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A note on pseudo resolvents

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ABSTRACT. Let $E \neq \{0\}$ be a Banach lattice. From elementary operator theory we know that for any bounded operator $\mathbf{T}$ mapping E into itself, the resolvent $\mathbf{R}(\lambda, \mathbf{T})$ of $\mathbf{T}$ satisfies the resolvent equation $\mathbf{R}(\lambda, \mathbf{T}) - \mathbf{R}(\mu, \mathbf{T}) = (\mu - \lambda)\mathbf{R}(\lambda, \mathbf{T})\mathbf{R}(\mu, \mathbf{T})$. The converse of the above statement in general is not true. In this note, we study the natural inverse problem. We investigate under what conditions a pseudo resolvent on E is the resolvent of a uniquely determined positive operator on E. Furthermore, we determine necessary and sufficient conditions for a pseudo resolvent to be the resolvent of a uniquely defined positive irreducible operator.

1 Introduction

In this note we provide necessary and sufficient conditions for a family of bounded operators satisfying the resolvent equation to be the resolvent of a uniquely defined positive irreducible operator on the Banach lattice. In the following we briefly summarize basic concepts and fundamental results.

Let $E \neq \{0\}$ be a Banach lattice, the subset $E_+ := \{x \in E | x \geqslant 0\}$ is called the *positive cone* of E, elements $x \in E_+$ are called *positive*, and any nontrivial element $x \in E_+$ will be denoted by the notation $x > 0$. A linear operator $\mathbf{S}$ mapping E into itself is called *positive* if $\mathbf{S}(E_+) \subset E_+$. We use $L(E)$ to denote the Banach space of all bounded linear operators mapping E into itself. A subset A of E is *solid* if $|x| \leqslant |y|$, and $y \in A$, implies $x \in A$. A solid subspace is called an *ideal*. A *principal ideal* is an ideal generated by a single element x and is denoted by E_x. It can be shown that if $x > 0$, then $E_x = \bigcup_{n=1}^{\infty} n[-x, x]$. Any $x \geqslant 0$ is called a *quasi-interior positive element* of E if its principal ideal E_x is dense in E, i.e., $\overline{E}_x = E$. A linear operator $\mathbf{S}: E \to E$ is called *ideal irreducible* if $\{0\}$ and E are the only $\mathbf{S}$-invariant closed ideals. We let $r(\mathbf{S})$ be the spectral radius of $\mathbf{S}$.

Let D be a nonempty open subset of $\boldsymbol{C}$, and let $R: D \to L(E)$ be a function satisfying

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$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu) \qquad \text{for all } \lambda,\ \mu \in D. \tag{1.1}$$

Such a mapping is called a *pseudo resolvent* on E. Discussion of Pseudo resolvents can be found in [1, 2, 3].

2 Main result

In the following we let $\mathbf{S}$ be a positive linear operator on E, $\rho(\mathbf{S})$ the resolvent set of $\mathbf{S}$ and let $\mathbf{R}(\lambda, \mathbf{S}) := (\lambda\mathbf{I} - \mathbf{S})^{-1}$, $\lambda \in \rho(\mathbf{S})$, denote the resolvent of $\mathbf{S}$. It follows from positive operator theory that $\lambda > r(\mathbf{S})$ if and only if $\mathbf{R}(\lambda, \mathbf{S}) \geqslant \mathbf{0}$. Moreover $\mathbf{S}$ is irreducible if and only if $\mathbf{R}(\lambda, \mathbf{S})x$ is a quasi-interior positive element of E for each $\lambda > r(\mathbf{S})$ and $x > 0$, as the ideal generated by $\mathbf{R}(\lambda, \mathbf{S})x$ is invariant under $\mathbf{S}$ [4].

It is known that a pseudo resolvent in general need not be the resolvent of a linear operator. Necessary and sufficient conditions for when a pseudo resolvent is the resolvent of a uniquely determined closed linear operator on Banach space appeared in [3]. Our main objective is to characterize conditions under which there exists a unique positive irreducible operator $\mathbf{T}$ such that $R: D \to L(E)$ is the resolvent of $\mathbf{T}$. We begin by stating some properties of pseudo resolvents.

