} [(D^{\mu}R_{\alpha})(-y)]f(y)\,dy.$$

With the aid of Remarks 3.3, 3.4, 4.1 and 4.2, Theorem 4.1 gives the present corollary.

Since $\lim_{r \to 0} r^{-\ell}K(r) = 0$, Corollary 4.1 implies that $U_{\alpha}f$ is ℓ times differentiable at the origin. On the other hand, Corollary 4.1 says that

$$U_\alpha f(x) - P_\ell(x) = o(K(|x|)) \qquad \text{as } x \to 0.$$

We next show that this holds locally uniformly in the following sense.

THEOREM 4.2. *Let $p = n/\alpha > 1$, and f be a nonnegative measurable function on R^n satisfying* (1.1) *and*

$$\int_{R^n} \Phi_p(f(y))dy < \infty. \tag{4.2}$$

If $\varphi^*(1) < \infty$, then

$$U_\alpha f(x) - U_\alpha f(z) = o(\varphi^*(|x - z|))$$

when $|x - z| \to 0$ and x, z are in a compact set in R^n.

PROOF. First nore that $\omega(r) = 1$ and $\ell = 0$ in this case, and hence

$$K(r) \sim \varphi^*(r)$$

because of Remark 4.2. Moreover, if $0 < \beta < \min\{1, \alpha\}$ and $2|x - z| < a < 1$, then Lemmas 3.5, 3.6 and 4.1 establish

$$|U_\alpha f(x) - U_\alpha f(z)| \leq M|x - z|G_a(x) + M|x - z|^\beta + MK(|x - z|)F_a(x),$$

where

$$G_a(x) = \int_{R^n - B(x,a)} |x - y|^{\alpha - n - 1} f(y)dy$$

and

$$F_a(x) = \left(\int_{B(x,a)} \Phi_p(f(y))dy\right)^{1/p}.$$

Since

$$\lim_{r \to 0} \sup_{x \in R^n} \int_{B(x,r)} |g(y)|dy = 0$$

for any integrable function g on R^n, for given $\varepsilon > 0$ there exists $a_0 > 0$ such that $F_{a_0}(x) < \varepsilon$ for all x. On the other hand, since $G_{a_0}(x)$ is continuous on R^n, it is bounded on a compact set. Hence, noting that $\lim_{r \to 0} r^\gamma [\varphi^*(r)]^{-1} = 0$ whenever $\gamma > 0$, for any compact set E in R^n we can find $\delta > 0$ so small that

$$|U_\alpha f(x) - U_\alpha f(z)| \leq \varepsilon \varphi^*(|x - z|)$$

whenever $x \in E$ and $|x - z| < \delta$. Thus the present theorem is obtained.

REMARK 4.4. Maz'ya proved Theorem 4.2 for Sobolev functions u for which (4.2) is satisfied with f replaced by $|\text{grad}\, u|$ (see [2, Theorem 5.4]).

REMARK 4.5. Theorem 4.2 can be extended to higher differences of order ℓ, in view of Corollary 4.1.

Here we discuss the best possibility of Corollary 4.1 (Theorem 4.2) as to the order of infinity in case $\alpha p = n$ and $\omega(r) = 1$.

PROPOSITION 4.1. *Assume* $\varphi^*(1) < \infty$. *Then, for any* $\varepsilon > 0$, *there exists a nonnegative measurable function* f *on* R^n *satisfying* (4.2) *with* $p = n/\alpha$ *such that* $U_\alpha f(0) < \infty$ *and*

$$\lim_{x \to 0} [K(|x|)]^{-\varepsilon-1} \{U_\alpha f(x) - U_\alpha f(0)\} = -\infty.$$

PROOF. Note that $K(r) \sim \varphi^*(r)$ in this case (cf. Remark 4.2). Let $0 < \varepsilon < p' - 1$ and $p' - 1 - \varepsilon < \delta < p' - 1$. We define

$$f(y) = [K(|y|)]^{-\delta} |y|^{-\alpha} [\varphi(|y|^{-1})]^{-p'/p} \qquad \text{for } y \in B = B(0, 1).$$

