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Oscillations of half-linear second order differential equations

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Abstract. Some oscillation criteria are given for the half-linear second order differential equation

$$[\Phi(u'(t))]' + c(t)\Phi(u(t)) = 0,$$

where $\Phi: \mathbb{R} \to \mathbb{R}$ is defined by $\Phi(s) = |s|^{p-2}s$ with a fixed number $p > 1$ and $c \in C([0, \infty), \mathbb{R})$. These results improve Willett's results.

1. Introduction

Define $\Phi: \mathbb{R} \to \mathbb{R}$ by $\Phi(s) = |s|^{p-2}s$, where $p > 1$ is a given number. Consider the half-linear second order differential equation

$$[\Phi(u'(t))]' + c(t)\Phi(u(t)) = 0, \tag{E}$$

where $c(t)$ is a continuous function on $[0, \infty)$. We observe that if $p = 2$, then equation (E) reduces to the linear equation

$$u''(t) + c(t)u(t) = 0. \tag{E_1}$$

Let u be a positive solution of (E). If $u' > 0$ or $u' < 0$, then (E) reduces to the following Euler-Lagrange equations:

$$\frac{d}{dt}[u'(t)]^{p-1} + c(t)u^{p-1}(t) = 0$$

or

$$\frac{d}{dt}[-u'(t)]^{p-1} - c(t)u^{p-1}(t) = 0,$$

respectively.

By a solution of (E) we mean a function $u \in C^1[0, \infty)$ such that $\Phi(u') \in C^1[0, \infty)$, satisfying equation (E). Elbert [1] established the existence, uniqueness and extension to $[0, \infty)$ of solutions to the initial value problem for (E). We say that a nontrivial solution u of (E) is oscillatory if for any $N > 0$ there exists $t > N$ such that $u(t) = 0$, otherwise, it is nonoscillatory.

From the uniqueness of the trivial solution of (E), the zeros of any nontrivial solution of (E) is isolated.

There is a striking similarity in the oscillatory property between the half-linear equation (E) and the corresponding linear equation (E_1). For example, Sturm's comparison and separation theorems for (E_1) extend in a natural way to (E), see Elbert [1, 2], Li and Yeh [6, 7] and Mirzov [8, 9]. Thus we know that all the nontrivial solutions of (E) are oscillatory or nonoscillatory. In the former case we say that (E) is oscillatory and in the latter we say that (E) is nonoscillatory.

In [12], Willett obtained some oscillation criteria for (E_1) by using weighted average. The purpose of this paper is to generalize Willett's results [12] to equation (E).

For more recent results, we refer to Kusano and Yoshida [3], Kusano, Naito and Ogata [4], Kusano and Naito [5], Li and Yeh [7], and Pino and Manasevich [10].

2. Preliminary lemmas

Let $(1/p) + (1/q) = 1$, and let $\mathfrak{I}$ be the set of all nonnegative locally integrable functions f on $[0, \infty)$ satisfying the condition

$$\limsup_{t\to\infty} \left\{\int^t f(s)ds\right\}^{q-1-k} \{F_k(\infty) - F_k(t)\} > 0 \qquad \text{for some } k \in [0, q-1), \tag{2.1}$$

where

$$F_k(t) = \int^t f(t) \frac{(\int^s f(\xi)d\xi)^k}{(\int^s f^p(\xi)d\xi)^{q-1}} ds.$$

If $F_k(\infty) = \infty$ in (2.1), then $f \in \mathfrak{I}$. Let $\mathfrak{I}_0$ be the set of all nonnegative locally integrable functions f on $[0, \infty)$ satisfying

$$\lim_{t\to\infty} \frac{\int^t f^p(s)ds}{(\int^t f(s)ds)^p} = 0. \tag{2.2}$$

In order that either (2.1) or (2.2) can be satisfied by a nonnegative function f, it is necessary that

$$\int^\infty f(s)ds = \infty. \tag{2.3}$$

On the other hand, every bounded nonnegative locally integrable function f satisfying (2.3) belongs to $\mathfrak{I}_0$, and $\mathfrak{I}_0 \subset \mathfrak{I}$. Since all nonnegative polynomials are in $\mathfrak{I}_0$, $\mathfrak{I}_0$ contains some unbounded functions.

Members of the classes $\mathfrak{J}$ and $\mathfrak{J}_0$ will be called weight functions.

