HIROSHIMA MATH. J.
26 (1996), 605–621

Existence of Dirichlet infinite harmonic measures on the Euclidean unit ball

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(Received June 26, 1995)

ABSTRACT. It is shown that there exist p-Dirichlet infinite p-harmonic measures on the unit ball in the Euclidean space of dimension $n \geqq 2$ even if $1 < p < 2$. The same is also proved to be true if the p-harmonicity is generalized to the so-called $\mathscr{A}$-harmonicity of exponent p.

1. Introduction

The purpose of this paper is to give an affirmative answer to a problem originally posed by Ohtsuka [11, Chap. VIII] whether there exists a p-harmonic measure on the unit ball in the n-dimensional Euclidean space $\boldsymbol{R}^n$ $(n \geqq 2)$ with an infinite p-Dirichlet integral for the exponent $1 < p < 2$. Actually Ohtsuka raised the question in terms of extremal distances in an equivalent to but superfacially different from the above formulation. However in this paper we will confine ourselves to the frame of harmonic measures as mentioned above.

We will discuss the problem in a broader potential theoretic setting of $\mathscr{A}$-harmonicity than that of mere p-harmonicity. Following the monograph [2] of Heinonen, Kilpeläinen and Martio (see also Maz'ya [6]), we say that $\mathscr{A}$ is a strictly monotone elliptic operator on the Euclidean space $\boldsymbol{R}^n$ of dimension $n \geqq 2$ with exponent $1 < p \leqq n$ if $\mathscr{A}$ is a mapping of $\boldsymbol{R}^n \times \boldsymbol{R}^n$ to $\boldsymbol{R}^n$ satisfying the following five conditions (2)–(6) for some constants $0 < \alpha \leqq \beta < \infty$:

(2) the function $h \mapsto \mathscr{A}(x, h)$ is continuous for almost every fixed $x \in \boldsymbol{R}^n$, and the function $x \mapsto \mathscr{A}(x, h)$ is measurable for all fixed $h \in \boldsymbol{R}^n$;

for almost every $x \in \boldsymbol{R}^n$ and for all $h \in \boldsymbol{R}^n$

$$\mathscr{A}(x, h) \cdot h \geqq \alpha |h|^p, \tag{3}$$

1991 *Mathematics Subject Classification.* Primary 31B35; Secondary 31C45, 30F25.

Key words and phrases. Dirichlet integral, harmonic measure.

This work was partly supported by Grant-in-Aid for Scientic Research, No. 06640227, Japanese Ministry of Education, Science and Culture.

$$|\mathscr{A}(x,h)| \leqq \beta|h|^{p-1}, \tag{4}$$

$$(\mathscr{A}(x,h_1)-\mathscr{A}(x,h_2))\cdot(h_1-h_2)>0 \tag{5}$$

whenever $h_1 \neq h_2$, and

$$\mathscr{A}(x,\lambda h)=|\lambda|^{p-2}\lambda\mathscr{A}(x,h) \tag{6}$$

for all $\lambda \in \boldsymbol{R}\backslash\{0\}$, where $\boldsymbol{R}$ is the field of real numbers. Here $|x|$ indicates the length of a vector $x=(x^1,\dots,x^n)$ in $\boldsymbol{R}^n$. The class of all operators $\mathscr{A}$ on $\boldsymbol{R}^n$ satisfying (2)–(6) with exponent $1<p\leqq n$ will be denoted by $\mathscr{A}_p(\boldsymbol{R}^n)$.

Using an $\mathscr{A}\in\mathscr{A}_p(\boldsymbol{R}^n)$ we consider a quasilinear elliptic partial differential equation

$$-\operatorname{div}\mathscr{A}(x,\nabla u(x))=0 \tag{7}$$

on $\boldsymbol{R}^n$. A function u on an open subset G of $\boldsymbol{R}^n$ is a weak solution of (7) if $u \in \operatorname{loc} W_p^1(G)$ and

$$\int_G \mathscr{A}(x,\nabla u(x))\cdot\nabla\varphi(x)dx=0$$

for every $\varphi\in C_0^\infty(G)$, where $W_p^1(G)$ is the Sobolev space on G consisting of functions $f\in L_p(G)=L_p(G;\boldsymbol{R})$ with distributional gradients $\nabla f\in L_p(G)=L_p(G;\boldsymbol{R}^n)$ and $dx=dx^1\cdots dx^n$. A weak solution u of (7) (possibly modified on a set of zero measure dx) is actually continuous. We say that a function u is *$\mathscr{A}$-harmonic* on G if u is a continuous weak solution of (7) on G. We denote by $H_{\mathscr{A}}(G)$ the class of all $\mathscr{A}$-harmonic functions on G. The simplest and the most typical operator $\mathscr{A}$ in $\mathscr{A}_p(\boldsymbol{R}^n)$ is the p-Laplacian $\mathscr{A}(x,h)=|h|^{p-2}h$ so that the corresponding elliptic equation is the p-Laplace equation

$$-\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x))=0.$$

In this case we use the term *p-harmonic* instead of $\mathscr{A}$-harmonic and the notation $H_p(G)$ in place of $H_{\mathscr{A}}(G)$.