In view of Lemma 2.3, for $\gamma > 0$,

$$s^\gamma K(s)^{-1} < M t^\gamma K(t)^{-1} \qquad \text{whenever } 0 < s < t, \tag{4.3}$$

so that we see that

$$\varphi(f(y)) = \varphi([K(|y|)]^{-\delta} |y|^{-\alpha} [\varphi(|y|^{-1})]^{-p'/p}) \le \varphi(M|y|^{-(\gamma\delta+\alpha)}) \le M\varphi(|y|^{-1})$$

for $y \in B$. Consequently we establish

$$\begin{aligned}
\int_B \Phi_p(f(y))dy &= \int_B ([K(|y|)]^{-\delta} |y|^{-\alpha} [\varphi(|y|^{-1})]^{-p'/p})^p \\
&\quad \times \varphi([K(|y|)]^{-\delta} |y|^{-\alpha} [\varphi(|y|^{-1})]^{-p'/p}) dy \\
&\le M \int_B [K(|y|)]^{-\delta p} |y|^{-\alpha p} [\varphi(|y|^{-1})]^{-p'+1} dy \\
&\le M \int_B [\varphi^*(|y|)]^{-\delta p} |y|^{-n} [\varphi(|y|^{-1})]^{-p'/p} dy \\
&= M \int_0^1 \{[\varphi^*(r)]^{p'}\}^{-\delta p/p'} \{[\varphi^*(r)]^{p'}\}' dr \\
&= M \int_0^{t^*} t^{-\delta p/p'} dt < \infty,
\end{aligned}$$

with $t^* = [\varphi^*(1)]^{p'}$. Thus it follows that f satisfies (4.2). Similarly, we have

$$\begin{aligned} U_\alpha f(0) &= \int_B |y|^{\alpha-n} f(y)dy \\ &= \int_B |y|^{\alpha-n}[K(|y|)]^{-\delta}|y|^{-\alpha}[\varphi(|y|^{-1})]^{-p'/p}dy \\ &\le \int_B [\varphi^*(|y|)]^{-\delta}|y|^{-n}[\varphi(|y|^{-1})]^{-p'/p}dy \\ &= M\int_0^1 \{[\varphi^*(t)]^{p'}\}^{-\delta/p'}\{[\varphi^*(t)]^{p'}\}' dt \\ &= M\int_0^{t^*} t^{-\delta/p'}dt < \infty. \end{aligned}$$

We write

$$U_2(x) = -\int_{B(0,|x|/2)} |y|^{\alpha-n}f(y)dy + \int_{B(0,|x|/2)} |x-y|^{\alpha-n}f(y)dy = -I + J.$$

Letting $r^* = [\varphi^*(|x|/2)]^{p'}$, we have as above

$$I \ge M\int_0^{r^*} t^{-\delta/p'}dt = M[\varphi^*(|x|/2)]^{-\delta+p'} \ge M[K(|x|)]^{-\delta+p'},$$

so that

$$\lim_{x\to 0}[K(|x|)]^{-\varepsilon-1}I = \infty. \tag{4.4}$$

On the other hand, letting $r = |x| < 1$, we have

$$\begin{aligned} J &= \int_{B(0,|x|/2)} |x-y|^{\alpha-n}[K(|y|)]^{-\delta}|y|^{-\alpha}[\varphi(|y|^{-1})]^{-p'/p}dy \\ &\le M|x|^{\alpha-n}\int_{B(0,|x|/2)} [K(|y|)]^{-\delta}|y|^{-\alpha}[\varphi(|y|^{-1})]^{-p'/p}dy \\ &= Mr^{\alpha-n}\int_0^{r/2}[K(t)]^{-\delta}t^{-\alpha+n}[\varphi(t^{-1})]^{-p'/p}t^{-1}dt \\ &\le Mr^{\alpha-n}\int_0^{r}[K(t)]^{-\delta}t^{-\alpha+n}[\varphi(t^{-1})]^{-p'/p}t^{-1}dt \\ &\le Mr^{\alpha-n}[K(r)]^{-\delta}r^{-\alpha+n}[\varphi(r^{-1})]^{-p'/p} \\ &= M[K(r)]^{-\delta}[\varphi(r^{-1})]^{-p'/p}. \end{aligned}$$

