HIROSHIMA MATH. J.
**25** (1995), 541–559

# Invariant nuclear space of a second quantization operator

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(Received April 7, 1994)
(Revised August 5, 1994)

**Abstract.** Let $S'(\mathbf{R})$ be the dual of the Schwartz space $S(\mathbf{R})$, $\mathbf{A}$ a self-adjoint operator in $L^2(\mathbf{R})$ and $d\Gamma(\mathbf{A})^*$ the adjoint operator of $d\Gamma(\mathbf{A})$ which is the second quantization operator of $\mathbf{A}$. It is proven that under a suitable condition on $\mathbf{A}$ there exists a nuclear subspace $S$ of a fundamental space $S_{\mathbf{A}}$ of Hida's type on $S'(\mathbf{R})$ such that $d\Gamma(\mathbf{A})S \subset S$ and $e^{-td\Gamma(\mathbf{A})}S \subset S$, which enables us to show that a stochastic differential equation arising from the central limit theorem for spatially extended neurons:

$$dX(t) = dW(t) - d\Gamma(\mathbf{A})^* X(t)dt,$$

has a unique solution on the dual space $S'$ of $S$, where $W(t)$ is an $S'$-valued Wiener process.

## 1. Introduction

Concerning with infinite dimensional geometry and analysis, several types of fundamental spaces on infinite dimensional topological vector spaces have attracted several authors ([1], [4], [9], [11], [13], [14]). As it has been known by [5], the nuclearity of the space gives us the regularization theorem which guarantees the existence of a strong solution of the stochastic differential equation. However, [7] tried to construct a unique weak solution of a Segal-Langevin type stochastic differential equation on a suitable space of infinite dimensional generalized functionals which is not nuclear, and the fundamental spaces used in the Malliavin calculus are known not to be nuclear [2]. With this background, we consider spaces of Hida's type which are nuclear.

Let $(S_{\mathbf{A}})$ be a fundamental space of Hida's type and $d\Gamma(\mathbf{A})$ the second quantization operator. Inspired by the works [11], [12], we construct a fundamental space which is invariant under the semi-group $e^{-td\Gamma(\mathbf{A})}$ and is nuclear and smaller than $(S_{\mathbf{A}})$ even if $(S_{\mathbf{A}})$ is not nuclear. This enables us to obtain a unique strong solution of the stochastic differential equation

$$dX(t) = dW(t) - d\Gamma(\mathbf{A})^* X(t)dt, \tag{1.1}$$

which is a special case of the types considered in [7].

First we begin with giving some notations and explanations. Let $E$ be a real locally convex topological vector space and $E'$ the topological dual space of $E$. We denote by $\langle\,,\rangle$ the pairing of $E$ and $E'$, and by $|\cdot|_E$ the norm of $E$ when $E$ is a Hilbert space. Let $\mathscr{H}$ be a separable real Hilbert space densely and continuously embedded in $E$. Then identifying $\mathscr{H}'$ with $\mathscr{H}$, we have

$$E' \subset \mathscr{H} \subset E. \tag{1.2}$$

Let $\mu$ be the countably additive Gaussian measure on $E$ whose characteristic functional is given by

$$\int_E \exp\left[i\langle x, \xi\rangle\right] d\mu(x) = \exp\left[-\frac{1}{2}|\xi|_{\mathscr{H}}^2\right], \qquad \xi \in E'. \tag{1.3}$$

If we replace $E$ by $S'(\mathbf{R})$, $(E, \mu)$ is called the white noise space [11].

Now we state our main result. Let $\mathbf{A}$ be a self-adjoint operator in Hilbert space $\mathscr{H}$ and $L^2(E, \mu)$ the space of square integrable functions with respect to $\mu$. Further we denote by $(S_{\mathbf{A}})$ a fundamental space of Hida's type determined by $\mathbf{A}$ and denote by $d\Gamma(\mathbf{A})$ the second quantization operator of $\mathbf{A}$, which will be precisely defined later. From now on we denote the domain of a linear operator $\mathbf{T}$ defined densely in $\mathscr{H}$ by $\mathscr{D}(\mathbf{T})$ and set $C^\infty(\mathbf{T}) = \bigcap_{n=1}^\infty \mathscr{D}(\mathbf{T}^n)$. We always consider $\mathscr{D}(\mathbf{T}^n)$ as a Hilbert space equipped with the inner product $(\mathbf{T}^n\cdot, \mathbf{T}^n\cdot)_{\mathscr{H}}$. We mean by $\mathbf{A} \geq \lambda$, $\lambda \in \mathbf{R}$, that $(\mathbf{A}f, f)_{\mathscr{H}} \geq \lambda(f, f)_{\mathscr{H}}$ for all $f \in \mathscr{D}(\mathbf{A})$.

THEOREM 1.1. *Let* $\mathbf{A}$ *be a self-adjoint operator in* $\mathscr{H}$. *Suppose that* $\mathbf{A} \geq 1 + \varepsilon$, *for some* $\varepsilon > 0$ *and there exist a self-adjoint operator* $\mathbf{B}$ *in* $\mathscr{H}$ *and natural numbers* $p$ *and* $q$, *satisfying the following conditions*;

(1) $\mathscr{D}(\mathbf{B}^p) \subset \mathscr{D}(\mathbf{A})$,
(2) *the identity map of* $\mathscr{D}(\mathbf{B}^q)$ *into* $\mathscr{H}$ *is a Hilbert Schmidt operator*,
(3) $\mathbf{A}C^\infty(\mathbf{B}) \subset C^\infty(\mathbf{B})$.

*Then there exists a nuclear subspace* $S$ *of* $L^2(E, \mu)$ *such that*

$$d\Gamma(\mathbf{A})S \subset S.$$

*Further, suppose that*

(4) $e^{-t\mathbf{A}}C^\infty(\mathbf{B}) \subset C^\infty(\mathbf{B})$. *Then*

$$e^{-td\Gamma(\mathbf{A})}S \subset S.$$

If there exists a positive self-adjoint operator $\mathbf{B}$ and $p, q \in \mathbf{N}$ satisfying the conditions (1) $\sim$ (4) of Theorem 1.1, those conditions (1) $\sim$ (4) still hold when we replace $\mathbf{B}$ by $\mathbf{B}+2\mathbf{I}$. Then, since $\mathbf{B}+2\mathbf{I} \geq 1+\varepsilon$ for some $\varepsilon > 0$, the desired space $S$ in Theorem 1.1 will be given by $(S_{\mathbf{B}+2\mathbf{I}})$.

## 2. Fundamental space of Hida's type

Before defining a fundamental space of Hida's type, we introduce the following notation. Let $\mathscr{H}$ be a separable Hilbert space. For $f_i \in \mathscr{H}$, $i = 1, 2, \cdots, n$, we denote the tensor product of them by

$$f_1 \otimes f_2 \otimes \cdots \otimes f_n, \tag{2.1}$$

and define the symmetric tensor product of them by

$$f_1 \hat{\otimes} f_2 \hat{\otimes} \cdots \hat{\otimes} f_n = \frac{1}{n!} \sum_{\sigma \in \Xi_n} f_{\sigma(1)} \otimes f_{\sigma(2)} \otimes \cdots \otimes f_{\sigma(n)}, \tag{2.2}$$

where $\Xi_n$ is the symmetric group of degree $n$.

Let $\mathscr{G}$ and $\mathscr{F}$ be the sets of finite linear combinations of terms of (2.1) and (2.2) types, respectively. For $f_i, g_i \in \mathscr{H}$, $i = 1, 2, \cdots, n$, we first set

$$(f_1 \otimes f_2 \otimes \cdots \otimes f_n, g_1 \otimes g_2 \otimes \cdots \otimes g_n)_{\mathscr{H}^{\otimes n}} = (f_1, g_1)_{\mathscr{H}} (f_2, g_2)_{\mathscr{H}} \cdots (f_n, g_n)_{\mathscr{H}}, \tag{2.3}$$

and

$$\begin{aligned}(f_1 \hat{\otimes} f_2 \hat{\otimes} \cdots \hat{\otimes} f_n, g_1 \hat{\otimes} g_2 \hat{\otimes} \cdots \hat{\otimes} g_n)_{\mathscr{H}^{\hat{\otimes} n}} \\ = \left(\frac{1}{n!}\right)^2 \sum_{\sigma, \tau \in \Xi_n} (f_{\sigma(1)}, g_{\tau((1))})_{\mathscr{H}} (f_{\sigma(2)}, g_{\tau(2)})_{\mathscr{H}} \cdots (f_{\sigma(n)}, g_{\tau(n)})_{\mathscr{H}}.\end{aligned} \tag{2.4}$$