If (E) is nonoscillatory and $u(t)$ is a nontrivial solution of (E), then there exists a number $T \geq 0$ such that $u(t)$ has no zero on $[T, \infty)$, and hence

$$v(t) = \frac{\Phi(u'(t))}{\Phi(u(t))}$$

satisfies the generalized Riccati equation

$$v'(t) + c(t) + (p-1)|v(t)|^q = 0 \quad \text{on } [T, \infty). \tag{2.4}$$

Conversely, if there exists a function v satisfying (2.4), then

$$u(t) = \exp\left(\int_T^t \Phi^{-1}(v(s))ds\right)$$

satisfies (E) on $[T, \infty)$. It follows from Sturm's separation theorem [1] that (E) is nonoscillatory. Therefore, we obtain the following

THEOREM 2.1. *Equation* (E) *is nonoscillatory if and only if there exist a number* $T \geq 0$ *and a function* $v \in C^1$ *satisfying* (2.4) *on* $T, \infty)$.

Clearly, (2.4) is equivalent to the integral equation

$$v(t) = v(s) - (p-1)\int_s^t |v(\tau)|^q d\tau - \int_s^t c(\tau)d\tau \qquad \text{for } t \geq s \geq T. \tag{2.5}$$

For $f \in \mathfrak{J}$, define

$$A_f(s, t) = \frac{\int_s^t f(\tau)\int_s^\tau c(\mu)d\mu d\tau}{\int_s^t f(\tau)d\tau}.$$

LEMMA 2.2. *Assume that* $v(t)$ *satisfies* (2.4) *on* $[T, \infty)$ *for some* $T > 0$. *If there exists* $f \in \mathfrak{J}$ *such that*

$$\liminf_{t \to \infty} A_f(\cdot, t) > -\infty, \tag{2.6}$$

then $\int^\infty |v(s)|^q ds < \infty$.

PROOF. Let $A(s, t) = A_f(s, t)$ and assume that

$$\int^\infty |v(s)|^q ds = \infty. \tag{2.7}$$

Multiplying (2.5) by $f(t)$ and integrating it from ξ to t, we obtain

$$\int_\xi^t f(s)v(s)ds \tag{2.8}$$

$$= v(\xi)\int_\xi^t f(s)ds - \int_\xi^t f(s)\int_\xi^s c(\tau)d\tau ds - (p-1)\int_\xi^t f(s)\int_\xi^s |v(\tau)|^q d\tau ds$$

$$= v(\xi)\int_\xi^t f(s)ds - A(\xi, t)\int_\xi^t f(s)ds - (p-1)\int_\xi^t f(s)\int_\xi^s |v(\tau)|^q d\tau ds$$

$$= [v(\xi) - A(\xi, t)]\int_\xi^t f(s)ds - (p-1)\int_\xi^t f(s)\int_\xi^s |v(\tau)|^q d\tau ds,$$

where $t \geq \xi \geq T$. From (2.5), we have

$$v(\xi) = v(T) - \int_T^\xi c(s)ds - (p-1)\int_T^\xi |v(s)|^q ds.$$

Since $f \in \mathfrak{J}$, (2.3) holds. This implies that

$$A(\xi, t) = \frac{\int_T^t f(s)ds}{\int_\xi^t f(s)ds} A(T, t) - \int_T^\xi c(s)ds - \frac{\int_T^\xi f(s)\int_T^s c(\tau)d\tau ds}{\int_\xi^t f(s)ds}$$
$$= \frac{\int_T^t f(s)ds}{\int_\xi^t f(s)ds} A(T, t) - \int_T^\xi c(s)ds + o(1) \qquad \text{as } t \to \infty.$$

Thus,

$$v(\xi) - A(\xi, t) = v(T) - \frac{\int_T^t f(s)ds}{\int_\xi^t f(s)ds} A(T, t) - (p-1)\int_T^\xi |v(s)|^q ds + o(1) \qquad \text{as } t \to \infty. \tag{2.9}$$

Since $f \in \mathfrak{J}$, there exists a positive number $\lambda > 0$ such that

$$\frac{\lambda^{1-q}}{p-1} < (q-1-k)\limsup_{t\to\infty}\left\{\int^t f(s)ds\right\}^{q-1-k}\{F_k(\infty) - F_k(t)\}, \tag{2.10}$$

where k is defined as in (2.1). It follows from (2.6), (2.7) and (2.9) that there exist two numbers a and b with $b \geq a \geq T$ such that

$$v(a) - A(a, t) \leq -\lambda \qquad \text{for all } t \geq b. \tag{2.11}$$

Let

$$z(t) = \int_a^t f(s)v(s)ds.$$

Then Hölder's inequality implies

$$\int_a^s |v(\tau)|^q d\tau \geq \frac{|z(s)|^q}{(\int_a^s f^p(\tau)d\tau)^{q-1}}.$$