In view of Lemma 2.2, we have

$$[K(r)]^{p'} \geq \int_{r^2}^{r} [\varphi(t^{-1})]^{-p'/p} t^{-1}\, dt \geq [\varphi(r^{-2})]^{-p'/p} \int_{r^2}^{r} t^{-1}\, dt$$
$$\geq M [\varphi(r^{-1})]^{-p'/p} \log \frac{1}{r} \qquad (M > 0),$$

so that

$$J \leq M [K(|x|)]^{-\delta+p'} [\log (1/|x|)]^{-1}.$$

Moreover, by Lemma 3.2, we have

$$|U_3(x)| \leq M \int_{B(0,2|x|)-B(0,|x|/2)} |x-y|^{\alpha-n} f(y)\, dy$$
$$= M \int_{B(0,2|x|)-B(0,|x|/2)} |x-y|^{\alpha-n} [K(|y|)]^{-\delta} |y|^{-\alpha} [\varphi(|y|^{-1})]^{-p'/p}\, dy$$
$$\leq M [K(|x|)]^{-\delta} |x|^{-\alpha} [\varphi(|x|^{-1})]^{-p'/p} \int_{B(0,2|x|)-B(0,|x|/2)} |x-y|^{\alpha-n}\, dy$$
$$\leq M [K(|x|)]^{-\delta} [\varphi(|x|^{-1})]^{-p'/p}$$
$$\leq M [K(|x|)]^{-\delta+p'} [\log (1/|x|)]^{-1}.$$

Similarly, by Lemmas 3.3 and 2.4, we have

$$|U_1(x)| \leq M |x| \int_{R^n - B(0,2|x|)} |y|^{\alpha-n-1} f(y)\, dy$$
$$= M |x| \int_{B(0,1)-B(0,2|x|)} |y|^{\alpha-n-1} [K(|y|)]^{-\delta} |y|^{-\alpha} [\varphi(|y|^{-1})]^{-p'/p}\, dy$$
$$= M |x| \int_{2|x|}^{1} [K(t)]^{-\delta} [\varphi(t^{-1})]^{-p'/p} t^{-2}\, dt$$
$$\leq M |x| [K(|x|)]^{-\delta} \int_{2|x|}^{1} [\varphi(t^{-1})]^{-p'/p} t^{-2}\, dt$$
$$\leq M [K(|x|)]^{-\delta} [\varphi(|x|^{-1})]^{-p'/p}$$
$$\leq M [K(|x|)]^{-\delta+p'} [\log (1/|x|)]^{-1}.$$

Thus it follows that

$$U_\alpha f(x) - U_\alpha f(0) \leq -[K(|x|)]^{-\delta+p'} (1 - M [\log (1/|x|)]^{-1}),$$

which together with (4.4) yields

$$\lim_{x \to 0} [K(|x|)]^{-\varepsilon-1} \{U_\alpha f(x) - U_\alpha f(0)\} = -\infty.$$

Thus f has all the required properties.

5. Fine limits

For a set $E \subset R^n$ and an open set $G \subset R^n$, we define

$$C_{\alpha,\Phi_p}(E; G) = \inf_g \int_G \Phi_p(g(y))\,dy,$$

where the infimum is taken over all nonnegative measurable functions g on R^n such that g vanishes outside G and $U_\alpha g(x) \geq 1$ for every $x \in E$ (cf. Meyers [3]).

In what follows, we collect elementary properties of this capacity (cf. [8, Lemma 2.2]).

LEMMA 5.1. *$C_{\alpha,\Phi_p}(\cdot\,; G)$ is countably subadditive.*

LEMMA 5.2. *Let G and G' be bounded open sets in R^n. If F is a compact subset of $G \cap G'$, then there exists $M > 0$ such that*

$$C_{\alpha,\Phi_p}(E; G) \leq M C_{\alpha,\Phi_p}(E; G') \qquad \textit{for any } E \subset F.$$

LEMMA 5.3. *Let G and G' be bounded open sets in R^n. If $C_{\alpha,\Phi_p}(E; G) = 0$, then $C_{\alpha,\Phi_p}(E \cap G'; G') = 0$.*

LEMMA 5.4. *Let G and G' be bounded open sets in R^n. If $C_{\alpha,\Phi_p}(E; G) = 0$, $E \subset G$, then, for any positive nonincreasing function ω on $(0, \infty)$, there exists a nonnegative measurable function f on G such that $U_\alpha f \not\equiv \infty$, $U_\alpha f = \infty$ on E and $\int_G \Phi_p(f(y))\omega(\rho(y))\,dy < \infty$, where $\rho(y)$ denotes the distance of y from the boundary ∂G.*