The greatest $\mathscr{A}$-harmonic minorant $u\wedge v$ on G, if it exists, of two $\mathscr{A}$-harmonic functions u and v on G is the $\mathscr{A}$-harmonic function $u\wedge v$ on G characterized by the following two conditions: (i) $u\wedge v\leqq u$ and $u\wedge v\leqq v$ on G; (ii) if there is an $\mathscr{A}$-harmonic function h on G such that $h\leqq u$ and $h\leqq v$ on G, then $h\leqq u\wedge v$ on G. A function w is said to be an *$\mathscr{A}$-harmonic measure* on G in the sense of Heins [3] if w is $\mathscr{A}$-harmonic on G and satisfies

$$w\wedge(1-w)=0 \tag{8}$$

on G. An $\mathscr{A}$-harmonic measure w on G always satisfies $0\leqq w\leqq 1$ on G;

$w \equiv 0$ and $w \equiv 1$ are $\mathscr{A}$-harmonic measures on G; when G is a region, an $\mathscr{A}$-harmonic measure w on G is nonconstant if and only if $0 < w < 1$ on G.

Our main concern in this paper is the *p-Dirichlet integral*

$$D_p(w) = D_p(w; B^n) = \int_{B^n} |\nabla w(x)|^p \, dx \leqq \infty$$

of each $\mathscr{A}$-harmonic measure w on the unit ball $B^n = \{x \in \boldsymbol{R}^n : |x| < 1\}$ with $\mathscr{A} \in \mathscr{A}_p(\boldsymbol{R}^n)$. We say that w is *p-Dirichlet finite* (*infinite*, resp.) if $D_p(w) < \infty$ ($D_p(w) = \infty$, resp.). In this regard we recall the following result (cf. [9] and also Herron-Koskela [4]):

9. THEOREM. *If $2 \leqq p \leqq n$, then every nonconstant $\mathscr{A}$-harmonic measure on the unit ball B^n is p-Dirichlet infinite for every $\mathscr{A}$ in $\mathscr{A}_p(\boldsymbol{R}^n)$; if $1 < p < 2$, then there exist nonconstant p-Dirichlet finite $\mathscr{A}$-harmonic measures on B^n for every $\mathscr{A}$ in $\mathscr{A}_p(\boldsymbol{R}^n)$.*

In view of this result we are naturally led to ask the question in the case $1 < p < 2$ whether there are p-Dirichlet infinite $\mathscr{A}$-harmonic measures on the unit ball B^n for every $\mathscr{A}$ in $\mathscr{A}_p(\boldsymbol{R}^n)$, which is the main theme of this paper and exactly identical with the problem of Ohtsuka stated at the beginning of this introduction for $\mathscr{A}(x, h) = |h|^{p-2}h$. To discuss this question we will construct a certain open set on the unit sphere $S^{n-1} = \partial B^n$ as an image of an open set in $\boldsymbol{R}^{n-1} \times \{0\}$ by the stereographic projection. Recall that the stereographic projection s of $\boldsymbol{R}^{n-1} \times \{0\}$ to $S^{n-1} \backslash \{N\}$ from the north pole $N = (0, \dots, 0, 1) \in S^{n-1}$ is given by

$$s(x) = \frac{2x}{|x|^2 + 1} + \frac{|x|^2 - 1}{|x|^2 + 1} N \qquad (x \in \boldsymbol{R}^{n-1} \times \{0\}),$$

which is a C^∞ diffeomorphism of $\boldsymbol{R}^{n-1} \times \{0\}$ onto $S^{n-1} \backslash \{N\}$. Considering the one point compactification $\bar{\boldsymbol{R}}^{n-1}$ of $\boldsymbol{R}^{n-1}$, s can be continued to a diffeomorpism of $\bar{\boldsymbol{R}}^{n-1} \times \{0\}$ onto S^{n-1}.