PROPOSITION 2.1 [3, p. 36]. *Let D be a nonempty open subset of $\boldsymbol{C}$ and $R: D \to L(E)$ be a pseudo resolvent on E. Then $R(\lambda)R(\mu) = R(\mu)R(\lambda)$ for all λ, $\mu \in D$. The null space $\mathcal{N}(R(\lambda))$ and the range $\mathcal{R}(R(\lambda))$ are independent of $\lambda \in D$.*

PROPOSITION 2.2 [2, p. 299]. *Let $R: D \to L(E)$ be a pseudo resolvent on E. The following statements are true.*

(1) *For any $\lambda_0 \in D$ and any $\mu \in D$ with $|\mu - \lambda_0| < \|R(\lambda_0)\|^{-1}$, we have $R(\mu) = \sum_{n=0}^{\infty} (\lambda_0 - \mu)^n R(\lambda_0)^{n+1}$.*

(2) *$\lambda \to R(\lambda)$ is a locally holomorphic function defined on $D \subset \boldsymbol{C}$ with values in $L(E)$.*

It follows from Proposition 2.2 that every pseudo-resolvent has a unique maximal extension.

The next result gives necessary and sufficient conditions for when a pseudo resolvent is the resolvent of a positive operator. The ideal of the proof is similar to that of [3, p. 37] in characterizing when a pseudo resolvent is the resolvent of a closed operator.

THEOREM 2.1. *Let $R: D \to L(E)$ be a pseudo resolvent on E. Then R is the resolvent of a uniquely defined positive linear operator $\mathbf{T}$ on E if and only if*

(1) $\mathscr{R}(R(\lambda)) = E$, $\mathscr{N}(R(\lambda)) = \{0\}$ *for some* $\lambda \in D$,

(2) *There exists an* $M > 0$ *such that*

$$\boldsymbol{B} := D \cap \{\lambda | \lambda > M\} \neq \varnothing \tag{2.1}$$

and

$$R(\lambda_0)^{-1} \geq 0 \quad and \quad \lambda_0 R(\lambda_0)x \geq x \qquad for\ x \geq 0\ and\ some\ \lambda_0 \in \boldsymbol{B}. \tag{2.2}$$

PROOF. Observe that condition (1) implies $R(\lambda)$ is invertible for all $\lambda \in D$ by Proposition 2.1, and (2.2) implies that $R(\lambda_0)$ is positive.

($\Rightarrow$) Suppose a pseudo resolvent $R: D \to L(E)$ is the resolvent of a uniquely determined positive operator $\mathbf{T}$ on E. Then $\mathscr{R}(R(\lambda)) = E$ and $\mathscr{N}(R(\lambda)) = \{0\}$ for all $\lambda \in D$. Furthermore, since the positivity of $\mathbf{T}$ implies the continuity of $\mathbf{T}$, if we choose any $M > r(\mathbf{T}) \geq 0$, then $\boldsymbol{B} = D \cap \{\lambda | \lambda > M\} \neq \varnothing$ (in fact $\{\lambda | \lambda > M\} \subset D$) and for all $\lambda > M$, $R(\lambda) \geq \mathbf{0}$ and hence

$$\lambda R(\lambda)x = \lambda(\lambda \mathbf{I} - \mathbf{T})^{-1}x = \lambda \sum_{n=0}^{\infty} \frac{\mathbf{T}^n x}{\lambda^{n+1}} \geq x \qquad \text{for all } x \in E_+.$$

($\Leftarrow$) Conversely, suppose R is a pseudo resolvent on E satisfying conditions (1) and (2). Then $R(\lambda)$ is invertible for all $\lambda \in D$. We now choose any $\lambda_0 \in \boldsymbol{B}$ satisfying (2.2) and define

$$\mathbf{T} := \lambda_0 \mathbf{I} - R(\lambda_0)^{-1}. \tag{2.3}$$

We claim that (2.2) implies

$$\lambda_0 x \geq R(\lambda_0)^{-1}x \qquad \text{for all } x \in E_+. \tag{2.4}$$

Indeed, let $x > 0$ be arbitrary, $\lambda_0 x - R(\lambda_0)^{-1}x = \lambda_0 R(\lambda_0)y - R(\lambda_0)^{-1}R(\lambda_0)y$ for a unique $y \in E_+$ such that $R(\lambda_0)y = x$ as $R(\lambda_0) \geq \mathbf{0}$ is one to one and onto E by assumption. So that $\lambda_0 x - R(\lambda_0)^{-1}x = \lambda_0 R(\lambda_0)y - y \geq 0$ by (2.2) and we conclude that $\lambda_0 x \geq R(\lambda_0)^{-1}x$. Therefore, $\mathbf{T}x = \lambda_0 x - R(\lambda_0)^{-1}x \geq 0$ for all $x \in E_+$, i.e., $\mathbf{T}$ is a positive operator on E.