Then the inner products $(\cdot, \cdot)_{\mathscr{H}^{\otimes n}}$ on $\mathscr{G}$ and $(\cdot, \cdot)_{\mathscr{H}^{\hat{\otimes} n}}$ on $\mathscr{F}$ are naturally extended for the linear combinations. Let $\mathscr{H}^{\otimes n}$ and $\mathscr{H}^{\hat{\otimes} n}$ be the completions of $\mathscr{G}$ and $\mathscr{F}$ with respect to the inner products $(\cdot, \cdot)_{\mathscr{H}^{\otimes n}}$ and $(\cdot, \cdot)_{\mathscr{H}^{\hat{\otimes} n}}$. Clearly

$$\mathscr{H}^{\hat{\otimes} n} \subset \mathscr{H}^{\otimes n}, \tag{2.5}$$

and

$$\begin{aligned}(f_1 \hat{\otimes} f_2 \hat{\otimes} \cdots \hat{\otimes} f_n, g_1 \hat{\otimes} g_2 \hat{\otimes} \cdots \hat{\otimes} g_n)_{\mathscr{H}^{\otimes n}} \\ = (f_1 \hat{\otimes} f_2 \hat{\otimes} \cdots \hat{\otimes} f_n, g_1 \hat{\otimes} g_2 \hat{\otimes} \cdots \hat{\otimes} g_n)_{\mathscr{H}^{\hat{\otimes} n}}.\end{aligned} \tag{2.6}$$

We define the Wick ordering, denoted by $:x^{\otimes n}:$, for $x \in E$, according to the case where $E = S'(\mathbf{R})$. First the Wick product $:\langle x, \xi_1 \rangle \langle x, \xi_2 \rangle \cdots \langle x, \xi_n \rangle:$ of random variables $\langle x, \xi_k \rangle$, $x \in E$, $\xi_k \in E'$, $k = 1, 2, \cdots, n$, with respect to the probability space $(E, \mu)$ is defined by the following recursion relation [6];

$$:\langle x, \xi_1 \rangle := \langle x, \xi_1 \rangle,$$

$$:\langle x, \xi_1 \rangle \langle x, \xi_2 \rangle \cdots \langle x, \xi_n \rangle := \langle x, \xi_1 \rangle :\langle x, \xi_2 \rangle \cdots \langle x, \xi_n \rangle:$$
$$- \sum_{k=2}^{n} \int_E \langle x, \xi_1 \rangle \langle x, \xi_k \rangle \, d\mu(x) :\langle x, \xi_2 \rangle \cdots \langle x, \check{\xi}_k \rangle \cdots \langle x, \xi_n \rangle:, \qquad n \geq 2,$$

where $\langle x, \check{\xi}_k \rangle$ means that the term $\langle x, \xi_k \rangle$ is excluded in the product. Using the Wick product, we define the Wick ordering $:x^{\otimes n}:$ by

$$\langle :x^{\otimes n}:, \xi_1 \hat{\otimes} \xi_2 \hat{\otimes} \cdots \hat{\otimes} \xi_n \rangle \equiv :\langle x, \xi_1 \rangle \langle x, \xi_2 \rangle \cdots \langle x, \xi_n \rangle:. \tag{2.7}$$

Let $\{e_i : i = 0, 1, 2, \cdots\}$ be a complete orthonormal system in $\mathscr{H}$ taken from $E'$. The well known Wiener-Ito theorem states that the space $L^2(E, \mu)$ has the following orthogonal decomposition

$$L^2(E, \mu) = \bigoplus_{n=0}^{\infty} \mathbf{K}_n,$$

where $\mathbf{K}_n$ consists of $n$-homogeneous chaos, i.e. each $\varphi$ in $\mathbf{K}_n$ has the formal expression

$$\varphi(x) = \langle :x^{\otimes n}:, \hat{f}_n \rangle, \qquad \hat{f}_n \in \mathscr{H}^{\hat{\otimes} n}.$$

In fact if $\hat{f}_n$ is represented by $\sum_{i_1, i_2, \ldots, i_n = 0}^{\infty} a_{i_1, i_2, \ldots, i_n} e_{i_1} \hat{\otimes} e_{i_2} \hat{\otimes} \cdots \hat{\otimes} e_{i_n}$, then the right hand side of the above expression is given by

$$\sum_{i_1, i_2, \ldots, i_n = 0}^{\infty} a_{i_1, i_2, \ldots, i_n} \langle :x^{\otimes n}:, e_{i_1} \hat{\otimes} e_{i_2} \hat{\otimes} \cdots \hat{\otimes} e_{i_n} \rangle.$$

Thus each $\psi \in L^2(E, \mu)$ can be represented uniquely in the following way;

$$\psi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, \hat{f}_n \rangle, \qquad \mu - a.e. \ x \in E. \tag{2.8}$$

Moreover, we have [11]

$$|\psi|^2_{L^2(E,\mu)} = \sum_{n=0}^{\infty} n! \, |\hat{f}_n|^2_{\mathscr{H}^{\hat{\otimes} n}}. \tag{2.9}$$

Let $\mathbf{A}$ be a positive self-adjoint operator in $\mathscr{H}$. Then there exists a unique positive self-adjoint operator $\Gamma(\mathbf{A})$ in $L^2(E, \mu)$ such that (see [12])

$$\Gamma(\mathbf{A}) 1 = 1$$

and for $\xi_i \in \mathscr{D}(\mathbf{A})$, $i = 1, 2, \cdots, n$,

$$\Gamma(\mathbf{A}) :\langle x, \xi_1 \rangle \langle x, \xi_2 \rangle \cdots \langle x, \xi_n \rangle:$$
$$= :\langle x, \mathbf{A}\xi_1 \rangle \langle x, \mathbf{A}\xi_2 \rangle \cdots \langle x, \mathbf{A}\xi_n \rangle:. \tag{2.10}$$

We denote by $\mathscr{P}_{\mathbf{A}}$ the collection of all polynomials of the form

$$\omega(x)=P(\langle x,\xi_1\rangle,\langle x,\xi_2\rangle,\cdots,\langle x,\xi_m\rangle),\qquad \xi_i\in C^\infty(\mathbf{A}),$$

where $P(t_1,\cdots,t_m)$ is a polynomial of $(t_1,\cdots,t_m)$. For each $p\in\mathbf{R}$ we define a semi-norm ${}_{\mathbf{A}}\|\cdot\|_{2,p}$ by

$$_{\mathbf{A}}\|\omega\|_{2,p}^2=\int_E|\Gamma(\mathbf{A})^p\omega(x)|^2\,d\mu(x).\tag{2.11}$$

It is not difficult to see that $\Gamma(\mathbf{A})^p=\Gamma(\mathbf{A}^p)$. By (2.7), each $\omega$ in $\mathscr{P}_{\mathbf{A}}$ has the following expression;

$$\omega(x)=\sum_{n=0}^{\infty}\langle :x^{\otimes n}:,\hat{g}_n\rangle,\qquad \hat{g}_n\in C^\infty(\mathbf{A})^{\hat{\otimes}n},$$

where $C^\infty(\mathbf{A})^{\hat{\otimes}n}=\overbrace{C^\infty(\mathbf{A})\,\hat{\otimes}\,C^\infty(\mathbf{A})\,\hat{\otimes}\cdots\hat{\otimes}\,C^\infty(\mathbf{A})}^{n\,\text{times}}$ is the set of finite linear combinations of the form $\xi_1\,\hat{\otimes}\,\xi_2\,\hat{\otimes}\cdots\hat{\otimes}\,\xi_n$ with $\xi_i\in C^\infty(\mathbf{A})$, $i=1,2,\cdots,n$. We note that there exists a natural number $k(\omega)$ such that $\hat{g}_n=0$ for $n\geq k(\omega)$. Since

$$\hat{g}_n=\sum_{i=1}^{m(n)}a_i(n)\xi_{i_1}\,\hat{\otimes}\,\xi_{i_2}\,\hat{\otimes}\cdots\hat{\otimes}\,\xi_{i_n},\quad \xi_{i_k}\in C^\infty(\mathbf{A}),\ k=1,2,\cdots,n,$$

by (2.9), ${}_{\mathbf{A}}\|\cdot\|_{2,p}^2$ can be also represented as

$$_{\mathbf{A}}\|\omega\|_{2,p}^2=\sum_{n=0}^{\infty}n!\,|(\mathbf{A}^p)^{\otimes n}\hat{g}_n|_{\mathscr{H}^{\hat{\otimes}n}}^2,\tag{2.12}$$

where

$$(\mathbf{A}^p)^{\otimes n}=\mathbf{A}^p\otimes\mathbf{A}^p\otimes\cdots\otimes\mathbf{A}^p.$$

For $p\geq 0$, $(S_{\mathbf{A}})_p$ is the completion of $\mathscr{P}_{\mathbf{A}}$ with respect to the semi-norm ${}_{\mathbf{A}}\|\cdot\|_{2,p}$. We define the fundamental space $(S_{\mathbf{A}})$ of the Hida distributions on $E$ by

$$(S_{\mathbf{A}})=\bigcap_{p\geq 0}(S_{\mathbf{A}})_p.\tag{2.13}$$

Let $(S_{\mathbf{A}})_{-p}$ be the topological dual space of $(S_{\mathbf{A}})_p$. Then we have

$$(S_{\mathbf{A}})'=\bigcup_{p\geq 0}(S_{\mathbf{A}})_{-p}.\tag{2.14}$$

There are several criteria for nuclearity of a fundamental space of Hida's type, such as ([2], [10]). Here we show a sufficient condition.

