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Tessellation automata on free groups

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Introduction

Tessellation automata on finitely generated free groups are investigated. Given a finitely generated free group G, we can construct hyperbolic tessellation on its Cayley graph, which is a tree. The vertex set of the graph is G itself. On each vertex we place an automaton with a finite set of states Q. Each of these automata is influenced by its neighbors, the number of which equals twice the number of generators of the free group G. With these local interactions we construct a dynamical system on the space Q^G. We call the elements of Q^G *configurations* as usual. In this way we obtain a cellular automaton on hyperbolic tessellation. We refer such automata as "tesselletion automata on free groups." In this paper we clarify relations among period preservability, injectivity and surjectivity of parallel maps. We also show the equivalence of finite orderedness and strong Poisson stability.

Historically, tessellation automata theory began with the work of Von Neumann [7]. Then Moore [6] showed the *Garden of Eden* theorem which states that violation of local injectivity implies existence of a Garden of Eden pattern. A Garden of Eden pattern is a partial configuration which cannot be reproduced in any environments. This shows an obstruction to *self-reproducing* property. Amoroso, Cooper and Patt [1] clarified the concept of a *Garden of Eden configuration.* Sato and Honda [8] investigated the relations among period preservability, Poisson stability and finite orderedness of parallel maps based on dynamical system theory. All these works, being very fruitful, were done in the framework of Euclidean tessellations. The aim of this paper is to extend cellular automata theory to non-Euclidean tessellations.

In section 1 we define tessellation automata on free groups and introduce group actions on them. In section 2 we define *periods* of configurations by using lattice of subgroups. In section 3 we state and prove the main theorems on injectivity, surjectivity, Poisson stability, strong Poisson stability and period preservability of parallel maps. In section 4 we state and prove the main theorem on finite orderedness and Poisson stability.

1. Tessellation automata on free groups

Although we describe our theory in the rank 2 case, the following arguments does not depend on the number of generators.

Let G be a free group generated by 2 elements a, b. G acts as a transformation group on the set G itself by operating from the left or from the right.

DEFINITION 1.1 For any $x, y \in G$ if one of the following equations holds, we say x and y are *directly connected.*

$$x = ya, \qquad x = ya^{-1}$$
$$x = yb, \qquad x = yb^{-1}$$

This binary relation on the set G is symmetric (but not transitive). Figure 1 shows the connectivity of the space G. Usually this graph is called the *Cayley graph* of G. This graph can be viewed as a fractal pattern. Figure 1 contains only level 5 construction for the sake of clarity to vision. With local interaction of automata, infomation travels along paths in this graph. Each vertex represents an element of G. The center of the figure correspond to a unit element e of G. The word $abab^{-1}$, for example, corresponds to a vertex which can be reached starting from the center vertex traveling along a path described below. The path begins at the center vertex. Then the path proceeds *right*, *up*, *right*, *down*. The words should be read from left to right. Systematic treatments for graphs with group actions can be found in Dick and Dunwoody [3].

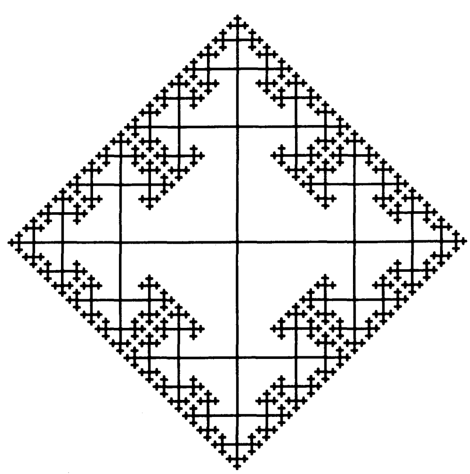

Fig. 1 Cayley graph of G

We define a metric on the space G. Let x, y be any pair of elements of G. If $x = y$, we define $\rho(x, y) = 0$. If $x \neq y$, let z be the shortest word such that $x = yz$. We define $\rho(x, y)$ as the length of the word z. It is easily verified that ρ is a metric on the space G. We denote by $B_n(x)$ the ball of radius n with center $x \in G$, that is,

$$B_n(x) = \{y \in G \mid \rho(x, y) \leq n\}.$$

When $x = e$, we will write B_n for short.