Take two sequences $(a_k) = (a_k : 1 \leqq k < K + 1)$ and $(b_k) = (b_k : 1 \leqq k < K + 1)$ of real numbers a_k and b_k such that

$$(10) \qquad 0 < a_{k+1} < b_k < a_k < 1 \qquad (1 \leqq k < K)$$

so that (a_k) and (b_k) are finite sequences of K terms if $1 \leqq K < \infty$ and infinite sequences if $K = \infty$. In the latter case we moreover assume that

$$\lim_{k \to \infty} a_k = 0.$$

With these two sequences (a_k) and (b_k) we associate the sequence $(A_k) =$

$(A_k : 1 \leqq k < K+1)$ of "intervals" A_k in $S^{n-1}\backslash\{N\}$ given by

$$s^{-1}(A_k) = (b_k, a_k) \times (-1/2, 1/2) \times \cdots \times (-1/2, 1/2) \times \{0\}. \tag{11}$$

We then consider the open subset A in $S^{n-1} = \partial B^n$ associated with sequences (a_k) and (b_k) given by

$$A = A((a_k), (b_k)) = \bigcup_{k=1}^{K} A_k. \tag{12}$$

We denote by $\omega(A, B^n; \mathscr{A})$ the *$\mathscr{A}$-harmonic measure of A with respect to B^n* in the sense of Martio ([5], [2, Chap. 11]). It may also be defined by

$$\omega(A, B^n; \mathscr{A}) = \sup \mathscr{V}(A, B^n; \mathscr{A}) \tag{13}$$

where $\mathscr{V}(A, B^n; \mathscr{A})$ denotes the class of all functions v in the class $C(B^n \cup S^{n-1}) \cap H_{\mathscr{A}}(B^n)$ such that $v|S^{n-1} \leqq 1$ and $v|(S^{n-1}\backslash A) \leqq 0$ (cf. [2, 11.1 and 11.5]). Clearly $\omega = \omega(A, B^n; \mathscr{A})$ is $\mathscr{A}$-harmonic and $0 \leqq \omega \leqq 1$ on B^n. Concerning the boundary behavior of ω at $S^{n-1} = \partial B^n$ we have (cf. [2, 11.6])

$$\begin{cases} \lim_{y \to x} \omega(A, B^n; \mathscr{A})(y) = 1 & (x \in A), \\ \lim_{y \to x} \omega(A, B^n; \mathscr{A})(y) = 0 & (x \in S^{n-1}\backslash \bar{A}). \end{cases} \tag{14}$$

We will later see in Proposition 45 that $\omega(A, B^n; \mathscr{A})$ is actually an $\mathscr{A}$-harmonic measure in the sense of Heins for $\mathscr{A}$ in $\mathscr{A}_p(\mathbf{R}^n)$ with $1 < p \leqq 2$.

The purpose of this paper is, as already stated, to give an affirmative answer to the problem of Ohtsuka on the existence of p-Dirichlet infinite p-harmonic measures (in the sense of Heins) on the unit ball B^n for $1 < p < 2$ by proving the following result.

15. MAIN THEOREM. *If $K < \infty$ or if $K = \infty$ and either the sequence $(|a_k - b_k| : 1 \leqq k < \infty)$ or $(|a_{k+1} - b_k| : 1 \leqq k < \infty)$ converges to zero so rapidly as to satisfy the condition*

$$\min\left(\sum_{k=1}^{\infty} |a_k - b_k|^{2-p}, \sum_{k=1}^{\infty} |a_{k+1} - b_k|^{2-p}\right) < \infty, \tag{16}$$

then the $\mathscr{A}$-harmonic measure $\omega(A, B^n; \mathscr{A})$ is p-Dirichlet finite for every $\mathscr{A}$ in $\mathscr{A}_p(\mathbf{R}^n)$ with each $1 < p < 2$ and $n \geqq 2$. If both of the sequences $(|a_k - b_k| : 1 \leqq k < \infty)$ and $(|a_{k+1} - b_k| : 1 \leqq k < \infty)$ converge to zero so slowly as to satisfy the condition

$$\sum_{k=1}^{\infty} \min(|a_k - b_k|^{2-p}, |a_{k+1} - b_k|^{2-p}) = \infty, \tag{17}$$

then the $\mathscr{A}$-harmonic measure $\omega(A, B^n; \mathscr{A})$ is p-Dirichlet infinite for every $\mathscr{A}$ in $\mathscr{A}_p(\boldsymbol{R}^n)$ with each $1 < p < 2$ and $n \geqq 2$.

This result was obtained in [10] for the special case of $n = 2$. The proof of this theorem will be given later in 48 after a series of preparations starting from 23. If we take $K = \infty$ and $b_k = (a_k + a_{k+1})/2$ $(1 \leqq k < \infty)$, then the above result takes the following more applicable form also obtained by Herron and Koskela [4] in the case $n = 2$.