For a nonnegative function χ on the interval $(0, 1]$, consider the generalized doubling condition:

$$(\chi) \qquad \chi(r) \leq M\chi(s) \qquad \text{whenever } 0 < r/2 \leq s \leq 2r \leq 1.$$

For monotone functions, (χ) is just the doubling condition as mentioned before. For $r > 0$ and $E \subset R^n$, set

$$rE = \{rx : x \in E\}.$$

LEMMA 5.5 (cf. [8, Lemma 2.3]). *Let χ_i, $i = 1, 2, 3$, be positive functions on* $(0, 1]$ satisfying condition (χ). *If f is a nonnegative function satisfying*

$$\int_{B(0,1)} \Phi_p(\chi_1(|y|)[\chi_2(|y|)]^\alpha f(y))[\chi_2(|y|)]^{-n}\chi_3(|y|)\,dy < \infty, \tag{5.1}$$

then there exists a set $E \subset R^n$ *such that*

(i) $\lim_{x \to 0, x \in R^n - E} \chi_1(|x|)U(x) = 0;$

(ii) $\sum_{j=1}^{\infty} \chi_3(2^{-j}) C_{\alpha,\Phi_p}([\chi_2(2^{-j})]^{-1}E_j; [\chi_2(2^{-j})]^{-1}B_j) < \infty,$

where $E_j = \{x \in E : 2^{-j} \leq |x| < 2^{-j+1}\}$, $B_j = \{x \in R^n : 2^{-j-1} < |x| < 2^{-j+2}\}$ *and*

$$U(x) = \int_{B(0,2|x|) - B(0,|x|/2)} |x - y|^{\alpha-n} f(y)\,dy.$$

PROOF. For a sequence $\{a_j\}$ of positive numbers, consider

$$E_j = \{x \in R^n : 2^{-j} \leq |x| < 2^{-j+1},\ U(x) \geq a_j^{-1}[\chi_1(|x|)]^{-1}\}$$

and

$$E = \bigcup_{j=1}^{\infty} E_j.$$

If $x \in E_j = \{x \in E : 2^{-j} \leq |x| < 2^{-j+1}\}$, then

$$\chi_1(|x|)U(x) \leq \chi_1(|x|) \int_{B_j} |x - y|^{\alpha-n} f(y)\,dy$$
$$\leq M t_j \int_{r_j B_j} |r_j x - z|^{\alpha-n} f(r_j^{-1} z)\,dz,$$

where $r_j = [\chi_2(2^{-j})]^{-1}$ and $t_j = [\chi_1(2^{-j})] r_j^{-\alpha}$. Hence it follows from the definition of C_{α,Φ_p} that

$$C_{\alpha,\Phi_p}(r_j E_j; r_j B_j) \leq \int_{r_j B_j} \Phi_p(M a_j t_j f(r_j^{-1} z))\,dz$$
$$= \int_{B_j} \Phi_p(M a_j t_j f(y)) r_j^n\,dy.$$

Now it suffices to choose $\{a_j\}$ so that $\lim_{j \to \infty} a_j = \infty$ but

$$\sum_j \chi_3(2^{-j}) \int_{B_j} \Phi_p(M a_j t_j f(y)) r_j^n\,dy < \infty$$

(see the proof of Lemma 2.3 in [8]).

THEOREM 5.1. *Set* $\kappa(r)=r^{\ell+1}\kappa_1(r)+r^{\ell}\kappa_2(r)$. *Assume that* $\ell<\alpha$, $\lim_{r\to 0}\kappa(r)=0$ *and*

$$\begin{aligned} &\kappa_1(0)=\infty && \text{in case } \alpha-\ell-1>0,\\ &\lim_{r\to 0} r^{\alpha-\delta}[r^{\ell+1}\kappa_1(r)]^{-1}=0 && \text{for some } \delta>0 \text{ in case } \alpha-\ell-1\le 0,\\ &\lim_{r\to 0} r^{\alpha-\delta}[r^{\ell}\kappa_2(r)]^{-1}=0 && \text{for some } \delta \text{ such that } 0<\delta<\alpha-\ell. \end{aligned}$$

Further, let $\kappa_4(r)=[r^{n-\alpha p}\eta(r)]^{-1/p}$. *If* f *is a nonnegative measurable function on* R^n *satisfying* (1.2), (1.6) *and*

$$\int_{B(0,1)} \Phi_p([\kappa_4(|y|)]^{-1}f(y))[\kappa_4(|y|)]^p\omega(|y|)\,dy<\infty, \tag{5.2}$$

then there exists a set $E\subset R^n$ *such that*

(i) $\lim_{x\to 0,x\in R^n-E}[\kappa(|x|)]^{-1}U_{\alpha,\ell}f(x)=0$;

(ii) $\sum_{j=1}^{\infty} 2^{j(n-\alpha p)}[\varphi(2^j)]^{-1}C_{\alpha,\Phi_p}(E_j;B_j)<\infty$.