Let $Q = \{0, 1, \ldots, s-1\}$. A configuration over Q is a map from G to Q. The set of all configurations over Q is denoted by $C(Q)$ or simply C if there can be no confusion. A local map is a map f from Q^5 to Q such that

$$f(0, 0, 0, 0, 0) = 0.$$

A *parallel map* $\hat{f}: C(Q) \to C(Q)$ induced by a local map f is defined as follows. For $\alpha \in C(Q)$, $x \in G$,

$$\hat{f}(\alpha)(x) = f(\alpha(x), \alpha(xa), \alpha(xa^{-1}), \alpha(xb), \alpha(xb^{-1})).$$

The pair of G and $\hat{f}$ forms a discrete time dynamical system. Being viewed as a cellular automaton, the state of a cell at time t is determined by the states of neighboring cells at time $t-1$.

We define a metric d on C. Let α, β be any elements in C. If $\alpha = \beta$, define $d(\alpha, \beta) = 0$. If $\alpha \neq \beta$, let k be the least nonnegative integer such that $\alpha | B_k \neq \beta | B_k$. We set $d(\alpha, \beta) = (1 + k)^{-1}$. It is easily verified that d is a metric on C. The topology induced by this metric coinsides with the product topology of Q endowed with the discrete topology. The well known Tichonov's theorem assures that C is a compact space.

G acts on "the space" G from the left or from the right. Via this action we define the action of $g^*(g \in G)$ on $\alpha \in C$ as follows. For $x \in G$

$$[g^*\alpha](x) = \alpha(g^{-1}x).$$

It is clear that the map $g^*: C \to C$ is bijective. So if we identify g and g^*, G is regarded as a transformation group acitng on the configuration space C.

PROPOSITION 1.2 *The action of G on C has the following properties.*

(1) *The action of any $g \in G$ is commutative with any parallel map $\hat{f}$. To be precise, for $\alpha \in C$, $x \in G$*

$$[g^*\hat{f}\alpha](x) = [\hat{f}g^*\alpha](x).$$

(2) *For any $g \in G$, the map $g^*: C \to C$ is continuous.*

PROOF. (1) Both sides of the formula equal to

$$f(\alpha(g^{-1}x), \alpha(g^{-1}xa), \alpha(g^{-1}xa^{-1}), \alpha(g^{-1}xb), \alpha(g^{-1}xb^{-1})).$$

(2) Let l be the length of g as a word in G. Let $U_k(\alpha)$ be any ball of radius $(1+k)^{-1}$ with center α in C. Then we have the following

$$g^*(U_{k+l}(\alpha)) \subseteq U_k(g^*(\alpha)).$$

This shows the continuity of the map $g^*: C \to C$. ■

2. Defining Periods via subgroups

In this section we introduce the notion of *period*. We need subgroups of G of finite index. There are infinitely many such subgroups. It is known that all subgroups of free groups are also free. See Schreier [9] or Kurosh [4].

We give some examles here.

EXAMPLE 2.1 Denote by $e(a, x)$ the algebraic sum of the exponents of a in the expression of the element $x \in G$ as a word. Since G is a free group, $e(a, x)$ is uniquely determined by x. $e(b, x)$ is defined in the same manner. We partition G into n disjoint classes

$$H_j = \{x \in G \mid e(a, x) \equiv j \ (mod\ n)\}.$$

Clealy H_0 is a subgroup of G of index n. This construction is very simple and it shows that there are infinitely many subgroups of G of finite index. It also shows that there exists a normal subgroup of G of arbitrarily given index.

EXAMPLE 2.2 The subgroup generated by the element a is not a normal subgroup of G. Its index is infinite.

For free groups the set of coset representatives of a subgroup can be chosen to form a tree in the Cayley graph. See Kurosh [4]. We call such a set *fundamental domain* or *fundamental set* of the subgroup as in the case of Fuchsian groups. Figure 2 shows an example of a fundamental set of the subgroup

$$\{x \in G \mid e(a, x) \equiv e(b, x) \equiv 0 \ (mod\ 2)\}.$$

The subgraph emphasized by bold lines represents a fundamental set.

Let H be any subgroup of G. H acts on the configuration space $C(Q)$ in the sense described in the previous section.

DEFINITION 2.3 If $h^*\alpha = \alpha$ for all $h \in H$, α is said to be H *invariant*.