It follows from (2.8) and (2.11) that

$$z(t) \le -\lambda \int_a^t f(s)ds - (p-1)\int_a^t f(s)|z(s)|^q \left(\int_a^s f^p(\tau)d\tau\right)^{1-q} ds := -G(t). \tag{2.12}$$

Thus,

$$G'(t) = \lambda f(t) + (p-1) f(t)|z(t)|^q \left(\int_a^t f^p(s)ds\right)^{1-q} \tag{2.13}$$

and

$$0 \le \lambda \int_a^t f(s)ds \le G(t) \le |z(t)|. \tag{2.14}$$

It follows from (2.12), (2.13) and (2.14) that

$$\begin{aligned} G'(t)G^{k-q}(t) &\ge G'(t)G^k(t)|z(t)|^{-q} \\ &\ge (p-1)\lambda^k f(t)\left(\int_a^t f(s)ds\right)^k \left(\int_a^t f^p(s)ds\right)^{1-q}. \end{aligned}$$

For $k < q-1$, if we integrte the above inequality from $t(\ge b)$ to ∞, then

$$\frac{1}{q-1-k} G^{k-q+1}(t) \ge (p-1)\lambda^k \{F_k(\infty) - F_k(t)\}.$$

It follows from (2.14) that

$$\frac{\lambda^{1-q}}{p-1} \ge (q-1-k)\left\{\int_a^t f(s)ds\right\}^{q-1-k} \{F_k(\infty) - F_k(t)\},$$

which contradicts (2.10). This contradiction completes our proof.

LEMMA 2.3. *Assume that $v(t)$ satisfies* (2.4) *on* $[T, \infty)$ *for some* $T > 0$. *If* $\int^\infty |v(s)|^q ds < \infty$, *then, for any* $f \in \mathfrak{I}_0$, $\lim_{t\to\infty} A_f(\cdot, t)$ *exists.*

PROOF. As in the proof of Lemma 2.2, (2.8) holds. This implies that

$$A(\xi, t) = v(\xi) - \frac{\int_\xi^t f(s)v(s)ds}{\int_\xi^t f(s)ds} - (p-1)\frac{\int_\xi^t f(s)\int_\xi^s |v(\tau)|^q d\tau ds}{\int_\xi^t f(s)ds}. \tag{2.15}$$

Since $f \in \mathfrak{I}_0$, (2.3) holds. Thus,

$$\lim_{t\to\infty} \frac{\int_\xi^t f(s)\int_\xi^s |v(\tau)|^q d\tau ds}{\int_\xi^t f(s)ds} = \int_\xi^\infty |v(s)|^q ds < \infty.$$

By Hölder's inequality,

$$0 \le \lim_{t\to\infty} \frac{|\int_\xi^t f(s)v(s)ds|}{\int_\xi^t f(s)ds} \le \lim_{t\to\infty} \frac{(\int_\xi^t f^p(s)ds)^{1/p}(\int_\xi^t |v(s)|^q ds)^{1/q}}{\int_\xi^t f(s)ds} = 0.$$

Hence, by (2.15), $\lim_{t\to\infty} A(\xi, t)$ exists and

$$\lim_{t\to\infty} A(\xi, t) = v(\xi) - (p-1)\int_\xi^\infty |v(s)|^q ds.$$

Lemma 2.4. *Assume that $B(s)$ and $Q(s, t)$ are nonnegative continuous functions for $T \le s, t < \infty$. If*

$$\int_t^\infty Q(s, t)B^q(s)ds \le p^{-q}B(t) \qquad \text{for } t \ge T, \tag{2.16}$$

then the equation

$$v(t) = B(t) + (p-1)\int_t^\infty Q(s, t)|v(s)|^q ds \qquad \text{for } t \ge T, \tag{2.17}$$

has a continuous solution $v(t)$.