REMARK 5.1. In view of [8, Lemma 7.3], we see that

$$C_{\alpha,\Phi_p}(A_j;B_j)\sim 2^{-j(n-\alpha p)}\varphi(2^j),$$

where $A_j=B(0,2^{-j+1})-B(0,2^{-j})$.

PROOF OF THEOREM 5.1. From Corollaries 3.1, 3.2 and 3.3, it follows that

$$\lim_{x\to 0}[\kappa(|x|)]^{-1}U_1(x)=0,$$
$$\lim_{x\to 0}[\kappa(|x|)]^{-1}U_2(x)=0.$$

In view of Lemma 3.2,

$$|U_3(x)|\le M\int_{B(0,2|x|)-B(0,|x|/2)}|x-y|^{\alpha-n}f(y)\,dy=MU(x).$$

Now let

$$\chi_1(r)=[\kappa_4(r)]^{-1},\qquad \chi_2(r)=1$$

and

$$\chi_3(r)=[\kappa_4(r)]^p\omega(r)=[r^{n-\alpha p}\varphi(r^{-1})]^{-1}.$$

We then apply Lemma 5.5 to find a set E satisfying (ii) and

$$\lim_{x \to 0, x \in R^n - E} [\kappa_4(|x|)]^{-1} U_3(x) = 0.$$

Since $[\kappa(r)]^{-1} \leq M[\kappa_4(r)]^{-1}$ by Remark 3.1 or 3.2, we obtain the required fine limit result.

LEMMA 5.6. *If*

$$\int_{B(0,1)} \Phi_p([\kappa_4(|y|)]^{-\gamma})[\kappa_4(|y|)]^p \omega(|y|)\,dy < \infty \tag{5.3}$$

for some $\gamma > 1$, *then* (5.2) *holds for any nonnegative measurable function* f *on* R^n *satisfying* (1.2).

PROOF. To show this fact, consider the sets

$$E_1 = \{y \in B(0, 1): [\kappa_4(|y|)]^{-1} f(y) \geq f(y)^{1+\delta}\},$$
$$E_2 = \{y \in B(0, 1): [\kappa_4(|y|)]^{-1} f(y) < f(y)^{1+\delta}\}$$

for $\delta > 0$ such that $\gamma = 1 + 1/\delta$. Then

$$\int_{E_1} \Phi_p([\kappa_4(|y|)]^{-1} f(y))[\kappa_4(|y|)]^p \omega(|y|)\,dy$$
$$\leq \int_{E_1} \Phi_p([\kappa_4(|y|)]^{-\gamma})[\kappa_4(|y|)]^p \omega(|y|)\,dy < \infty.$$

On the other hand, we have

$$\int_{E_2} \Phi_p([\kappa_4(|y|)]^{-1} f(y))[\kappa_4(|y|)]^p \omega(|y|)\,dy$$
$$= \int_{E_2} \varphi([\kappa_4(|y|)]^{-1} f(y)) f(y)^p \omega(|y|)\,dy$$
$$\leq \int_{E_2} \varphi(f(y)^{1+\delta}) f(y)^p \omega(|y|)\,dy$$
$$\leq M \int_{B(0,1)} \Phi_p(f(y))\omega(|y|)\,dy < \infty.$$

LEMMA 5.7. *Let* $\omega(r) = r^\beta$. *If* $-n < \beta \leq \alpha p - n$, *then* (5.3) *holds for some* $\gamma > 1$.