Every $\alpha \in C(Q)$ is left unchanged by the action of the trivial subgroup $\{e\}$ of G. So for any $\alpha \in C(Q)$ there exists at least one subgroup of G that leaves

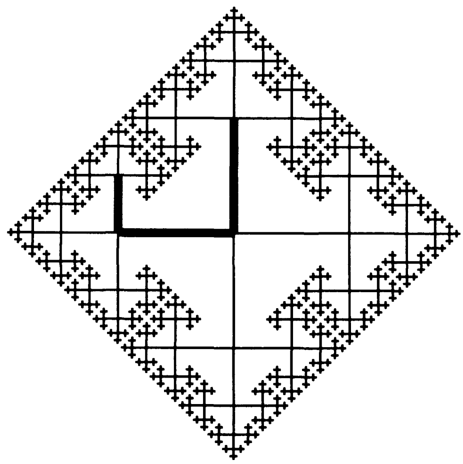

Fig. 2 An example of a fundamental set

α unchanged. If we have two such subgroups H_1, H_2, the smallest subgroup of G that includes both H_1, H_2 has the same property. There exists a maximal subgroup in the family of subgroups under whose action α is left unchanged. This maximal subgroup is uniquely determined.

Definition 2.4 Let $F=\{H_i \mid i\in\Lambda\}$ be the family of all subgroups of G that leave a configuration $\alpha\in C$ unchanged. The *period* of α is the maximal subgroup in the family F, and is denoted by $\omega(\alpha)$. A configuration $\alpha\in C(Q)$ is said to be a *periodic configuration* if $\omega(\alpha)$ is of finite index. The set of all periodic configurations is denoted by $C_P(Q)$ or simply C_P.

We introduce the second notion of period, namely *stable period.*

Definition 2.5 Let $=\{H_i \mid i\in\Lambda\}$ be the family of all normal subgroups of G that leave a configuration $\alpha\in C$ unchanged. The *stable period* of α is the maximal normal subgroup in the family F, and is denoted by $\Omega(\alpha)$.

Let F be the family of normal subgroups in the sbove definition. For $H_1, H_2\in F$, let H be the the smallest subgroup of G that includes both H_1 and H_2. H is also a normal subgroup of G and acts on α trivially. Thus the above definition makes sense.

The following proposition relates two notions of period.

Proposition 2.6 *For any $g\in G$, $\alpha\in C(Q)$, the following two equations hold.*

(1) $\omega(g^*\alpha)=g\omega(\alpha)g^{-1}$,

(2) $\Omega(g^*\alpha)=\Omega(\alpha)$.

PROOF. (1) Let $h \in \omega(\alpha)$, $x \in G$. If replace x by $gh^{-1}g^{-1}$ in the following defining equation of g^*

$$g^*\alpha(x) = \alpha(g^{-1}),$$

we have

$$g^*\alpha(ghg^{-1})^{-1}x) = \alpha(g^{-1}x).$$

Thus we have $(ghg^{-1})^*g^*\alpha = g^*\alpha$. This shows $g\omega(\alpha)g^{-1} \subseteq \omega(g^*\alpha)$.

The converse argument is possible almost in the same way.

(2) is a special case of (1). ■

A configuration $\alpha \in C(Q)$ might be called a *stably periodic configuration* if $\Omega(\alpha)$ is of finite index. But this notion is redundant, since we can always find a normal subgroup of finite index contained in a given subgroup of finite index.

3. Period preservability and Poisson stability

In this section we prove the equivalence of *injectivity*, *Poisson stability* and *strong Poisson stability* in the space C_P.

DEFINITION 3.1 Let $\hat{f}$ be a parallel map. $\hat{f}$ is said to be *period preserving on* C if for all $\alpha \in C$ if for all $\alpha \in C$, $\omega(\hat{f}\alpha) = \omega(\alpha)$. Likewise $\hat{f}$ is said to be *period preserving on* C_P if for all $\alpha \in C_P$, $\omega(\hat{f}\alpha) = \omega(\alpha)$.

DEFINITION 3.2 Let $M \subseteq C$. $\hat{f}$ is said to be *surjective on* M if $\hat{f}(M) = M$. $\hat{f}$ is said to be *injective on* M if for $\alpha, \beta \in M$, $\hat{f}(\alpha) = \hat{f}(\beta)$ implies $\alpha = \beta$.

The following proposition is simple and essential.

PROPOSITION 3.3 *$\omega(\alpha) \leq \omega(\hat{f}(\alpha))$ where $A \leq B$ means A is a subgroup (not necessarily proper) of B.*

PROOF. Let $h \in \omega(\alpha)$. From the definition of ω, $h^*\alpha = \alpha$. From proposition 1.2, $h^*\hat{f}\alpha = \hat{f}h^*\alpha = \hat{f}\alpha$. Thus we have $h \in \omega(\hat{f}\alpha)$. ■

From the above proposition we conclude that C_P is an invariant set of any parallel maps.