18. COROLLARY. *If the sequences $(a_k : 1 \leqq k < \infty)$ and $(b_k : 1 \leqq k < \infty)$ moreover satisfy the condition*

$$b_k = (a_k + a_{k+1})/2 \qquad (1 \leqq k < \infty), \tag{19}$$

then the $\mathscr{A}$-harmonic measure $\omega(A, B^n; \mathscr{A})$ is p-Dirichlet finite for every $\mathscr{A}$ in $\mathscr{A}_p(\boldsymbol{R}^n)$ with each $1 < p < 2$ and $n \geqq 2$ if and only if

$$\sum_{k=1}^{\infty} |a_k - b_k|^{2-p} < \infty. \tag{20}$$

We are now in a position to give an affirmative answer to the problem of Ohtsuka as an application of the above Corollary 18 and hence of Main theorem 15 by giving the following example.

21. EXAMPLE. *Choose decreasing zero sequences $(a_k : 1 \leqq k < \infty)$ and $(b_k : 1 \leqq k < \infty)$ so as to satisfy the conditions* (19) *and*

$$a_k - b_k = b_k - a_{k+1} = k^{-1/(2-p)} \tag{22}$$

for all sufficiently large k. Then the $\mathscr{A}$-harmonic measure $\omega(A, B^n; \mathscr{A})$ is p-Dirichlet infinite for every $\mathscr{A}$ in $\mathscr{A}_p(\boldsymbol{R}^n)$ with each $1 < p < 2$ and $n \geqq 2$.

23. Trace

The Sobolev space $W_p^1(G)$ $(1 < p \leqq n)$ is a Banach space equipped with the norm

$$\|f; W_p^1(G)\| = \|f; L_p(G)\| + \|\nabla f; L_p(G)\|,$$

where G is an open set in $\boldsymbol{R}^n$ $(n \geqq 2)$. The Sobolev null space $W_{p,0}^1(G)$ is the closure of $C_0^\infty(G)$ in $W_p^1(G)$ with respect to the above norm.

There exists a unique continuous linear operator γ of $W_p^1(B^n)$ into $L_p(S^{n-1})$ such that $\gamma f = f | S^{n-1}$ for every f in $C(B^n \cup S^{n-1}) \cap W_p^1(B^n)$, where $L_p(S^{n-1})$ is considered with respect to the surface element $d\sigma$ on S^{n-1}. The function γf defined a.e. on S^{n-1} and belonging to $L_p(S^{n-1})$ is referred to as the *trace* on S^{n-1} of f in $W_p^1(B^n)$. It is seen that the expression

$$(24) \qquad (\gamma f)(\zeta) = \lim_{r \uparrow 1} f(r\zeta)$$

holds for a.e. ζ in S^{n-1} (cf. e.g. [7, p. 47] and [8, pp. 180–181]).

Concerning the kernel $\operatorname{Ker}\gamma = \gamma^{-1}(0)$ and the image $\operatorname{Im}\gamma = \gamma(W_p^1(B^n))$ of $\gamma: W_p^1(B^n) \to L_p(S^{n-1})$ we have the following fundamental results. First, $\operatorname{Ker}\gamma$ characterizes the Sobolev null space (cf. e.g. [8, p. 187]):

$$(25) \qquad W_{p,0}^1(B^n) = \ker\gamma = \{f \in W_p^1(B^n) : \gamma f = 0\}.$$

Second, we denote $\operatorname{Im}\gamma = \gamma(W_p^1(B^n))$ by $\Lambda_p(S^{n-1})$. It is seen that the space $\Lambda_p(S^{n-1})$ forms a Banach space under the norm

$$(26) \qquad \|\varphi; \Lambda_p(S^{n-1})\| = \|\varphi; L_p(S^{n-1})\| + \left(\iint_{S^{n-1}\times S^{n-1}} \frac{|\varphi(\xi) - \varphi(\eta)|^p}{|\xi - \eta|^{p+n-2}} d\sigma(\xi)d\sigma(\eta)\right)^{1/p},$$

where $d\sigma$ is the area element on S^{n-1}. The theorem of Gagliardo [1] assures the existence of a constant $C_1 = C_1(n,p) > 1$ such that

$$(27) \qquad C_1^{-1}\|\varphi; \Lambda_p(S^{n-1})\| \leqq \inf_{\gamma f = \varphi} \|f; W_p^1(B^n)\| \leqq C_1\|\varphi; \Lambda_p(S^{n-1})\|$$

for every φ in $\Lambda_p(S^{n-1})$. The quantity $\|\varphi; \Lambda_p(S^{n-1})\|$ will be referred to as the *Gagliardo norm* of φ in this paper.