PROOF. We see from Lemma 2.3 that

$$M^{-1} r^{-(n-\alpha p+\beta)/p} r^\delta \leq \kappa_4(r) \leq M r^{-(n-\alpha p+\beta)/p}, \qquad 0 < r < 1,$$

for $\delta > 0$. Hence we find that

$$\Phi_p([\kappa_4(r)]^{-\gamma}) \le M r^{\gamma(n-\alpha p+\beta)} r^{-\delta'}$$

for $\gamma > 1$ and $\delta' > 0$. Consequently it follows that

$$\int_{B(0,1)} \Phi_p([\kappa_4(|y|)]^{-\gamma})[\kappa_4(|y|)]^p \omega(|y|)\,dy$$

$$\le M \int_{B(0,1)} |y|^{(\gamma-1)(n-\alpha p+\beta)} |y|^{-\delta'} |y|^{\beta}\,dy$$

$$= M \int_0^1 r^{(\gamma-1)(n-\alpha p+\beta)-\delta'+\beta+n} r^{-1}\,dr < \infty$$

for some $\gamma > 1$ and $\delta' > 0$, because $\lim_{\gamma\to 1,\delta'\to 0} \{(\gamma-1)(n-\alpha p+\beta)-\delta'+\beta+n\} = \beta + n > 0$. Thus the present lemma is obtained.

COROLLARY 5.1. *Let f be a nonnegative measurable function on R^n satisfying* (1.1) *and*

$$\int_{R^n} \Phi_p(f(y))|y|^{\beta}\,dy < \infty$$

for $-n < \beta \le \alpha p - n$. If ℓ is the nonnegative integer such that $\ell \le \alpha - (n+\beta)/p < \ell + 1$ and $\kappa(1) < \infty$, then there exist a set $E \subset R^n$ and a polynomial P_ℓ of degree at most ℓ for which (ii) *of Theorem* 5.1 *holds and*

$$\lim_{x\to 0, x\in R^n - E} [\kappa(|x|)]^{-1}[U_\alpha f(x) - P_\ell(x)] = 0.$$

REMARK 5.2. Meyers [4] dealt with L^q-mean limits for Taylor expansion of Bessel potentials of L^p-functions. In this connection, it will be expected that

$$\lim_{r\to 0} [\kappa(r)]^{-1}\left(r^{-n}\int_{B(0,r)} |U_\alpha f(x) - P_\ell(x)|^q\,dx\right)^{1/q} = 0$$

holds in our case.

The following is a special case of Lemma 5.5.

LEMMA 5.8. *Let χ be a positive function on* (0, 1] *satisfying* (χ). *If f is a nonnegative function satisfying*

$$\int_{B(0,1)} \Phi_p(\chi(|y|)|y|^{\alpha} f(y))|y|^{-n}\,dy < \infty, \tag{5.4}$$

then there exists a set $E \subset R^n$ such that

(i) $\lim_{x\to 0, x\in R^n - E} \chi(|x|)U(x) = 0;$

(ii′) $\sum_{j=1}^{\infty} C_{\alpha,\Phi_p}(2^j E_j;\, B_0) < \infty$.

With the aid of Lemma 5.8, we can establish the following result which is useful for the study of radial limits.

THEOREM 5.2. *Let κ be as in Theorem 5.1, and χ be a positive function on* $(0, 1]$ *satisfying condition* (χ) *and*

$$\chi(r) \le M[\kappa(r)]^{-1}. \tag{5.5}$$

If f is a nonnegative measurable function on R^n satisfying (1.2), (1.6) *and* (5.4), *then there exists a set $E \subset R^n$ for which* (ii′) *of Lemma* 5.8 *is satisfied and*

$$\lim_{x\to 0,\, x\in R^n - E} \chi(|x|)\, U_{\alpha,\ell} f(x) = 0.$$

6. Radial limits

Before discussing the existence of radial limits of Riesz potentials, we prepare two lemmas concerning the capacity C_{α,Φ_p}.

A mapping $T\colon G \to G'$ is said to be bi-Lipschitzian if there exists $A > 1$ such that

$$A^{-1}|x - y| \le |Tx - Ty| \le A|x - y| \qquad \text{for all } x, y \in G.$$

The following result can be proved easily by the definition of C_{α,Φ_p}.

LEMMA 6.1. *Let T be a bi-Lipschitzian mapping from G onto TG. Then*

$$C_{\alpha,\Phi_p}(TE;\, TG) \le M C_{\alpha,\Phi_p}(E;\, G) \qquad \textit{for any } E \subset G,$$

where M is a positive constant which may depend on A (the Lipschitz constant of T).

For a set $E \subset R^n$, we denote by $\tilde{E}$ the set of all $\xi \in \partial B(0, 1)$ such that $r\xi \in E$ for some $r > 0$. By using Lemma 5.8 and applying the methods in the proof of Lemma 5 in [6], we can prove the following lemma.