PROPOSITION 2.1. *Let* $\mathbf{Z}$ *be a positive self-adjoint operator in* $\mathscr{H}$ *with eigenvalues* $\lambda_k > 1$, $k = 0, 1, \cdots$, *such that* $\sum_{k=0}^{\infty} \lambda_k^{-\gamma} < +\infty$ *for some* $\gamma > 0$. *Then* $(S_{\mathbf{Z}})$ *is a nuclear space.*

*Proof.* Here we mimic a proof from [11]. It is sufficient to show that for any $p \geq 0$, there exist an $s > 0$ such that the inclusion map $\iota: (S_{\mathbf{Z}})_{p+2s} \to (S_{\mathbf{Z}})_p$ is a nuclear operator. For the notational simplicity, we prove the assertion in the case where $p = 0$. This is equivalent to show that the inclusion map $\iota: L^2(E, \mu) \to (S_{\mathbf{Z}})_{-s}$ is a Hilbert-Schmidt operator. Let $\mathbf{N}_0 = \{0, 1, 2, \cdots\}$ and let $\mathbf{I}_n$ be the set of all ordered $n$-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_1 \leq \cdots \leq \alpha_n$ in $\mathbf{N}_0^n$. For $\alpha \in \mathbf{I}_n$, define

$$n_k(\alpha) = \#\{j: \alpha_j = k\}, \qquad n(\alpha)! = \prod_{k=0}^{\infty} n_k(\alpha)!.$$

Let $\{e_k: k = 0, 1, 2, \cdots\}$ be a complete orthonormal system associated with eigenfunctions of $\mathbf{Z}$ such that $\mathbf{Z}e_k = \lambda_k e_k$, $k = 0, 1, 2, \cdots$. For each $\alpha \in \mathbf{I}_n$ we set

$$\begin{aligned} \mathrm{H}_\alpha(x) &= 1 \qquad \text{for } n = 0, \\ \mathrm{H}_\alpha(x) &= (n(\alpha)!)^{-\frac{1}{2}} \prod_{k=0}^{\infty} :\langle x, e_k \rangle^{n_k(\alpha)}: \qquad \text{for } n \neq 0 \\ &= (n(\alpha)!)^{-\frac{1}{2}} \prod_{k=0}^{\infty} \langle :x^{\otimes n_k(\alpha)}:, e_k^{\hat{\otimes} n_k(\alpha)} \rangle. \end{aligned}$$

Moreover we have

$$\begin{aligned} \Gamma(\mathbf{Z})^r \mathrm{H}_\alpha(x) &= (n(\alpha)!)^{-\frac{1}{2}} \prod_{k=0}^{\infty} \langle :x^{\otimes n_k(\alpha)}:, (\mathbf{Z}^r)^{\otimes n_k(\alpha)} e_k^{\hat{\otimes} n_k(\alpha)} \rangle \\ &= \prod_{k=0}^{\infty} (\lambda_k^r)^{n_k(\alpha)} \mathrm{H}_\alpha(x), \qquad \text{for } r \in \mathbf{R} \end{aligned}$$

and hence

$$\begin{aligned} \sum_\alpha {}_{\mathbf{Z}}\|\iota \mathrm{H}_\alpha\|_{2,-s}^2 &= \sum_\alpha |\Gamma(\mathbf{Z})^{-s} \mathrm{H}_\alpha|_{L^2(E,\mu)}^2 \\ &= \sum_\alpha \left\{ \frac{1}{\prod_{k=0}^{\infty} \lambda_k^{n_k(\alpha)}} \right\}^{2s} \\ &\leq \sum_{n=0}^{\infty} \left[ \sum_{k_1, \ldots, k_n = 0}^{\infty} \left\{ \frac{1}{\lambda_{k_1} \cdots \lambda_{k_n}} \right\}^{2s} \right] \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{\infty} \lambda_k^{-2s} \right\}^n < \infty, \end{aligned}$$

provided that $\sum_{k=0}^{\infty} \lambda_k^{-2s} < 1$, which is valid for sufficiently large $s$ by the assumptions $\sum_{k=0}^{\infty} \lambda_k^{-\gamma} < +\infty$ and $\lambda_k > 1$. This, together with the fact that $\{\mathrm{H}_\alpha(x) : \alpha \in \mathbf{I}_n, n = 0, 1, 2, \cdots\}$ forms a complete orthonormal system of $L^2(E, \mu)$, implies the inclusion map $\iota : (S_{\mathbf{Z}})_{p+2s} \to (S_{\mathbf{Z}})_p$ is a Hilbert-Schmidt operator.

EXAMPLE 2.1 ([4], [11]). *Let* $\mathbf{Z}$ *be the operator*

$$\mathbf{Z} = -\left(\frac{d}{dx}\right)^2 + x^2 + 1. \tag{2.15}$$

*Then*

$$\mathbf{Z} e_n = (2n + 2) e_n, \; n = 0, 1, 2, \cdots$$

*where* $\{e_n : n = 0, 1, 2, \cdots\}$ *is the complete orthonormal system consisting of Hermite functions in* $L^2(\mathbf{R})$. *If we take* $E = S'(\mathbf{R})$, *then* $(S_{\mathbf{Z}})$ *becomes a nuclear space by Proposition* 2.1 *and originally it is called the fundamental space of the Hida distributions.*

## 3. Proof of Theorem 1.1

Before proving Theorem 1.1, we define the second quantization operator $d\Gamma(\mathbf{A})$ of a self-adjoint operator $\mathbf{A}$ as

$$\begin{aligned} & d\Gamma(\mathbf{A}) : \langle x, \xi_1 \rangle \cdots \langle x, \xi_n \rangle : \\ &= \sum_{i=1}^{n} : \langle x, \xi_1 \rangle \cdots \langle x, \mathbf{A}\xi_i \rangle \cdots \langle x, \xi_n \rangle : \\ &= \langle : x^{\otimes n} :, \mathrm{N}_{\mathbf{A}}(\xi_1 \hat{\otimes} \cdots \hat{\otimes} \xi_n) \rangle, \end{aligned} \tag{3.1}$$

where

$$\mathrm{N}_{\mathbf{A}} = \mathbf{A} \otimes I \otimes \cdots \otimes I + I \otimes \mathbf{A} \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes \mathbf{A}.$$

Let $\mathbf{B}$ be a self-adjoint operator which satisfies the conditions in Theorem 1.1 and set $\tilde{\mathbf{B}} = \mathbf{B} + 2\mathrm{I}$. Consider the fundamental space $S = (S_{\tilde{\mathbf{B}}})$ of the Hida distributions on $E$. Take any $\omega(x) \in \mathscr{P}_{\tilde{\mathbf{B}}}$ then the following expression holds,

$$\omega(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, \hat{h}_n \rangle, \qquad \hat{h}_n \in C^{\infty}(\tilde{\mathbf{B}})^{\hat{\otimes} n}. \tag{3.2}$$

By the assumption (1) of Theorem 1.1, $\omega(x) \in \mathscr{D}(d\Gamma(\mathbf{A}))$, so that

$$d\Gamma(\mathbf{A})\omega(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, \mathrm{N}_{\mathbf{A}} \hat{h}_n \rangle. \tag{3.3}$$

Define the Hilbert space $\mathscr{H}_{\tilde{\mathbf{B}}^l}$ for any natural number $l$ by

$$\mathscr{H}_{\tilde{\mathbf{B}}^l} = \mathscr{D}(\tilde{\mathbf{B}}^l) \equiv \{h \in \mathscr{H} : |\tilde{B}^l h|_{\mathscr{H}} < \infty\},$$

where the inner product of the Hilbert space $\mathscr{H}_{\tilde{\mathbf{B}}^l}$ is given by $(\cdot, \cdot)_{\mathscr{H}_{\tilde{\mathbf{B}}^l}} = (\tilde{\mathbf{B}}^l \cdot, \tilde{\mathbf{B}}^l \cdot)_{\mathscr{H}}$.

Noticing that $C^\infty(\mathbf{B}) = \bigcap_{n=1}^\infty \mathscr{D}(\mathbf{B}^n)$ is a complete metric space equipped with a countable system of semi norms $|\cdot|_n = |\mathbf{B}^n \cdot|_{\mathscr{H}}$, $n = 1, 2, \cdots$, for any natural number $k$, we have, by Baire's category theorem, a natural number $l_1 (\geq k)$ and a constant $c_1 \geq 1$ such that

$$|\tilde{\mathbf{B}}^k \mathbf{A} h|_{\mathscr{H}} \leq c_1 |\tilde{\mathbf{B}}^{l_1} h|_{\mathscr{H}}, \qquad h \in C^\infty(\tilde{\mathbf{B}}). \tag{3.4}$$

For any $1 \leq n \leq k$ and any $\xi \in C^\infty(\mathbf{B})$ such that $|\xi|_{\mathscr{H}} \leq 1$, the assumption (2) of Theorem 1.1 implies that $(\mathbf{B}^n \mathbf{A} h, \xi)_{\mathscr{H}} = (h, \mathbf{A}\mathbf{B}^n \xi)_{\mathscr{H}}$ is continuous in $h$ on $C^\infty(\mathbf{B})$, so that noticing that

$$|\mathbf{B}^k \mathbf{A} h|_{\mathscr{H}} = \sup \{(\mathbf{B}^k \mathbf{A} h, \xi)_{\mathscr{H}};\ \xi \in C^\infty(\mathbf{B}),\ |\xi|_{\mathscr{H}} \leq 1\},$$

we see that the norm $|\mathbf{B}^n \mathbf{A} h|_{\mathscr{H}}$ is lower-semicontinuous in $h$ on $C^\infty(\mathbf{B})$. The assumption (3) of Theorem 1.1 yields that

$$\{h \in C^\infty(\mathbf{B});\ |\mathbf{B}^n \mathbf{A} h|_{\mathscr{H}} < \infty\} = \bigcup_{m=0}^{\infty} \{h \in C^\infty(\mathbf{B});\ |\mathbf{B}^n \mathbf{A} h|_{\mathscr{H}} \leq m\} \supset C^\infty(\mathbf{B}).$$