The following arguments are based on the theory of dynamical systems. See Bhatia and Szegö [2].

DEFINITION 3.4 Let $\alpha \in C$ and $\hat{f}$ be a parallel map. α is said to be *Poissson stable* with respect to $\hat{f}$ if there exists a sequence of intergers

$n_1 < n_2 < \cdots$ such that

$$\lim_{i\to\infty} \hat{f}^{n_i}(\alpha) = \alpha.$$

Let $M \in C$. $\hat{f}$ is said to be *M Poisson stable* if every point in M is Poisson stable with respect to $\hat{f}$. α is said to be *strongly Poisson stable* with respect to $\hat{f}$ if there exists a nonnegative integer n_α such that $\hat{f}^{n_\alpha}(\alpha) = \alpha$ where n_α depends on α. $\hat{f}$ is said to be *M strongly Poisson stable* if every point in M is strongly Poisson stable with respect to $\hat{f}$.

THEOREM 3.5 *Let $\hat{f}$ be a parallel map. $\hat{f}$ is injective on C_P if and only if $\hat{f}$ is period preserving and surjective on C_P.*

PROOF. Let $\hat{f}$ be injective on C_P. Suppose that there exists $\alpha \in C_P$ such that $\omega(\hat{f}(\alpha)) \neq \omega(\alpha)$. Let

$$E = \{\beta \in C_P \mid \omega(\beta) \geq \omega(\hat{f}(\alpha))\}.$$

The set E is finite because α is periodic. From Proposition. 3.3 (2) $\hat{f}(E) \subseteq E$. Injectivity of $\hat{f}$ and finiteness of E yield $\hat{f}(E) = E$. Recall that $\alpha \notin E$ and $\hat{f}(\alpha) \in E$. This contradicts the injectivity of $\hat{f}$ on C_P. Thus injectivity for C_P implies period preservability.

Next we will show injectivity implies surjectivity. For any $\alpha \in C_P$ we set $E_\alpha = \{\beta \in C_P \mid \omega(\beta) = \omega(\alpha)\}$. E_α is again a finite set. $\hat{f} \mid E_\alpha$ can be regarded as an element of the permutation group of E_α. As this group is finite, there exists a nonnegative integer n such that $\hat{f}^n(\alpha) = \alpha$. If we take $\beta = \hat{f}^{n-1}(\alpha)$, we have $\hat{f}(\beta) = \alpha$. This means $\hat{f}$ is surjective on C_P.

Conversely, let $\hat{f}$ be period preserving and surjective on C_P. Suppose there exist $\alpha, \beta \in C_P$ such that $\alpha \neq \beta$ and $\hat{f}(\alpha) = \hat{f}(\beta)$. Then $\alpha, \beta \in E_\alpha$. Injectivity and surjectivity of $\hat{f} \mid E_\alpha$ are equivalent because E_α is a finite set. This leads to a contradiction. Thus period preservability and surjectivity on C_P imply injectivity on C_P. ■

THEOREM 3.6 *Let $\hat{f}$ be a parallel map. The following statements are equivalent.*

(1) *$\hat{f}$ is injective on C_P.*
(2) *$\hat{f}$ is strongly C_P Poisson stable.*
(3) *$\hat{f}$ is C_P Poisson stable.*

PROOF. (1) $\Rightarrow$ (2): From theorem 3.5 $\hat{f}$ is period preserving and surjective on C_P. For any $\alpha \in C_P$ let $E_\alpha = \{\beta \in C_P \mid \omega(\beta) = \omega(\alpha)\}$. As before $\hat{f} \mid E_\alpha$ is regarded as an element of the permutaion group acting on the finite set E_α. Let k be the order of $\hat{f} \mid E_\alpha$. Then we have

$$\alpha = \hat{f}^k(\alpha) = \hat{f}^{2k}(\alpha) = \hat{f}^{3k}(\alpha) = \cdots.$$

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1): Suppose $\hat{f}$ is not injective on C_P. From theorem 3.5 $\hat{f}$ is not period preserving or not surjctive on C_P.