Proof. Let $v_1(t) = B(t)$ and define

$$v_{k+1}(t) = B(t) + (p-1)\int_t^\infty Q(s, t)|v_k(s)|^q ds \qquad \text{for } k = 2, 3, \ldots.$$

Then, by (2.16),

$$\begin{aligned} v_2(t) &= B(t) + (p-1)\int_t^\infty Q(s, t)B^q(s)ds \\ &\le B(t) + (p-1)p^{-q}B(t) \\ &\le pB(t), \end{aligned}$$

and $v_1(t) \le v_2(t)$. Suppose $v_1 \le v_2 \le \cdots \le v_n \le pB(t)$, where n is a positive integer. Now

$$\begin{aligned} v_n(t) &= B(t) + (p-1)\int_t^\infty Q(s, t)|v_{n-1}(s)|^q ds \\ &\le B(t) + (p-1)\int_t^\infty Q(s, t)|v_n(s)|^q ds = v_{n+1}(t) \\ &\le B(t) + (p-1)p^q\int_t^\infty Q(s, t)B^q(s)ds \\ &\le B(t) + (p-1)p^q \cdot p^{-q}B(t) \end{aligned}$$

$$= pB(t), \qquad \text{for } t \geq T.$$

Thus, the sequence $\{v_n\}$ is increasing and bounded above by $pB(t)$. Hence, $\{v_n\}$ converges uniformly to a continuous function $v(t)$, which is a solution of (2.17). Thus, the proof is complete.

LEMMA 2.5. *Assume that $B(t)$ and $Q(s, t)$ are nonnegative continuous functions for $T \leq s, t < \infty$. If there exists $\varepsilon > 0$ such that*

$$\int_t^\infty Q(s, t)B^q(s)ds \geq p^{-q}(1+\varepsilon)B(t) \not\equiv 0 \qquad \textit{for } t \geq T, \tag{2.18}$$

then the inequality

$$v(t) \geq B(t) + (p-1)\int_t^\infty Q(s, t)|v(s)|^q ds, \qquad t \geq T, \tag{2.19}$$

does not have a continuous solution for $v(t)$.

PROOF. Suppose to the contrary that $v(t)$ is a continuous function satisfying (2.19) for $t \geq T$. Then $v(t) \geq B(t) \geq 0$, which implies $v^q(t) \geq B^q(t) \geq 0$ for $t \geq T$. Thus

$$v(t) \geq B(t) + (p-1)\int_t^\infty Q(s, t)B^q(s)ds \geq \{1 + (p-1)(1+\varepsilon)p^{-q}\}B(t)$$

$$\text{for } t \geq T.$$

Continuing in this way, we obtain $v(t) \geq a_n B(t)$, where $a_1 = 1$, $a_n < a_{n+1}$ and

$$a_{n+1} = 1 + (p-1)a_n^q p^{-q}(1+\varepsilon) \qquad \text{for } n \geq 1. \tag{2.20}$$

We claim that $\lim_{n\to\infty} a_n = \infty$. Assume, to the contrary, that $\lim_{n\to\infty} a_n = \lambda < \infty$. Thus $\lambda \geq 1$. It follows from (2.20) that

$$\lambda = 1 + (p-1)(1+\varepsilon)\lambda^q p^{-q}. \tag{2.21}$$

But it is easy to see that (2.21) is not possible for $\lambda \geq 1$. This contradiction proves $\lim_{n\to\infty} a_n = \infty$. Then $B(t) \equiv 0$ for $t \geq T$, which contradicts (2.18). This contradiction completes our proof.

3. Oscillation criteria

In this section, we shall derive some oscillation criteria for solutions of equation (E).

THEOREM 3.1. *If there exists $f \in \mathfrak{I}$ such that* (2.6) *holds, then either* (E) *is oscillatory, or $\lim_{t\to\infty} A_g(\cdot, t)$ exists for all $g \in \mathfrak{I}_0$.*

PROOF. Suppose (E) is nonoscillatory. Then, by Theorem 2.1, there exist a $T \geq 0$ and a function $v \in C^1$ such that (2.4) holds on $[T, \infty)$. It follows from Lemma 2.2 that $\int^{\infty} |v(s)|^q ds < \infty$. Thus, by Lemma 2.3, $\lim_{t\to\infty} A_g(\cdot, t)$ exists for all $g \in \mathfrak{I}_0$. Hence, we complete the proof.

COROLLARY 3.2. *If there exist two nonnegative bounded functions f and g on $[T, \infty)$ satisfying $\int^{\infty} f(s)ds = \int^{\infty} g(s)ds = \infty$ such that*

$$\lim_{t\to\infty} A_f(T, t) < \lim_{t\to\infty} A_g(T, t),$$

then equation (E) *is oscillatory.*

PROOF. Let α and β be numbers satisfying

$$\lim_{t\to\infty} A_f(T, t) < \alpha < \beta < \lim_{t\to\infty} A_g(T, t).$$

Let $h(t) = g(t)$ for $T \leq t < t_1$, where t_1 is determined such that $A_g(T, t_1) \geq \beta$ and $\int_T^{t_1} g(s)ds \geq 1$. Let $h(t) = f(t)$ for $t_1 \leq t \leq t_2$, where t_2 is determined such that $A_h(T, t_2) \leq \alpha$ and $\int_T^{t_2} h(s)ds \geq 2$. This is possible because