Baire's category theorem then asserts that for some $a > 0$ the closed set $\Delta_a = \{h \in C^\infty(\mathbf{B});\ |\mathbf{B}^n \mathbf{A} h|_{\mathscr{H}} \leq a\}$ contains a ball of the form

$$h_0 + \{h \in C^\infty(\mathbf{B});\ |h|_{l(n)} \leq b\}$$

with $h_0 \in C^\infty(\mathbf{B})$ and $b > 0$. If $|h|_{l(n)} \leq b$, $h_0 - h \in \Delta_a$. Since $|\mathbf{B}^n \mathbf{A} h|_{\mathscr{H}}$ is symmetric, so $-h_0 + h \in \Delta_a$. Since $h_0 + h \in \Delta_a$, we have

$$|\mathbf{B}^n \mathbf{A}(2h)|_{\mathscr{H}} = |\mathbf{B}^n \mathbf{A}(-h_0 + h + h_0 + h)|_{\mathscr{H}} \leq 2a.$$

Thus we get for $1 \leq n \leq k$,

$$|\mathbf{B}^n \mathbf{A} h|_{\mathscr{H}} \leq a b^{-1} |\mathbf{B}^{l(n)} h|_{\mathscr{H}}, \qquad h \in C^\infty(\mathbf{B}).$$

For $p, q$ given in Theorem 1.1, set $p \vee q = \max\{p, q\}$. Then the lower-semicontinuity of $|\mathbf{A} h|_{\mathscr{H}}$ on $\mathscr{D}(\mathbf{B}^{p \vee q})$ is proved similarly by the assumptions (1) and (2) of Theorem 1.1, so that by Baire's category theorem again we get

$$|\mathbf{A} h|_{\mathscr{H}} \leq \text{const.}\, |\mathbf{B}^{p \vee q} h|_{\mathscr{H}}, \qquad h \in \mathscr{D}(\mathbf{B}^{p \vee q}).$$

Since $C^\infty(\tilde{\mathbf{B}}) \subset C^\infty(\mathbf{B}) \subset \mathscr{D}(\mathbf{B}^{p \vee q})$, the two inequalities above hold for $h \in C^\infty(\tilde{\mathbf{B}})$, which together with

$$|\tilde{\mathbf{B}}^k \mathbf{A} h|_{\mathscr{H}} \leq \sum_{n=0}^{k} \frac{k!}{n!(k-n)!} 2^{k-n} |\mathbf{B}^n \mathbf{A} h|_{\mathscr{H}}$$

completes the proof of (3.4)

Here we prepare a lemma concerning with the operator norm of tensor product of linear operators on Hilbert space, which will be used later. Given Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ and a linear operator T from $\mathscr{H}_1$ to $\mathscr{H}_2$, we denote the operator norm of T by

$$\| \mathrm{T} \|_{\mathscr{H}_1 \to \mathscr{H}_2} = \sup_{x \in \mathscr{H}_1} \frac{|\mathrm{T}x|_{\mathscr{H}_2}}{|x|_{\mathscr{H}_1}}.$$

We note that (3.4) implies

$$\| \mathbf{A} \|_{\mathscr{H}_{\tilde{\mathbf{B}}^{l_1}} \to \mathscr{H}_{\tilde{\mathbf{B}}^k}} \leq c_1. \tag{3.5}$$

For simplicity we use the notation $\mathscr{H}_{\tilde{\mathbf{B}}^l}^{\hat{\otimes} n}$ instead of $(\mathscr{H}_{\tilde{\mathbf{B}}^l})^{\hat{\otimes} n}$.

Let $U_i$, $i = 1, 2, \cdots, n$ be bounded linear operators from $\mathscr{H}_{\tilde{\mathbf{B}}^l}$ to $\mathscr{H}_{\tilde{\mathbf{B}}^k}$ for any natural number $l$ and $k$ such that $l \geq k$. We have the following lemma, which is an extended version of the proposition of [12] on p. 299.

Lemma 3.1. *Let $U_i$, $i = 1, 2, \cdots, n$ be bounded linear operators from $\mathscr{H}_{\tilde{\mathbf{B}}^l}$ to $\mathscr{H}_{\tilde{\mathbf{B}}^k}$. Then*

$$\| U_1 \otimes U_2 \otimes \cdots \otimes U_n \|_{\mathscr{H}_{\tilde{\mathbf{B}}^l}^{\hat{\otimes} n} \to \mathscr{H}_{\tilde{\mathbf{B}}^k}^{\otimes n}} \leq \| U_1 \|_{\mathscr{H}_{\tilde{\mathbf{B}}^l} \to \mathscr{H}_{\tilde{\mathbf{B}}^k}} \| U_2 \|_{\mathscr{H}_{\tilde{\mathbf{B}}^l} \to \mathscr{H}_{\tilde{\mathbf{B}}^k}} \cdots \| U_n \|_{\mathscr{H}_{\tilde{\mathbf{B}}^l} \to \mathscr{H}_{\tilde{\mathbf{B}}^k}}.$$

Proof. Let $\{\theta_k : k = 0, 1, 2, \cdots\}$ be a complete orthonormal basis of $\mathscr{H}_{\tilde{\mathbf{B}}^l}$, $\{\zeta_k : k = 0, 1, 2, \cdots\}$ a complete orthonormal basis of $\mathscr{H}_{\tilde{\mathbf{B}}^k}$ and $\Sigma c_{k_1 \ldots k_n}(\theta_{k_1} \otimes \cdots \otimes \theta_{k_i} \otimes \zeta_{k_{i+1}} \otimes \cdots \otimes \zeta_{k_n})$ a finite sum in the space $\mathscr{H}_{\tilde{\mathbf{B}}^l}^{\otimes i} \otimes \mathscr{H}_{\tilde{\mathbf{B}}^l}^{\otimes n-i}$. We get

$$\begin{aligned}
&|\mathrm{I} \otimes \cdots \otimes \mathrm{I} \otimes U_i \otimes \mathrm{I} \otimes \cdots \otimes \mathrm{I} \\
&\quad (\textstyle\sum c_{k_1 \ldots k_n} \theta_{k_1} \otimes \cdots \otimes \theta_{k_i} \otimes \zeta_{k_{i+1}} \otimes \cdots \otimes \zeta_{k_n})|^2_{\mathscr{H}_{\tilde{\mathbf{B}}^l}^{\otimes i-1} \otimes \mathscr{H}_{\tilde{\mathbf{B}}^k}^{\otimes n-i+1}} \\
&= \textstyle\sum \sum c_{k_1 \ldots k_n} c_{k_1 \ldots k_{i-1} l_i k_{i+1} \ldots k_n} (U_i \theta_{k_i}, U_i \theta_{l_i})_{\mathscr{H}_{\tilde{\mathbf{B}}^k}} \\
&= \sum_{k_1 \ldots k_{i-1} k_{i+1} \ldots k_n} |\sum_{l_i} c_{k_1 \ldots k_{i-1} l_i k_{i+1} \ldots k_n} U_i \theta_{l_i}|^2_{\mathscr{H}_{\tilde{\mathbf{B}}^k}} \\
&\leq \| U_i \|^2_{\mathscr{H}_{\tilde{\mathbf{B}}^l} \to \mathscr{H}_{\tilde{\mathbf{B}}^k}} \sum_{k_1 \ldots k_{i-1} k_{i+1} \ldots k_n} |\sum_{l_i} c_{k_1 \ldots k_{i-1} l_i k_{i+1} \ldots k_n} \theta_{l_i}|^2_{\mathscr{H}_{\tilde{\mathbf{B}}^l}} \\
&= \| U_i \|^2_{\mathscr{H}_{\tilde{\mathbf{B}}^l} \to \mathscr{H}_{\tilde{\mathbf{B}}^k}} |\textstyle\sum c_{k_1 \ldots k_n} (\theta_{k_1} \otimes \cdots \otimes \theta_{k_i} \otimes \zeta_{k_{i+1}} \otimes \cdots \otimes \zeta_{k_n})|^2_{\mathscr{H}_{\tilde{\mathbf{B}}^l}^{\otimes i} \otimes \mathscr{H}_{\tilde{\mathbf{B}}^k}^{\otimes n-i}}
\end{aligned}$$

so that

$$\| \mathrm{I} \otimes \mathrm{I} \otimes U_i \otimes \mathrm{I} \otimes \cdots \otimes \mathrm{I} \|_{\mathscr{H}_{\tilde{\mathbf{B}}^l}^{\otimes i} \otimes \mathscr{H}_{\tilde{\mathbf{B}}^k}^{\otimes n-i} \to \mathscr{H}_{\tilde{\mathbf{B}}^l}^{\otimes i-1} \otimes \mathscr{H}_{\tilde{\mathbf{B}}^k}^{\otimes n-i+1}} \leq \| U_i \|_{\mathscr{H}_{\tilde{\mathbf{B}}^l} \to \mathscr{H}_{\tilde{\mathbf{B}}^k}}.$$