In the case $\hat{f}$ is not period preserving, there exists $\alpha \in C_P$ such that $\hat{f}(E_\alpha)$ is a proper subset of E_α. Thus there exists $\beta \in E_\alpha$ such that for every interger n, $\hat{f}^n(\beta) \neq \beta$. This β cannot be Poisson stable with respect to $\hat{f}$ because $d(\beta, \hat{f}^n(\beta)) \geq 1$ for all n.

In the case $\hat{f}$ is not surjective, there exists $\alpha \in C_P$ such that $\alpha \notin \hat{f}(C_P)$. If we consider the finite set E_α, the same conclusion can be derived as above.

In either case contradiction arises. ■

4. Parallel maps of finite order and Poisson stability

In this section we deal with the relation between finite-orderedness and Poisson stability.

DEFINITION 4.1 A configuration $\alpha \in C$ that has the following property is said to be a *transitive configuration.* For any $\beta \in C$, there exists a sequence $g_1, g_2, g_3, \ldots \in G$ such that $\lim_{n \to \infty} g_n^*(\alpha) = \beta$.

LEMMA 4.2 *There exists a transitive configuration in C.*

PROOF. Let B_n be a ball in G of radius n with center e as defined in section 2. An easy calculation shows that there are $2 \cdot 3^n - 1$ cells (elements of G) in this ball. The number of configurations restricted on this ball is $s^{2 \cdot 3^n - 1}$. We write this number as b_n. Let $g_{i,j} \in G$, $1 \leq i$, $1 \leq j \leq b_i$ be chozen as follows. For any pair of ordered pairs $(m, n) \neq (p, q)$, the translated balls $g_{m,n}^{-1} B_m$ and $g_{p,q}^{-1} B_p$ are disjoint.

All configurations defined on the ball B_n can be numbered from 1 to b_n. Let these be $\beta_1, \beta_2, \ldots, \beta_{b_n}$.

We define $\alpha \in C$ as follows. $\alpha \mid g_{i,j}^{-1} B_i = g_{i,j}^*(\beta_j)$, and on the complement of these balls the value of α is 0.

This α clearly has the required property. ■

DEFINITION 4.3 A configuration which has quiescent state on all but a finite number of cells is said be a *finite configuration.* We denote the set all finite configurations as C_F.

Clealy C_F is a countable dense subset of C and invariant under the actions of parallel maps and G.

DEFINITION 4.4 Let $\hat{f}$ be a parallel map. $\hat{f}$ is said to have *finite order* if there exists a positive integer n such that $\hat{f}^n = I$ where I denotes the identity map. The least such positive integer n is called the *order* of $\hat{f}$. Let $M \subseteq C$ be an invariant subset of $\hat{f}$. $\hat{f}$ is said to have *finite order on* M if $\hat{f}^n | M = I | M$ for some positive interger n.

THEOREM 4.5 *Let $\hat{f}$ be a parallel map. The folowing statements are equivalent.*

(1) *$\hat{f}$ has a finite order.*
(2) *$\hat{f}$ has a finite order on C_F.*
(3) *$\hat{f}$ is strongly C Poisson stable.*

PROOF. Notice that C_F is a $\hat{f}$-invariant dense subseet of C.

For $\hat{f}$ is continuous, the equation $\hat{f}^n | M = I | M$ on a dense subset of C is equivalent to the equation $\hat{f}^n =$ identity. Thus (1) and (2) are equivalent.

(1) $\Rightarrow$ (3) is evident.

We will show (3) $\Rightarrow$ (1). From lemma 4.2 there exists a transitive configuration $\alpha \in C$. By assumption there exists a nonnegative integer n such that $\hat{f}^n(\alpha) = \alpha$. From the definition of α, for any $\beta \in C$ we have a sequence $g_1, g_2, \ldots \in G$ such that $\lim_{m \to \infty} g_m^*(\alpha) = \beta$. Since $\hat{f}$ is continuous, we have

$$\hat{f}^n(\beta) = \hat{f}^n(\lim_{m \to \infty} g_m^*(\alpha)) = \lim_{m \to \infty} g_m^* \hat{f}^n(\alpha) = \beta.$$

Thus $\hat{f}$ has finite order. ■

5. Conclusions

We introduced cellular automata on free groups, and succeded in constructing a theory analogous to that of Euclidean tessellations. We would like to extend the results here to wider class of cellular automata defined on graphs which have group actions.

We have not mentioned that period configurations C_P are dense in C. This is proved in the coming paper Yukita [10], and also the results in free groups are reproduced in Fuchsian group cases.

The question whether the *Garden of Eden Theorem* holds in the non-Euclidean theory is yet to be investigated.

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