28. Dirichlet problem

Let G be a bounded region in $\boldsymbol{R}^n$. We will mainly consider the case $G = B^n$ but G is supposed to be a general bounded region for a while. For any f in $W_p^1(G)$ there exists a *unique* u in the space $H_{\mathscr{A}}(G) \cap W_p^1(G)$ such that $u - f$ belongs to $W_{p,0}^1(G)$ (cf. Maz'ya [6]; see also [2, 3.17]). This fact can be reformulated as the Maz'ya decomposition of $W_p^1(G)$:

$$(29) \qquad W_p^1(G) = (H_{\mathscr{A}}(G) \cap W_p^1(G)) \oplus W_{p,0}^1(G),$$

i.e. any f in $W_p^1(G)$ can be expressed uniquely as the sum of the $\mathscr{A}$-harmonic part u in $H_{\mathscr{A}}(G) \cap W_p^1(G)$ and the "potential part" g in $W_{p,0}^1(G) : f = u + g$. We denote by $\pi_{\mathscr{A}}^G$ the projection operator of $W_p^1(G)$ to $H_{\mathscr{A}}(G) \cap W_p^1(G)$ determined by $\pi_{\mathscr{A}}^G f = u$. Although $\pi_{\mathscr{A}}^G$ is homogeneous but not linear in general, we see that $\pi_{\mathscr{A}}^G$ is *monotone* (cf. e.g. [9]), i.e. if $f_1 \geqq f_2$ a.e. on G for any

f_1 and f_2 in $W_p^1(G)$, then $\pi_{\mathscr{A}}^G f_1 \geqq \pi_{\mathscr{A}}^G f_2$ on G. We say that G is *$\mathscr{A}$-regular* if

$$\lim_{x \in G, x \to y} (\pi_{\mathscr{A}}^G f)(x) = f(y) \tag{30}$$

for any f in $C(\bar{G}) \cap W_p^1(G)$ and for every y in ∂G. If G is bounded by a finite number of mutually disjoint smooth closed hypersurfaces, then G is $\mathscr{A}$-regular (cf. e.g. [2, 6.31]). The ball is the most typical example of $\mathscr{A}$-regular regions.

We now restrict ourselves to the case $G = B^n$. We use the abbreviation $\pi = \pi_{\mathscr{A}} = \pi_{\mathscr{A}}^{B_n}$. In view of the relation (25) and the uniqueness of the Maz'ya decomposition (29) we can define the operator

$$\tau = \pi \circ \gamma^{-1} : \Lambda_p(S^{n-1}) \to H_{\mathscr{A}}(B^n) \cap W_p^1(B^n)$$

in the following sense: for any $\varphi \in \Lambda_p(S^{n-1})$ choose any f in $\gamma^{-1}(\varphi)$ and then set $\tau\varphi := u = \pi f$. Clearly the operator $\tau = \tau_{\mathscr{A}} = \tau_{\mathscr{A}}^{B_n}$ is *bijective*. Moreover we have the following result.

31. PROPOSITION. *The operator τ is monotone, i.e. if $\varphi_1 \geqq \varphi_2$ a.e. on S^{n-1} for any φ_1 and φ_2 in $\Lambda_p(S^{n-1})$, then $\tau\varphi_1 \geqq \tau\varphi_2$ everywhere on B^n.*

PROOF. Choose an arbitrary g_i in $W_p^1(B^n)$ with $\gamma g_i = \varphi_i$ $(i = 1, 2)$. In general we denote by $F \cup G$ the function given by $(F \cup G)(x) = \max(F(x), G(x))$ for any two functions F and G. Then $(g_1 - g_2) \cup 0$ belongs to $W_p^1(B^n)$ by the lattice property of $W_p^1(B^n)$. By (24) we see that

$$\gamma((g_1 - g_2) \cup 0) = (\gamma(g_1 - g_2)) \cup 0 = (\varphi_1 - \varphi_2) \cup 0 = \varphi_1 - \varphi_2.$$

If we set $f_2 = g_2$ and $f_1 = g_2 + (g_1 - g_2) \cup 0$, then $\gamma f_2 = \gamma g_2 = \varphi_2$ and

$$\gamma f_1 = \gamma g_2 + \gamma((g_1 - g_2) \cup 0) = \varphi_2 + (\varphi_1 - \varphi_2) = \varphi_1.$$

Then $\tau\varphi_1 = \pi f_1$, $\tau\varphi_2 = \pi f_2$ and $f_1 \geqq f_2$ on B^n imply that $\tau\varphi_1 \geqq \tau\varphi_2$ on B^n by the monotoneity of π. □

We say that a measurable function φ on S^{n-1} has an essential limit α at η in S^{n-1},

$$\alpha = \operatorname*{ess\,lim}_{\xi \to \eta} \varphi(\xi)$$

in notation, if

$$\lim_{\varepsilon \downarrow 0} \|\varphi - \alpha; L_\infty(S^{n-1} \cap B(\eta, \varepsilon))\| = 0,$$

where $B(\eta, \varepsilon)$ is the ball of radius $\varepsilon > 0$ centered at η. Besides the defining boundary behavior $\gamma(\tau\varphi) = \varphi$ of $\tau\varphi$, we have the following more precise boundary behavior of $\tau\varphi$ if an additional condition is imposed upon φ:

32. PROPOSITION. *If $\varphi \in L_\infty(S^{n-1}) \cap \Lambda_p(S^{n-1})$ is continuous at a point $\eta \in S^{n-1}$ in the sense that* $\operatorname{ess\,lim}_{\xi\to\eta} \varphi(\xi) = \varphi(\eta)$, *then $\tau\varphi$ has a boundary value $\varphi(\eta)$ at η.*

PROOF. We only have to show that $\lim_{x\in B^n, x\to\eta} (\tau\varphi)(x) = \varphi(\eta)$. Since $\tau(\varphi - \varphi(\eta)) = \tau\varphi - \varphi(\eta)$, we may suppose $\varphi(\eta) = \operatorname{ess\,lim}_{\xi\to\eta} \varphi(\xi) = 0$ to show the above identity. Let $|\varphi| \leqq M$ a.e. on S^{n-1} for a positive constant M and $\rho(x) = |x - \eta|$ on $\boldsymbol{R}^n$. Clearly ρ belongs to the class $C(B^n \cup S^{n-1}) \cap W_p^1(B^n)$ and $\tau(\rho|S^{n-1}) = \pi\rho$, or roughly $\tau\rho = \pi\rho$. Hence by (30) we have

$$\lim_{x\in B^{n-1}, x\to\eta} (\tau\rho)(x) = 0.$$

For any $\varepsilon > 0$ there is a δ such that $|\varphi(\xi)| < \varepsilon$ for a.e. ξ in $B(\eta, \delta) \cap S^{n-1}$. Since $(M/\delta)\rho(\xi) \geqq M$ for every ξ in $S^{n-1}\backslash B(\eta, \delta)$, we see that

$$-\frac{M}{\delta}\rho(\xi) - \varepsilon \leqq \varphi(\xi) \leqq \frac{M}{\delta}\rho(\xi) + \varepsilon$$

a.e. on S^{n-1}. By Proposition 31, we have

$$-\frac{M}{\delta}(\tau\rho)(x) - \varepsilon \leqq (\tau\varphi)(x) \leqq \frac{M}{\delta}(\tau\rho)(x) + \varepsilon \qquad (x \in B^n).$$

On letting x in B^n tend to η, we see by $(\tau\rho)(x) \to 0$ that

$$-\varepsilon \leqq \liminf_{x\to\eta} (\tau\varphi)(x) \leqq \limsup_{x\to\eta} (\tau\varphi)(x) \leqq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we finally conclude the required identity $\lim_{x\in B^n, x\to\eta} (\tau\varphi)(x) = 0$. □

33. Estimates of Gagliardo norms

We are interested in computing or estimating $\|1_A; \Lambda_p(S^{n-1})\|$ for the open set $A = A((a_k), (b_k))$ given by (12) where 1_A stands for the characteristic function of A on S^{n-1}. In this case (26) takes the form

$$\|1_A; \Lambda_p(S^{n-1})\| = \sigma(A)^{1/p} + \left(2\iint_{A\times A^c} |\xi - \eta|^{-p-n+2} d\sigma(\xi) d\sigma(\eta)\right)^{1/p},$$

where A^c is the complement of A in S^{n-1}. In view of this, for two measurable subsets X and Y in S^{n-1} we consider the set function

$$S(X, Y) = \iint_{X\times Y} |\xi - \eta|^{-p-n+2} d\sigma(\xi) d\sigma(\eta)$$

by which we have

$$\|1_A; A_p(S^{n-1})\| = \sigma(A)^{1/p} + (2S(A, A^c))^{1/p}. \tag{34}$$

To facilitate the estimation of the quantity $S(X, Y)$ for various sets X and Y, we introduce an auxiliary set function. We denote by $X\hat{}$ the counterimage in $\boldsymbol{R}^{n-1} \times \{0\}$ of a set $X \subset S^{n-1}$ by the stereographic projection s, i.e. $s(X\hat{}) = X \backslash \{N\}$ where $N = (0, \dots, 0, 1) \in S^{n-1}$. Then $X\hat{}$ is measurable $d\lambda$, where $d\lambda$ is the surface element on $\boldsymbol{R}^{n-1} \times \{0\}$ so that $d\lambda(x) = dx^1 \cdots dx^{n-1}$ for $x = (x^1, \dots, x^{n-1}, 0)$ in $\boldsymbol{R}^{n-1} \times \{0\}$, if and only if X is measurable $d\sigma$. In addition to the standard norm $|x| = \left(\sum_{i=1}^n |x^i|^2\right)^{1/2}$ of a vector $x = (x^1, \dots, x^n)$ in $\boldsymbol{R}^n$ we consider the norm

$$\|x\| = \|x; \ell_1(n)\| = \sum_{i=1}^n |x^i|$$

of x. Clearly $|x| \leqq \|x\| \leqq \sqrt{n}|x|$ for every x in $\boldsymbol{R}^n$. For measurable subsets X and Y in S^{n-1} we consider the following auxiliary set function

$$T(X, Y) = \iint_{X\hat{} \times Y\hat{}} \|x - y\|^{-p-n+2} d\lambda(x) d\lambda(y).$$