Hence

$$\|U_1 \otimes U_2 \otimes \cdots \otimes U_n\|_{\mathscr{H}^{\otimes n}_{\tilde{\mathbf{B}}^l} \to \mathscr{H}^{\otimes n}_{\tilde{\mathbf{B}}^k}} \leq \|U_1 \otimes \mathrm{I} \otimes \cdots \otimes \mathrm{I}\|_{\mathscr{H}_{\tilde{\mathbf{B}}^l} \otimes \mathscr{H}^{\otimes n-1}_{\tilde{\mathbf{B}}^k} \to \mathscr{H}^{\otimes n}_{\tilde{\mathbf{B}}^k}}$$

$$\times \|\mathrm{I} \otimes U_2 \otimes \mathrm{I} \otimes \cdots \otimes \mathrm{I}\|_{\mathscr{H}^{\otimes 2}_{\tilde{\mathbf{B}}^l} \otimes \mathscr{H}^{\otimes n-2}_{\tilde{\mathbf{B}}^k} \to \mathscr{H}_{\tilde{\mathbf{B}}^l} \otimes \mathscr{H}^{\otimes n-1}_{\tilde{\mathbf{B}}^k}} \cdots \|\mathrm{I} \otimes \cdots \otimes \mathrm{I} \otimes U_n\|_{\mathscr{H}^{\otimes n}_{\tilde{\mathbf{B}}^l} \to \mathscr{H}^{\otimes n-1}_{\tilde{\mathbf{B}}^l} \otimes \mathscr{H}_{\tilde{\mathbf{B}}^k}}$$

$$\leq \|U_1\|_{\mathscr{H}_{\tilde{\mathbf{B}}^l} \to \mathscr{H}_{\tilde{\mathbf{B}}^k}} \|U_2\|_{\mathscr{H}_{\tilde{\mathbf{B}}^l} \to \mathscr{H}_{\tilde{\mathbf{B}}^k}} \cdots \|U_n\|_{\mathscr{H}_{\tilde{\mathbf{B}}^l} \to \mathscr{H}_{\tilde{\mathbf{B}}^k}}. \tag{3.6}$$

On the other hand, by making use of (2.5) and (2.6) we have

$$\begin{aligned}
&\|U_1 \otimes U_2 \otimes \cdots \otimes U_n\|^2_{\mathscr{H}^{\hat{\otimes} n}_{\tilde{\mathbf{B}}^l} \to \mathscr{H}^{\hat{\otimes} n}_{\tilde{\mathbf{B}}^k}} \\
&= \sup_{\hat{f} \in \mathscr{H}^{\hat{\otimes} n}_{\tilde{\mathbf{B}}^l}} ((U_1 \otimes U_2 \otimes \cdots \otimes U_n)\hat{f}, (U_1 \otimes U_2 \otimes \cdots \otimes U_n)\hat{f})_{\mathscr{H}^{\otimes n}_{\tilde{\mathbf{B}}^k}} / (\hat{f}, \hat{f})_{\mathscr{H}^{\hat{\otimes} n}_{\tilde{\mathbf{B}}^l}} \\
&\leq \sup_{f \in \mathscr{H}^{\otimes n}_{\tilde{\mathbf{B}}^l}} ((U_1 \otimes U_2 \otimes \cdots \otimes U_n)f, (U_1 \otimes U_2 \otimes \cdots \otimes U_n)f)_{\mathscr{H}^{\otimes n}_{\tilde{\mathbf{B}}^k}} / (f, f)_{\mathscr{H}^{\otimes n}_{\tilde{\mathbf{B}}^l}} \\
&= \|U_1 \otimes U_2 \otimes \cdots \otimes U_n\|^2_{\mathscr{H}^{\otimes n}_{\tilde{\mathbf{B}}^l} \to \mathscr{H}^{\otimes n}_{\tilde{\mathbf{B}}^k}}.
\end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7) we obtain that

$$\|U_1 \otimes U_2 \otimes \cdots \otimes U_n\|_{\mathscr{H}^{\hat{\otimes} n}_{\tilde{\mathbf{B}}^l} \to \mathscr{H}^{\hat{\otimes} n}_{\tilde{\mathbf{B}}^k}} \leq \|U_1\|_{\mathscr{H}_{\tilde{\mathbf{B}}^l} \to \mathscr{H}_{\tilde{\mathbf{B}}^k}} \|U_2\|_{\mathscr{H}_{\tilde{\mathbf{B}}^l} \to \mathscr{H}_{\tilde{\mathbf{B}}^k}} \cdots \|U_n\|_{\mathscr{H}_{\tilde{\mathbf{B}}^l} \to \mathscr{H}_{\tilde{\mathbf{B}}^k}}.$$

■

Now we return to the proof of Theorem 1.1. Since $\mathrm{N}_{\mathbf{A}}\hat{h}_n$ is also symmetric, we have by (2.6),

$$\begin{aligned}
&|\Gamma(\tilde{\mathbf{B}})^k d\Gamma(\mathbf{A})\langle :x^{\otimes n}:, \hat{h}_n\rangle|^2_{L^2(E,d\mu)} = n!\,|((\tilde{\mathbf{B}}^k)^{\otimes n}\mathrm{N}_{\mathbf{A}})\hat{h}_n|^2_{\mathscr{H}^{\hat{\otimes} n}} \\
&= n!\,|\mathrm{N}_{\mathbf{A}}\hat{h}_n|^2_{\mathscr{H}^{\hat{\otimes} n}_{\tilde{\mathbf{B}}^k}} = n!\,|\mathrm{N}_{\mathbf{A}}\hat{h}_n|^2_{\mathscr{H}^{\otimes n}_{\tilde{\mathbf{B}}^k}} \\
&\leq n!\Big(\sum_{i=1}^n |(I \otimes \cdots \otimes I \otimes \overset{i}{\check{\mathbf{A}}} \otimes I \otimes \cdots \otimes I)\hat{h}_n|_{\mathscr{H}^{\otimes n}_{\tilde{\mathbf{B}}^k}}\Big)^2 \\
&\leq n!\Big(\sum_{i=1}^n \|I \otimes \cdots \otimes I \otimes \overset{i}{\check{\mathbf{A}}} \otimes I \otimes \cdots \otimes I\|_{\mathscr{H}^{\hat{\otimes} n}_{\tilde{\mathbf{B}}^{l_1}} \to \mathscr{H}^{\otimes n}_{\tilde{\mathbf{B}}^k}} |\hat{h}_n|_{\mathscr{H}^{\hat{\otimes} n}_{\tilde{\mathbf{B}}^{l_1}}}\Big)^2 \\
&= n!\Big(\sum_{i=1}^n \|I \otimes \cdots \otimes I \otimes \overset{i}{\check{\mathbf{A}}} \otimes I \otimes \cdots \otimes I\|_{\mathscr{H}^{\hat{\otimes} n}_{\tilde{\mathbf{B}}^{l_1}} \to \mathscr{H}^{\otimes n}_{\tilde{\mathbf{B}}^k}}\Big)^2 |(\tilde{\mathbf{B}}^{l_1})^{\otimes n}\hat{h}_n|^2_{\mathscr{H}^{\hat{\otimes} n}} \\
&\leq n!\,n^2 \|\mathbf{A}\|^2_{\mathscr{H}_{\tilde{\mathbf{B}}^{l_1}} \to \mathscr{H}_{\tilde{\mathbf{B}}^k}} |(\tilde{\mathbf{B}}^{l_1})^{\otimes n}\hat{h}_n|^2_{\mathscr{H}^{\hat{\otimes} n}} \qquad \text{(by Lemma 3.1)} \\
&\leq n!\,n^2 c_1^2 |(\tilde{\mathbf{B}}^{l_1})^{\otimes n}\hat{h}_n|^2_{\mathscr{H}^{\hat{\otimes} n}} \qquad \text{(by (3.5))}
\end{aligned}$$

and hence

$$\begin{aligned}
{}_{\tilde{\mathbf{B}}}\|d\Gamma(\mathbf{A})\omega\|^2_{2,k} &= |\Gamma(\tilde{\mathbf{B}})^k d\Gamma(\mathbf{A})\omega|^2_{L^2(E,d\mu)} \\
&= \sum_{n=0}^{\infty} |\Gamma(\tilde{\mathbf{B}})^k d\Gamma(\mathbf{A})\langle :x^{\otimes n}:, \hat{h}_n\rangle|^2_{L^2(E,d\mu)}
\end{aligned}$$

$$\leq \sum_{n=0}^{\infty} n!n^2 c_1^2 |(\tilde{\mathbf{B}}^{l_1})^{\otimes n}\hat{h}_n|^2_{\mathscr{H}^{\otimes n}}. \tag{3.8}$$

Since for the natural number $q$ given in the assumption (2) of Theorem 1.1, $\tilde{\mathbf{B}}^q \geq 2^q$, $(\tilde{\mathbf{B}}^q)^{-1} \leq 2^{-q}$, we choose a natural number $m_1$ such that $(\tilde{\mathbf{B}}^q)^{-m_1} \leq (2c_1)^{-1}$. Then the right hand side of (3.8) is dominated by

$$\sum_{n=0}^{\infty} n!n^2 c_1^2 \left(\frac{1}{2c_1}\right)^{2n} |(\tilde{\mathbf{B}}^{l_1+qm_1})^{\otimes n}\hat{h}_n|^2_{\mathscr{H}^{\otimes n}} \leq \sum_{n=0}^{\infty} n! |(\tilde{\mathbf{B}}^{l_1+qm_1})^{\otimes n}\hat{h}_n|^2_{\mathscr{H}^{\otimes n}}$$
$$= |\Gamma(\tilde{\mathbf{B}})^{l_1+qm_1}\omega|^2_{L^2(E,d\mu)} = {}_{\tilde{\mathbf{B}}}\|\omega\|^2_{2,l_1+qm_1}.$$

Therefore we have $d\Gamma(\mathbf{A})S \subset S$. As noted in Proposition 2.1, $S$ is a nuclear space. Thus the proof of the first half of Theorem 1.1 is completed.