By the definition of $\mathbf{T}$ in (2.3),

$$(\lambda_0 \mathbf{I} - \mathbf{T})R(\lambda_0) = R(\lambda_0)(\lambda_0 \mathbf{I} - \mathbf{T}) = \mathbf{I}, \tag{2.5}$$

and hence $\mathbf{R}(\lambda_0, \mathbf{T}) = R(\lambda_0)$. Furthermore, for any $\lambda \in D$,

$$\begin{aligned}
(\lambda \mathbf{I} - \mathbf{T})R(\lambda) &= [(\lambda - \lambda_0)\mathbf{I} + (\lambda_0 \mathbf{I} - \mathbf{T})]R(\lambda) \\
&= [(\lambda - \lambda_0)\mathbf{I} + (\lambda_0 \mathbf{I} - \mathbf{T})]R(\lambda_0)[\mathbf{I} - (\lambda - \lambda_0)R(\lambda)] \\
&= \mathbf{I} + (\lambda - \lambda_0)[R(\lambda_0) - R(\lambda) - (\lambda - \lambda_0)R(\lambda)R(\lambda_0)] \\
&= \mathbf{I} + \mathbf{0} \\
&= \mathbf{I}.
\end{aligned}$$

Similarly, $R(\lambda)(\lambda\mathbf{I}-\mathbf{T})=\mathbf{I}$ for all $\lambda \in D$. We then conclude that $R(\lambda)$ is the resolvent of $\mathbf{T}$, as $R: D \to L(E)$ is locally holomorphic on D and has a unique maximal extension on an open subset Δ of $\boldsymbol{C}$ satisfying (1.1) which can not be continued analytically beyond Δ [1, pp. 188–189].

Suppose now we choose a different $\tilde{\lambda} \in \boldsymbol{B}$ satisfying (2.2) and define

$$\mathbf{S} := \tilde{\lambda}\mathbf{I} - R(\tilde{\lambda})^{-1}. \tag{2.6}$$

By the same argument we can show that $\mathbf{S} \geq \mathbf{0}$, $\mathbf{R}(\lambda, \mathbf{S}) = R(\lambda)$ for all $\lambda \in D$. In particular, $(\tilde{\lambda}\mathbf{I}-\mathbf{S})^{-1} = (\tilde{\lambda}\mathbf{I}-\mathbf{T})^{-1}$, hence $(\tilde{\lambda}\mathbf{I}-\mathbf{S}) = (\tilde{\lambda}\mathbf{I}-\mathbf{T})$ and $\mathbf{S}=\mathbf{T}$. So that $\mathbf{T}$ is independent of the choice of $\lambda_0 \in \boldsymbol{B}$ satisfying (2.2).

On the other hand, if there exists an $M_2 > 0$ such that condition (2) holds, we then proceed with a similar argument. Indeed, we choose any $\lambda_2 \in D \cap \{\lambda | \lambda > M_2\}$ satisfying (2.2) and define $\mathbf{T}_2 := \lambda_2\mathbf{I} - R(\lambda_2)^{-1}$. Then $\mathbf{R}(\lambda, \mathbf{T}) = R(\lambda) = \mathbf{R}(\lambda, \mathbf{T}_2)$ for all $\lambda \in D$ and thus $\mathbf{T} = \mathbf{T}_2$. This shows that the positive operator $\mathbf{T}$ is uniquely determined by the pseudo resolvent satisfying conditions (1) and (2).

Let Q be the set of all quasi-interior positive elements of E.

PROPOSITION 2.3. *Let $R: D \to L(E)$ be a pseudo resolvent on E. Suppose there exists an $\alpha > 0$ such that*
(1) $(\alpha, \infty) \subset D$,
(2) $\lambda R(\lambda)x \geq x$ *for all* $\lambda > \alpha$, $x \in E_+$,
(3) $R(\lambda)x \in Q$ *for all* $\lambda > \alpha$, $x > 0$.
Then $\beta := \inf\{\alpha_\gamma | \alpha_\gamma > 0$ *satisfying* (1) (2) (3)$\}$ *also satisfies conditions* (1), (2) *and* (3).

PROOF. If $\beta = 0$, we are then in a position to show that $(0, \infty) \subset D$, $\lambda R(\lambda)x \geq x$ for all $\lambda > 0$, $x \in E_+$ and $R(\lambda)x$ is a quasi-interior positive element of E for every $x > 0$, $\lambda > 0$. Let $\lambda > 0$ be given, then there exists an $\alpha_\gamma > 0$ such that $0 < \alpha_\gamma < \lambda$. Since α_γ satisfies hypotheses (1), (2) and (3), we have $\lambda \in D$, $\lambda R(\lambda)x \geq x$ for all $x \geq 0$ and $R(\lambda)x$ is a quasi-interior positive element of E for every $x > 0$. Therefore we have shown that $\beta = 0$ satisfies conditions (1), (2) and (3) as $\lambda > 0$ were arbitrary.