Fix two regions Ω and ω on S^{n-1} given by

$$\begin{cases} \Omega = s(\{x \in \bar{\boldsymbol{R}}^{n-1} \times \{0\} : 2\sqrt{n} < |x| \leqq \infty\}), \\ \omega = s(\{x \in \bar{\boldsymbol{R}}^{n-1} \times \{0\} : |x| < \sqrt{n}\}). \end{cases} \tag{35}$$

Observe that $d\sigma(s(x))/d\lambda(x) = (2/(|x|^2 + 1))^{n-1}$ for every x in $\boldsymbol{R}^{n-1} \times \{0\}$ and

$$|s(x) - s(y)| = 2((|x|^2 + 1)(|y|^2 + 1))^{-1/2}|x - y|$$

for every x and y in $\boldsymbol{R}^{n-1} \times \{0\}$. Therefore, on setting

$$C_2 = C_2(n, p) = \max(2^{n-p} n^{(p+n-2)/2}, ((4n + 1)/2)^{n-p}),$$

we obtain

$$C_2^{-1} T(X, Y) \leqq S(X, Y) \leqq C_2 T(X, Y) \tag{36}$$

for every pair of measurable subsets X and Y in $S^{n-1} \backslash \Omega$.

Hereafter in this paper we always assume that $1 < p < 2$ unless otherwise is explicitly stated. We consider open intervals $(a, b; c)$ in $\boldsymbol{R}^{n-1} \times \{0\}$ given by

$$(a, b; c) = (a, b) \times (-c, c) \times \cdots \times (-c, c) \times \{0\} \subset \boldsymbol{R}^{n-1} \times \{0\},$$

where $-\infty \leqq a < b \leqq \infty$ and $0 < c \leqq \infty$. For the set $X = s((a,b;c)) \subset S^{n-1} \backslash \{N\}$, the number $|X| = b - a$ will be referred to as the *width* of X.

37. IDENTITY. *If* $A = s((b,a;1/2))$ *with* $-\infty < b < a < \infty$ *and* P *is either* $s((c,b;\infty))$ *with* $-\infty < c < b$ *or* $s((a,c;\infty))$ *with* $a < c < \infty$, *then*

$$T(A,P) = C_3(|A|^{2-p} + |P|^{2-p} - (|A| + |P|)^{2-p}),$$

where $C_3 = C_3(n,p) = 2^{n-2}(-\prod_{j=1}^{n}(p+n-2-j))^{-1}$.

PROOF. We first compute the iterated integral

$$J_i = \int_{-1/2}^{1/2} dx^i \int_{-\infty}^{\infty} dy^i \int_{-1/2}^{1/2} dx^{i-1} \int_{-\infty}^{\infty} dy^{i-1}$$

$$\cdots \int_{-1/2}^{1/2} dx^2 \int_{-\infty}^{\infty} \|x-y\|^{-p-n+2} dy^2$$

for $i = 2, \dots, n-1$ where $x = (x^1, \dots, x^{n-1}, 0)$ and $y = (y^1, \dots, y^{n-1}, 0)$. For each $x = (x^1, \dots, x^{n-1}, 0)$ we put

$$x_i = (x^1, \hat{x}^2, \dots, \hat{x}^i, x^{i+1}, \dots, x^{n-1}, 0)$$

for $i = 2, \dots, n-1$ where we mean $\hat{r} = 0$ for any real number r. The relations

$$J_i = k(i)\|x_i - y_i\|^{-p-n+1+i} \tag{38}$$

with $k(i) = 2^{i-1}\left(\prod_{j=2}^{i}(p+n-1-j)\right)^{-1}$ $(i = 2, \dots, n-1)$ can easily be seen to hold by induction on i. We assume that $P = s(a,c;\infty))$ in the following computation. By a similar consideration we can arrive at the same conclusion in the $P = s((c,b;\infty))$ case as well. Observe that

$$T(A,P) = \int_b^a dx^1 \int_a^c J_{n-1} dy^1.$$

Here by (38) we have $J_{n-1} = k(n-1)\|x_{n-1} - y_{n-1}\|^{-p} = k(n-1)|x^1 - y^1|^{-p}$. Therefore we obtain that