Now we will prove the second half of the theorem. Note that $e^{-td\Gamma(\mathbf{A})}$ is the self-adjoint operator defined on $L^2(E, \mu)$ such that

$$e^{-td\Gamma(\mathbf{A})} = \Gamma(e^{-t\mathbf{A}}). \tag{3.9}$$

This follows from the identity

$$\frac{d}{dt}\Gamma(e^{-t\mathbf{A}}){:}\langle x, \xi_1{}^1\rangle\cdots\langle x, \xi_n\rangle{:}= -d\Gamma(\mathbf{A})\Gamma(e^{-t\mathbf{A}}){:}\langle x, \xi_1\rangle\cdots\langle x, \xi_n\rangle{:}$$

which is verified by the induction on $n$. By (2.10), (3.2) and (3.9), we have the following expression;

$$e^{-td\Gamma(\mathbf{A})}\omega(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, (e^{-t\mathbf{A}})^{\otimes n}\hat{h}_n\rangle. \tag{3.10}$$

For any natural number $k$, by the assumption (4) of Theorem 1.1 and the manner similar to that in (3.4) we find a natural number $l_2(\geq k)$ and a constant $c_2 \geq 1$ such that

$$|\tilde{\mathbf{B}}^k e^{-t\mathbf{A}}h|_{\mathscr{H}} \leq c_2 |\tilde{\mathbf{B}}^{l_2}h|_{\mathscr{H}}, \qquad h \in C^\infty(\tilde{\mathbf{B}}). \tag{3.11}$$

Then we have

$$\begin{aligned}
&|\Gamma(\tilde{\mathbf{B}})^k e^{-td\Gamma(\mathbf{A})}\langle :x^{\otimes n}:, \hat{h}_n\rangle|^2_{L^2(E,d\mu)} \\
&= n!|(\tilde{\mathbf{B}}^k e^{-t\mathbf{A}})^{\otimes n}\hat{h}_n|^2_{\mathscr{H}^{\otimes n}} \\
&= n!|(e^{-t\mathbf{A}})^{\otimes n}\hat{h}_n|^2_{\mathscr{H}^{\otimes n}_{\tilde{\mathbf{B}}^k}} \\
&\leq n!\|(e^{-t\mathbf{A}})^{\otimes n}\|^2_{\mathscr{H}^{\otimes n}_{\tilde{\mathbf{B}}^{l_2}} \to \mathscr{H}^{\otimes n}_{\tilde{\mathbf{B}}^k}} |\hat{h}_n|^2_{\mathscr{H}^{\otimes n}_{\tilde{\mathbf{B}}^{l_2}}} \\
&= n!\|(e^{-t\mathbf{A}})^{\otimes n}\|^2_{\mathscr{H}^{\otimes n}_{\tilde{\mathbf{B}}^{l_2}} \to \mathscr{H}^{\otimes n}_{\tilde{\mathbf{B}}^k}} |(\tilde{\mathbf{B}}^{l_2})^{\otimes n}\hat{h}_n|^2_{\mathscr{H}^{\otimes n}} \\
&\leq n!\|e^{-t\mathbf{A}}\|^{2n}_{\mathscr{H}_{\tilde{\mathbf{B}}^{l_2}} \to \mathscr{H}_{\tilde{\mathbf{B}}^k}} |(\tilde{\mathbf{B}}^{l_2})^{\otimes n}\hat{h}^n|^2_{\mathscr{H}^{\otimes n}} \qquad \text{(by Lemma 3.1)}
\end{aligned}$$

$$\leq n!\, c_2^{2n} |(\tilde{\mathbf{B}}^{l_2})^{\otimes n} \hat{h}_n|^2_{\mathscr{H}^{\otimes n}} \qquad \text{(by (3.11))}$$

and

$$\begin{aligned} {}_{\tilde{\mathbf{B}}}\| e^{-td\Gamma(\mathbf{A})}\omega \|^2_{2,k} &= |\Gamma(\tilde{\mathbf{B}})^k e^{-td\Gamma(\mathbf{A})}\omega|^2_{L^2(E,d\mu)} \\ &= \sum_{n=0}^{\infty} |\Gamma(\tilde{\mathbf{B}})^k e^{-td\Gamma(\mathbf{A})} \langle :x^{\otimes n}:, \hat{h}_n \rangle|^2_{L^2(E,d\mu)} \\ &\leq \sum_{n=0}^{\infty} n!\, c_2^{2n} |(\tilde{\mathbf{B}}^{l_2})^{\otimes n} \hat{h}_n|^2_{\mathscr{H}^{\otimes n}}. \end{aligned} \tag{3.12}$$

In the same manner as the proof of the first half, we choose a natural number $m_2$ such that $(\tilde{\mathbf{B}}^q)^{-m_2} \leq c_2^{-1}$. Then the right hand side of (3.12) is dominated by

$$\sum_{n=0}^{\infty} n!\, c_2^{2n} \left(\frac{1}{c_2}\right)^{2n} |(\tilde{\mathbf{B}}^{l_2+qm_2})^{\otimes n} \hat{h}_n|^2_{\mathscr{H}^{\otimes n}} = {}_{\tilde{\mathbf{B}}}|\omega|^2_{2,l_2+qm_2},$$

which completes the proof of the second half of Theorem 1.1.

## 4. Inveriant nuclear space

In this section we discuss the conditions of Theorem 1.1. Especially the conditions (3) and (4) have been examined by several authors such as J. Fröhlich [3]. Let $\mathbf{A}$ and $\mathbf{B}$ be positive self-adjoint operators in the separable Hilbert space $\mathscr{H}$. In the sequel we denote by $c_i$, $i = 3, 4, \cdots$ positive constants.

### 4.1. Case where A and B are non-commutative.

LEMMA 4.1. *Let $\mathbf{D}$ and $\mathbf{B}$ be positive self-adjoint operators in $\mathscr{H}$ such that $C^\infty(\mathbf{B}) \subset C^\infty(\mathbf{D})$ and $\mathbf{D} \geq 1 + \varepsilon$ for some $\varepsilon > 0$. Suppose that $\mathbf{B}$ has a bounded inverse and for any natural number n, there exists a constant $c_3$ such that*

$$|\mathbf{B}^n \mathbf{D} f|_{\mathscr{H}} \leq c_3 |\mathbf{B}^n f|_{\mathscr{H}}, \qquad f \in C^\infty(\mathbf{B}).$$

*Then, for any natural number n, there exists a constant $c_4$ such that*

$$|\mathbf{B}^n e^{-t\mathbf{D}} f|_{\mathscr{H}} \leq c_4 |\mathbf{B}^n f|_{\mathscr{H}}, \qquad f \in C^\infty(\mathbf{B}).$$

PROOF. Since $\mathbf{B}^n \mathbf{D} \mathbf{B}^{-n}$ is a bounded operator, we have

$$\begin{aligned} |\mathbf{B}^n e^{-t\mathbf{D}} f|_{\mathscr{H}} &= |\mathbf{B}^n e^{-t\mathbf{D}} \mathbf{B}^{-n} \mathbf{B}^n f|_{\mathscr{H}} \\ &= |e^{-t\mathbf{B}^n\mathbf{D}\mathbf{B}^{-n}} \mathbf{B}^n f|_{\mathscr{H}} \\ &\leq c_4 |\mathbf{B}^n f|_{\mathscr{H}} \end{aligned}$$

■

Next, we havve the following proposition which is implied by Lemma 4.1.

Suppose that $\mathbf{D}$ is a positive self-adjoint operator in $\mathscr{H}$ and $\mathbf{D} \geq 1 + \varepsilon$ for some $\varepsilon > 0$. We choose and fix a complete orthonormal system $\{e_i : i = 0, 1, 2, \cdots\}$ of the Hilbert space $\mathscr{H}$ taken from $\mathscr{D}(\mathbf{D})$. Given $\{\lambda_i\}$ such that $\lambda_i \geq 1 + \varepsilon$, $\varepsilon > 0$, $\lambda_i \uparrow \infty$, define a positive self-adjoint operator $\mathbf{B}$ in $\mathscr{H}$ by $\mathbf{B}e_i = \lambda_i e_i$. Let $\{d_{i,j}\}_{i,j=0}^{\infty}$ be the infinite dimensional matrix such that

$$\mathbf{D}e_i = \sum_{j=0}^{\infty} d_{i,j} e_j.$$

PROPOSITION 4.1. *Suppose that*

$$M_n \equiv \sum_{i=0}^{\infty} \frac{\sum_{j=0}^{\infty} d_{i,j}^2 \lambda_j^{2n}}{\lambda_i^{2n}} < +\infty, \qquad \textit{for all } n \in \mathbf{N}. \tag{4.1}$$

*Then the conditions* (1), (2), (3), (4) *of Theorem* 1.1 *are satisfied for* $\mathbf{A} = \mathbf{D}$ *and* $\mathbf{B}$.