If $\beta > 0$, then by the property of infimum that for any $\lambda > \beta$ there exists an α_γ such that $\beta < \alpha_\gamma < \lambda$. Thus the argument can proceed as in the case $\beta = 0$.

THEOREM 2.2. *Let $R: D \to L(E)$ be a pseudo resolvent on E. Then R is the resolvent of a uniquely defined positive irreducible operator on E if and only if there exists an $\alpha > 0$ such that*
(1) $\mathscr{R}(R(\lambda)) = E$, $\mathscr{N}(R(\lambda)) = \{0\}$ *for some* $\lambda \in D$,
(2) $(\alpha, \infty) \subset D$,

(3) $\lambda R(\lambda)x \geq x$ *for* $x \geq 0$, $\lambda > \alpha$, *and* $R(\lambda_0)^{-1} \geq 0$ *for some* $\lambda_0 > \alpha$,
(4) $R(\lambda)x \in Q$ *for all* $x > 0$ *and* $\lambda > \alpha$.

PROOF. ($\Rightarrow$) Clearly, if R is the resolvent of a positive irreducible operator $\mathbf{T}$, we can choose $\alpha > r(\mathbf{T}) \geq 0$. It then follows from the proceeding discussion in Theorem 2.1 that (1) (2) (3) hold and (4) follows from the irreducibility of $\mathbf{T}$.

($\Leftarrow$) Conversely, suppose there exists an $\alpha > 0$ such that $R(\lambda)$ satisfies hypotheses (1) $\sim$ (4). Then let

$$\beta := inf\{\alpha_\gamma | \alpha_\gamma > 0 \text{ satisfying conditions (1)} \sim \text{(4)}\}. \tag{2.7}$$

We have as guaranteed by Proposition 2.3 that

$$(\beta, \infty) \subset D,$$

$$\lambda R(\lambda)x \geq x \qquad \text{for all } \lambda > \beta, \quad x \in E_+,$$

and

$$R(\lambda)x \in Q \qquad \text{for all } \lambda > \beta, \quad x > 0.$$

We choose a number $\lambda_0 > \beta$ so that $R(\lambda_0)^{-1} \geq 0$ and define

$$\mathbf{T} := \lambda_0 \mathbf{I} - R(\lambda_0)^{-1}.$$

By Theorem 2.1, we see that conditions (1), (2) and (3) imply that $\mathbf{T}$ is a positive operator which is uniquely determined by the given pseudo resolvent. It remains to show that $\mathbf{T}$ is irreducible.

(case i) Let $\beta = 0$. Since $(0, \infty) \subset \rho(\mathbf{T})$ and $\mathbf{T}$ is bounded, we see that $r(\mathbf{T}) = 0$. Hence $\mathbf{T}$ is irreducible as $R(\lambda)x$ is a quasi-interior positive element of E for all $x > 0$, $\lambda > 0$.

(case ii) If $\beta > 0$, then we conclude that $\beta \geq r(\mathbf{T})$. If $\beta > r(\mathbf{T})$, then $\mathbf{T}$ is a positive operator such that the resolvent coincides with the given pseudo resolvent. Hence we see that for any $\lambda \in (r(\mathbf{T}), \beta)$, $(\lambda, \infty) \subset D$ and $\lambda R(\lambda)x = \lambda(\lambda\mathbf{I} - \mathbf{T})^{-1}x \geq x$ for all $x \geq 0$. Furthermore, there exists $\mu > \beta$ such that $R(\mu)x$ is a quasi-interior positive element of E for all $x > 0$ by (4). Therefore for any $x > 0$, it follows from the resolvent equation

$$R(\lambda)x = R(\mu)x + (\mu - \lambda)R(\lambda)R(\mu)x$$

that $R(\lambda)x \geq R(\mu)x$ for all $x > 0$ as $\mu - \lambda > 0$ and $R(\lambda)$, $R(\mu) \geq 0$. Thus $E_{R(\mu)x} \subset E_{R(\lambda)x}$ and we conclude that $R(\lambda)x$ is also a quasi-interior positive element of E for all $x > 0$. Hence this contradicts the definition of β. Therefore $\beta = r(\mathbf{T})$ and $\mathbf{T}$ is thus irreducible. This completes the proof of our main result.

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