LEMMA 6.2. *There exists a positive constant M such that*

$$C_{\alpha,\Phi_p}(\tilde{E};\, B(0, 4)) \le M C_{\alpha,\Phi_p}(E;\, B(0, 4))$$

whenever $E \subset B(0, 2) - B(0, 1)$.

LEMMA 6.3. *Let χ be a positive function on* $(0, 1]$ *satisfying* (χ). *If f is a non-negative function satisfying* (5.4), *then there exists a set $E^* \subset \partial B(0, 1)$ such that $C_{\alpha,\Phi_p}(E^*;\, B(0, 2)) = 0$ and*

$$\lim_{r\to 0} \chi(r) U(r\xi) = 0 \qquad \textit{for any } \xi \in \partial B(0, 1) - E^*,$$

where U is as in Lemma 5.5.

PROOF. Take a set $E \subset R^n$ as in Lemma 5.8, and set

$$E^* = \bigcap_{k=1}^{\infty} \Big(\bigcup_{j=k}^{\infty} \tilde{E}_j\Big).$$

Then we have by the countable subadditivity (Lemma 5.1) and Lemma 6.2

$$C_{\alpha,\Phi_p}(E^*; B(0, 2)) = 0.$$

If $\xi \in \partial B(0, 1) - E^*$, then there exists k such that $\xi \notin \bigcup_{j=k}^{\infty} \tilde{E}_j$, so that $r\xi \notin \bigcup_{j=k}^{\infty} E_j$ for $0 < r < 2^{-k+1}$. Hence we see that

$$\lim_{r\to 0} \chi(r) U(r\xi) = 0.$$

Thus the proof of Lemma 6.3 is completed.

THEOREM 6.1. *If κ, χ and f are as in Theorem 5.2, then there exists a set $E^* \subset \partial B(0, 1)$ such that*

$$C_{\alpha,\Phi_p}(E^*; B(0, 2)) = 0$$

and

$$\lim_{r\to 0} \chi(r) U_{\alpha,\ell} f(r\xi) = 0 \qquad \textit{for every } \xi \in \partial B(0, 1) - E^*.$$

PROOF. As in the proof of Theorem 5.1, we see that

$$\lim_{x\to 0} [\kappa(|x|)]^{-1} \{U_1(x) + U_2(x)\} = 0.$$

On the other hand, in view of Lemma 6.3, we can find a set $E^* \subset \partial B(0, 1)$ such that $C_{\alpha,\Phi_p}(E^*; B(0, 2)) = 0$ and

$$\lim_{r\to 0} \chi(r) U_3(r\xi) = 0 \qquad \text{for any } \xi \in \partial B(0, 1) - E^*.$$

Hence it follows from (5.5) that

$$\lim_{r\to 0} \chi(r) U_{\alpha,\ell} f(r\xi) = 0 \qquad \text{for any } \xi \in \partial B(0, 1) - E^*.$$

Thus the proof of Theorem 6.1 is completed.

LEMMA 6.4. *If $-n < \beta \le \alpha p - n$, then (1.2) with $\omega(r) = r^\beta$ implies (5.4) with $\chi(r) = r^{(n-\alpha p+\beta)/p}$.*

PROOF. First note that

$$\int_{B(0,1)} \Phi_p(|y|^\alpha \chi(|y|)f(y))|y|^{-n}\,dy \le \int_{B(0,1)} \Phi_p(\chi(|y|)f(y))|y|^{\alpha p-n}\,dy.$$

We show that the second integral is finite. For this purpose, consider the sets

$$E_1 = \{y \in B(0,1) : \chi(|y|)f(y) \ge f(y)^{1+\delta}\},$$
$$E_2 = \{y \in B(0,1) : \chi(|y|)f(y) < f(y)^{1+\delta}\}$$

for $\delta > 0$. Then we see that

$$\int_{E_1} \Phi_p(\chi(|y|)f(y))|y|^{\alpha p-n}\,dy \le \int_{E_1} \Phi_p([\chi(|y|)]^{1+1/\delta})|y|^{\alpha p-n}\,dy < \infty,$$

since $\lim_{\delta\to\infty} \{(n-\alpha p+\beta)(1+1/\delta)+(\alpha p-n)+n\} = \beta+n>0$. On the other hand, we have

$$\begin{aligned}\int_{E_2} \Phi_p(\chi(|y|)f(y))|y|^{\alpha p-n}\,dy &= \int_{E_2} \varphi(\chi(|y|)f(y))f(y)^p|y|^\beta\,dy\\ &\le \int_{E_2} \varphi(f(y)^{1+\delta})f(y)^p|y|^\beta\,dy\\ &\le M\int_{B(0,1)} \Phi_p(f(y))|y|^\beta\,dy < \infty,\end{aligned}$$

so that Lemma 6.4 is obtained.