*Proof.* To prove (1), let $f = \sum a_i e_i$ be a finite sum. Then $f \in \mathscr{D}(\mathbf{D})$ and hence

$$\mathbf{D}f = \sum a_i \mathbf{D}e_i.$$

Therefore

$$\begin{aligned} |\mathbf{D}f|_{\mathscr{H}}^2 &= |\textstyle\sum a_i \mathbf{D}e_i|_{\mathscr{H}}^2 \\ &= |\textstyle\sum a_i \lambda_i \lambda_i^{-1} \mathbf{D}e_i|_{\mathscr{H}}^2 \\ &\leq (\textstyle\sum a_i^2 \lambda_i^2)(\sum \lambda_i^{-2} |\mathbf{D}e_i|_{\mathscr{H}}^2) \\ &\leq c_5 |\mathbf{B}f|_{\mathscr{H}}^2, \end{aligned}$$

where $c_5 = \sum \lambda_i^{-2} |\mathbf{D}e_i|_{\mathscr{H}}^2 = \sum \lambda_i^{-2} \sum_{j=0}^{\infty} d_{i,j}^2 \leq M_1$.

Since $\{\lambda_i^{-1} e_i : i = 0, 1, 2, \cdots\}$ forms a complete orthonormal system of $\mathscr{D}(\mathbf{B})$ and

$$\sum_{i=0}^{\infty} \left| \frac{1}{\lambda_i} e_i \right|_{\mathscr{H}}^2 = \sum_{i=0}^{\infty} \frac{1}{\lambda_i^2} \leq \sum_{i=0}^{\infty} \frac{|\mathbf{D}e_i|_{\mathscr{H}}^2}{\lambda_i^2} < +\infty,$$

then the identity map $\mathscr{D}(\mathbf{B}) \to \mathscr{H}$ becomes a Hilbert-Schamit operator. Of course $\mathbf{B}$ has a bounded inverse $\mathbf{B}^{-1}$ such that $\mathbf{B}^{-1} e_i = \lambda_i^{-1} e_i$.

Since for any integer $i$ and natural number $n$,

$$\sum_{j=0}^{\infty} d_{i,j}^2 \lambda_j^{2n} < \infty,$$

from (4.1), we have $\mathbf{D}e_i \in \mathscr{D}(\mathbf{B}^n)$, so that for any $f \in C^{\infty}(\mathbf{B})$ and for any fixed natural number N,

$$\begin{aligned}|\mathbf{B}^n \sum_{i=0}^{\mathrm{N}} (f, e_i)_{\mathscr{H}} \mathbf{D} e_i|_{\mathscr{H}}^2 &= |\sum_{i=0}^{\mathrm{N}} (f, e_i)_{\mathscr{H}} \mathbf{B}^n \mathbf{D} e_i|_{\mathscr{H}}^2 \\ &= |\sum_{i=0}^{\mathrm{N}} (f, e_i)_{\mathscr{H}} \lambda_i^n \lambda_i^{-n} \mathbf{B}^n \mathbf{D} e_i|_{\mathscr{H}}^2 \\ &\le (\sum_{i=0}^{\mathrm{N}} (f, e_i)_{\mathscr{H}}^2 \lambda_i^{2n})(\sum_{i=0}^{\mathrm{N}} \lambda_i^{-2n} |\mathbf{B}^n \mathbf{D} e_i|_{\mathscr{H}}^2) \\ &\le \mathrm{M}_n |\mathbf{B}^n f|_{\mathscr{H}}^2.\end{aligned}$$

This implies the condition (3) of Theorem 1.1.

Since $\mathbf{B}$ has a bounded inverse, Lemma 4.1 yields the condition (4) of Theorem 1.1. ■

PROPOSITION 4.2. *Suppose that* $\mathbf{D}$ *and* $\mathbf{B}$ *satisfy the assumptions of Proposition* 4.1 *and* $\mathbf{C}$ *is a positive self-adjoint operator in* $\mathscr{H}$ *such that* $\mathbf{C}$ *and* $\mathbf{B}$ *are commutative and* $\mathscr{D}(\mathbf{B}) \subset \mathscr{D}(\mathbf{C})$. *Then the conditions* (1), (2), (3), (4) *of Theorem* 1.1 *are satisfied for* $\mathbf{A} = \mathbf{C} + \mathbf{D}$ *and* $\mathbf{B}$.

PROOF. The condition (1) is obvious from $\mathscr{D}(\mathbf{B}) \subset \mathscr{D}(\mathbf{D})$ and $\mathscr{D}(\mathbf{B}) \subset \mathscr{D}(\mathbf{C})$. It has been already proved in Proposition 4.1 that the condition (2) is satisfied.

Since $\mathscr{D}(\mathbf{B}) \subset \mathscr{D}(\mathbf{C})$, there exists a constant $c_6$ such that

$$|\mathbf{C}f|_{\mathscr{H}} \le c_6 |\mathbf{B}f|_{\mathscr{H}}, \qquad f \in C^\infty(\mathbf{B}). \tag{4.2}$$

By the commutativity of $\mathbf{C}$ and $\mathbf{B}$ and (4.2), for any natural number $n$, we have

$$\begin{aligned}|\mathbf{B}^n \mathbf{A} f|_{\mathscr{H}}^2 &= |\mathbf{B}^n(\mathbf{C} + \mathbf{D})f|_{\mathscr{H}}^2 \\ &\le 2\{|\mathbf{B}^n \mathbf{C} f|_{\mathscr{H}}^2 + |\mathbf{B}\mathbf{D}f|_{\mathscr{H}}^2\} \\ &\le 2\{|\mathbf{C}\mathbf{B}^n f|_{\mathscr{H}}^2 + \mathrm{M}_n |\mathbf{B}^n f|_{\mathscr{H}}^2\} \\ &\le 2\{c_6 |\mathbf{B}^{n+1} f|_{\mathscr{H}}^2 + \mathrm{M}_n |\mathbf{B}^n f|_{\mathscr{H}}^2\} \\ &\le c_7 |\mathbf{B}^{n+1} f|_{\mathscr{H}}^2.\end{aligned}$$

To prove (4), it suffices to show that for some constant $0 < c_8 < \infty$

$$|\mathbf{B}^n e^{-t\mathbf{A}} f|_{\mathscr{H}} \le c_8 |\mathbf{B}^n f|_{\mathscr{H}}, \qquad f \in C^\infty(\mathbf{B}).$$

By Theorem 1.19 of Chap. IX in [8], we have an integral equation

$$e^{-t\mathbf{A}} f = e^{-t\mathbf{C}} f + \int_0^t e^{-(t-\tau)\mathbf{C}} \mathbf{D} e^{-\tau\mathbf{A}} f \, d\tau,$$

so that operating $\mathbf{B}^n$ on both sides of the above equality and using the commutativity of $\mathbf{C}$ and $\mathbf{B}$, the assumption of Proposition 4.1, we get

$$
\begin{aligned}
|\mathbf{B}^n e^{-t\mathbf{A}} f|_{\mathscr{H}} &\le |e^{-t\mathbf{C}} \mathbf{B}^n f|_{\mathscr{H}} + \int_0^t |e^{-(t-\tau)\mathbf{C}} \mathbf{B}^n \mathbf{D} e^{-\tau\mathbf{A}} f|_{\mathscr{H}} d\tau \\
&\le |\mathbf{B}^n f|_{\mathscr{H}} + \int_0^t |\mathbf{B}^n \mathbf{D} e^{-\tau\mathbf{A}} f|_{\mathscr{H}} d\tau \\
&\le |\mathbf{B}^n f|_{\mathscr{H}} + \sqrt{\mathbf{M}_n} \int_0^t |\mathbf{B}^n e^{-\tau\mathbf{A}} f|_{\mathscr{H}} d\tau.
\end{aligned}
$$

Gronwall's lamma then yields

$$|\mathbf{B}^n e^{-t\mathbf{A}} f|_{\mathscr{H}} \le e^{t\sqrt{\mathbf{M}_n}} |\mathbf{B}^n f|_{\mathscr{H}} =: c_8 |\mathbf{B}^n f|_{\mathscr{H}}.$$

This completes the proof of Proposition 4.2. ∎

We have a sufficient condition for (4) of Theorem 1.1. (See [3].)