$$\begin{aligned}
&A_h(T, t_2)\\
&= \frac{\int_T^{t_2} h(s)\int_T^s c(\mu)d\mu ds}{\int_T^{t_2} h(s)ds}\\
&= \frac{\int_T^{t_1} [g(s) - f(s)]\int_T^s c(\mu)d\mu ds}{\int_T^{t_1} g(s)ds + \int_{t_1}^{t_2} f(s)ds} + \frac{\int_T^{t_2} f(s)\int_T^s c(\mu)d\mu ds}{\int_T^{t_2} f(s)ds} \frac{\int_T^{t_2} f(s)ds}{\int_T^{t_1} g(s)ds + \int_{t_1}^{t_2} f(s)ds}\\
&= A_f(T, t_2)[1 + o(1)] + o(1) \qquad \text{as } t_2 \to \infty.
\end{aligned}$$

Continuing in this manner, we obtain a nonnegative, nonintegrable and bounded function $h(t)$ defined on $[T, \infty)$ such that

$$\limsup_{t\to\infty} A_h(T, t) \geq \beta > \alpha \geq \liminf_{t\to\infty} A_h(T, t).$$

Hence, by Theorem 3.1, (E) is oscillatory.

THEOREM 3.3. *If $B(t) = \int_t^{\infty} c(s)ds \geq 0$ and $\int_t^{\infty} B^q(s)ds \leq p^{-q}B(t)$ for $t \geq T > 0$, then* (E) *is nonoscillatory.*

PROOF. By Lemma 2.4, the equation

$$v(t) = B(t) + (p - 1)\int_t^{\infty} |v(s)|^q ds$$

has a continuous solution $v(t)$ on $[T, \infty)$. Then

$$v'(t) = -c(t) - (p-1)|v(t)|^q \qquad \text{for } t \geq T.$$

By Theorem 2.1, (E) is nonoscillatory.

THEOREM 3.4. *If* $c(t)$ *satisfies the following conditions*:
(i) *there is a* $T \geq 0$ *such that* $B(t) = \int_t^\infty c(s)ds \geq 0$ *for* $t \geq T$;
(ii) *there is an* $\varepsilon > 0$ *such that* $\int_t^\infty B^q(s)ds \geq p^{-q}(1+\varepsilon)B(t)$ *for* $t \geq T$,
then (E) *is oscillatory, where the constant* p^{-q} *in* (ii) *is best possible.*

PROOF. Suppose to the contrary that (E) is nonoscillatory. It follows from Theorem 2.1 that there exist a number T_1 and a function $v \in C^1$ satisfying (2.4) on $[T_1, \infty)$. Without loss of generality, let $T_1 = T$. It follows from (i) that $\liminf_{t\to\infty} A_f(T,t) = \int_T^\infty c(\mu)d\mu > -\infty$. This and Lemma 2.2 imply $\int_T^\infty |v(s)|^q ds < \infty$. Thus, it follows from (i) and (2.5) that $v(t)$ satisfies

$$v(t) = B(t) + (p-1)\int_t^\infty |v(s)|^q ds.$$

But, by (ii) and Lemma 2.5, the equation

$$w(t) = B(t) + (p-1)\int_t^\infty |w(s)|^q ds$$

does not have a continuous solution, which is a contradiction.

To see that the constant p^{-q} is best possible, we consider the half-linear differential equation

$$\frac{d}{dt}\Phi(u'(t)) + q^{-p}(t+1)^{-p}\Phi(u(t)) = 0. \tag{3.1}$$

Clearly, (3.1) has a solution $u(t) = (t+1)^{1/q}$. Hence, (3.1) is nonoscillatory. Since $c(t) = q^{-p}(t+1)^{-p}$,

$$\begin{aligned}\int_t^\infty \left(\int_s^\infty c(\tau)d\tau\right)^q ds &= q^{-pq}(p-1)^{-q-1}(t+1)^{-p+1} \\ &= q^{-p}(p-1)^{-q}\int_t^\infty c(s)ds \\ &= p^{-q}\int_t^\infty c(s)ds.\end{aligned}$$

This means that the number p^{-q} in Theorem 3.4 is best possible.

If $c(t)$ is a continuous positive function on $[0, \infty)$, then Theorems 3.3 and 3.4 reduce to Theorem 1.1 of [10].

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