COROLLARY 6.1. *Let f be a nonnegative measurable function on R^n satisfying* (1.1) *and*

$$\int_{R^n} \Phi_p(f(y))|y|^\beta\,dy < \infty$$

for $-n < \beta \le \alpha p - n$. If ℓ is the nonnegative integer such that $\ell \le \alpha - (n+\beta)/p < \ell + 1$ and $\kappa(1) < \infty$, then there exist a set $E^* \subset \partial B(0,1)$ and a polynomial P_ℓ of degree at most ℓ such that $C_{\alpha,\Phi_p}(E^*; B(0,2)) = 0$ and

$$\lim_{r\to 0} r^{(n-\alpha p+\beta)/p}[U_\alpha f(r\xi) - P_\ell(r\xi)] = 0 \qquad \textit{for any } \xi \in \partial B(0,1) - E^*.$$

REMARK 6.1. We show the sharpness of Lemma 6.3 as to the order $\chi(r)$. In fact, for a nonincreasing positive function $a(r)$ on $(0,\infty)$ such that $\lim_{r\to 0} a(r) = \infty$, we find a nonnegative function f satisfying (5.4) such that

$$\limsup_{r\to 0} a(r)\chi(r)U(rz) = \infty \qquad \text{for all } z \in \partial B(0,1).$$

To show this, let $A_j = B(0, 2r_j) - B(0, r_j)$, $2r_{j+1} < r_j$ and define

$$f(y) = \begin{cases} a(2r_j)^{-1/p} r_j^{-\alpha} [\chi(r_j)]^{-1} & y \in A_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then we see that

$$a(|x|)\chi(|x|)U(x) \geq Ma(2r_j)^{1/p'}, \qquad x \in A_j$$

and

$$\int_{B(0,1)} \Phi_p(|y|^\alpha \chi(|y|) f(y)) |y|^{-n} dy \leq M \sum_j \Phi_p(a(2r_j)^{-1/p}).$$

Now it suffices to choose $\{r_j\}$ so that the last sum is convergent.

REMARK 6.2. If $\lim_{r \to 0} r^\alpha \chi(r) = \infty$, then (5.4) implies the following condition of type (1.2):

$$\int \Phi_p(f(y)) [\chi(|y|)]^p |y|^{\alpha p - n} dy < \infty. \tag{6.1}$$

If in addition $\lim_{r \to 0} \varphi(r) = 0$, then we can find a nonnegative measurable function f satisfying (6.1) and

$$\limsup_{r \to 0} \chi(r) U(rz) = \infty \qquad \text{for any } z \in \partial B(0, 1). \tag{6.2}$$

For this purpose, take a sequence $\{r_j\}$ of positive numbers for which $2r_{j+1} < r_j$ and

$$\sum_{j=1}^{\infty} \varphi(b_j) < \infty,$$

where $b_j = [r_j^\alpha \chi(r_j)]^{-1}$. Next find a sequence $\{a_j\}$ of positive numbers such that $\lim_{j \to \infty} a_j = \infty$ and

$$\sum_{j=1}^{\infty} a_j^p \varphi(a_j b_j) < \infty.$$

Now consider the function

$$f(y) = \begin{cases} a_j b_j & \text{for } y \in B(0, 2r_j) - B(0, r_j), \\ 0 & \text{otherwise.} \end{cases}$$

Then we note that

$$\int \Phi_p(f(y)) [\chi(|y|)]^p |y|^{\alpha p - n} dy \leq M \sum_{j=1}^{\infty} a_j^p \varphi(a_j b_j) < \infty.$$

Moreover,

$$\chi(|x|)U(x) \geq M\chi(r_j)a_j b_j r_j^{\alpha} = Ma_j$$

for $x \in B(0, 2r_j) - B(0, r_j)$, from which (6.2) follows readily.

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