PROPOSITION 4.3. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be positive self-adjoint operators in* $\mathscr{H}$ *such that* $\mathbf{A} \ge 1 + \varepsilon$, *for some* $\varepsilon > 0$ *and the condition* (3) *of Theorem* 1.1 *is satisfied. Suppose that for any natural numbers n and k, there exist a natural number* $\alpha(n) \ge n$ *and a constant* $C_k^n > 0$ *such that*

$$|(\mathbf{A}^k \mathbf{B}^n - \mathbf{B}^n \mathbf{A}^k) f|_{\mathscr{H}} \le C_k^n |\mathbf{B}^{\alpha(n)} f|_{\mathscr{H}}, \qquad f \in C^\infty(\mathbf{B}),$$

*and*

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} C_k^n < \infty, \qquad \textit{for any } t > 0.$$

*Then for any natural number n, there exists a constant* $C^n(t) > 0$ *such that*

$$|\mathbf{B}^n e^{-t\mathbf{A}} f|_{\mathscr{H}} \le C^n(t) |\mathbf{B}^{\alpha(n)} f|_{\mathscr{H}}.$$

PROOF. For any natural number $n$ and $f \in C^\infty(\mathbf{B})$, we have

$$
\begin{aligned}
\left| \mathbf{B}^n \sum_{k=0}^{\mathrm{N}} \frac{(-t)^k}{k!} \mathbf{A}^k f \right|_{\mathscr{H}} &= \left| \sum_{k=0}^{\mathrm{N}} \frac{(-t)^k}{k!} \mathbf{B}^n \mathbf{A}^k f \right|_{\mathscr{H}} \\
&\le \left| \sum_{k=0}^{\mathrm{N}} \frac{(-t)^k}{k!} \mathbf{A}^k \mathbf{B}^n f \right|_{\mathscr{H}} + \left| \sum_{k=0}^{\mathrm{N}} \frac{(-t)^k}{k!} (\mathbf{A}^k \mathbf{B}^n - \mathbf{B}^n \mathbf{A}^k) f \right|_{\mathscr{H}} \\
&\le \left| \sum_{k=0}^{\mathrm{N}} \frac{(-t)^k}{k!} \mathbf{A}^k \mathbf{B}^n f \right|_{\mathscr{H}} + \sum_{k=0}^{\mathrm{N}} \frac{t^k}{k!} C_k^n |\mathbf{B}^{\alpha(n)} f|_{\mathscr{H}}.
\end{aligned}
$$

From the above estimate it is seen that

$$|\mathbf{B}^n e^{-t\mathbf{A}} f|_{\mathscr{H}} \le |e^{-t\mathbf{A}} \mathbf{B}^n f|_{\mathscr{H}} + \sum_{k=0}^{\infty} \frac{t^k}{k!} C_k^n |\mathbf{B}^{\alpha(n)} f|_{\mathscr{H}}$$

$$\leq \left(1 + \sum_{k=0}^{\infty} \frac{t^k}{k!} C_k^n\right) |\mathbf{B}^{\alpha(n)} f|_{\mathscr{H}} \qquad (\text{by } A > 1).$$

REMARK 4.1. *If* $\mathbf{D}$ *is a bounded operator in Lemma* 4.1, *then the conclusion is derived from Proposition* 4.3 *without* $\mathbf{B}$ *having a bounded inverse.*

### 4.2. Case where A and B are commutative.

Here we assume that $\mathbf{A}$ and $\mathbf{B}$ are positive self-adjoint operators defined in the separable Hilbert space $\mathscr{H}$.

By the spectral theorem we have the following spectral representations;

$$\mathbf{A} = \int_0^{\infty} v dE(v), \qquad \mathbf{B} = \int_0^{\infty} \lambda dF(\lambda).$$

We say that $\mathbf{A}$ and $\mathbf{B}$ are commutative if $E(v)$ and $F(\lambda)$ are commutative for all $v, \lambda \geq 0$.

PROPOSITION 4.4. *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are commutative and there exists the inverse* $\mathbf{B}^{-1}$ *of* $\mathbf{B}$ *such that* $\mathbf{B}^{-1}$ *is compact on* $\mathscr{H}$. *Then* $\mathbf{A}$ *has countable eigenvectors which form a complete orthonormal system of* $\mathscr{H}$.

PROOF. Since $\mathbf{B}^{-1}$ is compact and $\ker \mathbf{B}^{-1} = \{0\}$, we have

$$\mathbf{B}^{-1} = \sum_{n=1}^{\infty} \gamma_n P_n,$$

where $\{P_n\}$ are orthogonal projections on $\mathscr{H}$ satisfying

$$\dim P_n \mathscr{H} < +\infty$$

and

$$\mathscr{H} = \bigoplus_{n=1}^{\infty} P_n \mathscr{H}.$$

By the commutativity of $E(v)$ and $P_n$, $\mathbf{A}$ maps $P_n\mathscr{H}$ into $P_n\mathscr{H}$, which asserts Proposition 4.4. ∎

By Proposition 4.4, $\mathbf{A}$ has countable eigenvalues $\{v_i\}$ such that $\mathbf{A}e_i = v_i e_i$, $i = 0, 1, 2, \cdots$, where $\{e_i\}$ forms a complete orthonormal system of $\mathscr{H}$. Setting $\mathbf{B}e_i = \lambda_i e_i$, $i = 0, 1, 2, \cdots$, we have

COROLLARY 4.1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be as above. Supose that* $\mathbf{A} \geq 1 + \varepsilon$ *for some* $\varepsilon > 0$. *If for some natural number* $k$

$$\sum_{i=0}^{\infty} \frac{v_i^2}{\lambda_i^{2k}} = c_9 < +\infty,$$

*then the conditions* (1), (2), (3) and (4) *of Theorem* 1.1 *are satisfied for* **A** *and* **B**.

PROOF. Following the same argument as in the proof of Proposition 4.1, we see that for any finite sum $f = \sum a_i e_i$ and any integer $n \geq 0$,

$$|\mathbf{B}^n \mathbf{A} f|_{\mathscr{H}}^2 \leq c_9 |\mathbf{B}^{n+k} f|_{\mathscr{H}}^2,$$

which implies the conditions (1) and (3) of Theorem 1.1. Since $\nu_i > 1$, $i = 0, 1, 2, \cdots$, (2) is proved by the same argument as in Proposition 4.1. Since **A** and **B** are commutative, we have

$$|\mathbf{B}^n e^{-t\mathbf{A}} f|_{\mathscr{H}}^2 \leq |\mathbf{B}^n f|_{\mathscr{H}}^2,$$

which completes the proof of Corollary 4.1.

## 5. Strong solution for a Segal-Langevin type equation

In [7], they discussed a fluctuation phenomena for interacting, spatially extended neurons and as a limit equation, they found a suitable fundamental space $\mathscr{D}_E$ of functionals on $E$ and studied Segal-Langevin type stochastic differential equations including a class of the weak version of (1.1):

$$dX_F(t) = dW_F(t) + X_{-d\Gamma(\mathbf{A})F}(t)dt, \qquad F \in \mathscr{D}_E. \tag{5.1}$$

A stochastic process $X_F(t)$ indexed by elements in $\mathscr{D}_E$ is called a continuous $L(\mathscr{D}_E)$-process if for any fixed $F \in \mathscr{D}_E$, $X_F(t)$ is a real continuous process and $X_{\alpha F + \beta G}(t) = \alpha X_F(t) + \beta X_G(t)$ almost surely for real numbers $\alpha$, $\beta$ and elements $F, G \in \mathscr{D}_E$ and further $E[X_F(t)^2]$ is continuous on $\mathscr{D}_E$. $W_F(t)$ is an $L(\mathscr{D}_E)$-Wiener process such that for any fixed $F \in \mathscr{D}_E$, $W_F(t)$ is a real Wiener process.

Although the above $\mathscr{D}_E$ is not nuclear, appealing to the results in [7], we get a unique continuous $L(\mathscr{D}_E)$-process satisfying (5.1).

We consider the case where for **A** in (5.1) there exists a self-adjoint operator **B** satisfying all the conditions of Theorem 1.1. In this case, by Theorem 1.1, there is a nuclear space $S$ invariant under both $d\Gamma(\mathbf{A})$ and a strong continuous semigroup $T(t) = e^{-td\Gamma(\mathbf{A})}$. If we replace $\mathscr{D}_E$ by $S$ in (5.1), then by the regularization theorem [5] there exists an $S'$-valued Wiener process $W(t)$ such that $\langle W(t), F \rangle = W_F(t)$ almost surely and the strong form of the equation with $\mathscr{D}_E$ replaced by $S$ in (5.1) is the following stochastic differential equation on $S'$:

$$dX(t) = dW(t) - d\Gamma(\mathbf{A})^* X(t)dt.$$

Let $T(t)^*$ be the adjoint operator of $T(t)$. Since $S$ is nuclear, again by the regularization theorem, the stochastic integral $\int_0^t T(t-s)^* dW(s)$ is well defined from the weak form such that

$$\left\langle \int_0^t T(t-s)^* dW(s),\, F \right\rangle = \int_0^t \langle dW(s),\, T(t-s)F \rangle .$$

Since $T(t-s)F = F + \int_s^t T(\tau - s)(-d\Gamma(\mathbf{A}))F\,d\tau$, we get

$$\int_0^t T(t-s)^* dW(s) = W(t) + \int_0^t (-d\Gamma(\mathbf{A})^*)\left(\int_0^\tau T(\tau - s)^* dW(s)\right) d\tau .$$

Noticing that

$$\int_0^t (-d\Gamma(\mathbf{A})^*) T(\tau)^* X(0) d\tau = T(t)^* X(0) - X(0),$$

we see that

$$X(t) = T(t)^* X(0) + \int_0^t T(t-s)^* dW(s)$$

is a unique strong solution of (1.1) on $S'$.

## Acknowledgment

The authors would like to express heartful thank to Professor A. Inoue who kindly informed them the works of J. Fröhlich and their related ones.

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