$$T(A,P) = k(n-1) \int_b^a dx^1 \int_a^c (y^1 - x^1)^{-p} dy^1$$

and an easy computation leads to the desired conclusion. □

39. IDENTITY. *If* $A = s((b,a;1/2))$ *with* $-\infty < b < a < \infty$ *and* P *is either* $s((-\infty,b;\infty))$ *or* $s((a,\infty;\infty))$, *then*

$$T(A,P) = C_3|A|^{2-p}.$$

PROOF. We assume that $P = s((a, \infty; \infty))$. The $P = s((-\infty, b; \infty))$ case can be treated similarly. Using J_{n-1} in (38), we have

$$T(A, p) = \int_b^a dx^1 \int_a^\infty J_{n-1} dy^1 = k(n-1) \int_b^a dx^1 \int_a^\infty (y^1 - x^1)^{-p} dy^1$$
$$= C_3 |A|^{2-p}. \qquad \square$$

We denote by Q_i^+ the image by the stereographic projection s of the half-hyperplane $\{x = (x^1, \ldots, x^{n-1}, 0) : x^i > 1/2\}$ of the hyperplane $\boldsymbol{R}^{n-1} \times \{0\}$ and similarly by Q_i^- the image by s of $\{x = (x^1, \ldots, x^{n-1}, 0) : x^i < -1/2\}$ for $i = 2, \ldots, n-1$.

40. IDENTITY. *If* $A = s((b, a; 1/2))$ *with* $-\infty < b < a < \infty$, *then*

$$T(A, Q_i^{\pm}) = C_3 |A| \qquad (i = 2, \ldots, n-1).$$

PROOF. By the symmetry we have $T(A, Q_i^{\pm}) = T(A, Q_2^+)$ $(i = 2, \ldots, n-1)$ and therefore we only have to show that $T(A, Q_2^+) = C_3 |A|$. Consider

$$J = \int_{-1/2}^{1/2} dx^{n-1} \int_{-\infty}^{\infty} dy^{n-1} \int_{-1/2}^{1/2} dx^{n-2} \int_{-\infty}^{\infty} dy^{n-2}$$
$$\cdots \int_{-1/2}^{1/2} dx^3 \int_{-\infty}^{\infty} \|x - y\|^{-p-n+2} dy^3,$$

where $x = (x^1, \ldots, x^{n-1}, 0)$ and $y = (y^1, \ldots, y^{n-1}, 0)$. As we deduced the identities (38) we can also show that

$$J = k(n-2)(|x^1 - y^1| + |x^2 - y^2|)^{-p-1},$$

where $k(n-2)$ is identical with that in (38) for $i = n-2$. Using $k(n-2)p^{-1} = 2^{-1}k(n-1)$, we see that

$$T(A, Q_2^+) = \int_b^a dx^1 \int_{-\infty}^{\infty} dy^1 \int_{-1/2}^{1/2} dx^2 \int_{1/2}^{\infty} J\, dy^2$$
$$= k(n-2) \int_b^a dx^1 \int_{-\infty}^{\infty} dy^1 \int_{-1/2}^{1/2} dx^2 \int_{1/2}^{\infty} (|x^1 - y^1| + (y^2 - x^2))^{-p-1} dy^2$$
$$= k(n-2)p^{-1} \int_b^a dx^1 \int_{-\infty}^{\infty} dy^1 \int_{-1/2}^{1/2} (|x^1 - y^1| + (1/2 - x^2))^{-p} dx^2$$
$$= 2^{-1}k(n-1) \int_{-1/2}^{1/2} dx^2 \int_b^a dx^1 \int_{-\infty}^{\infty} (|x^1 - y^1| + (1/2 - x^2))^{-p} dy^1$$

$$= (p-1)^{-1}k(n-1)\int_{-1/2}^{1/2} dx^2 \int_b^a (1/2 - x^2)^{1-p} dx^1$$

$$= (p-1)^{-1}k(n-1)(b-a)\int_{-1/2}^{1/2} (1/2 - x^2)^{1-p} dx^2 = C_3|A|. \qquad \square$$

For any set X in S^{n-1} we denoted by X^c the complement $S^{n-1}\backslash X$ of X in S^{n-1} so that $(X^c)^\wedge = (X^\wedge)^c$ where $(X^\wedge)^c$ is the complement $\boldsymbol{R}^{n-1} \times \{0\}\backslash X^\wedge$ of $X^\wedge$ in $\boldsymbol{R}^{n-1} \times \{0\}$.

41. ESTIMATE. *If* $A = s((b,a;1/2))$ *with* $-\infty < b < a < \infty$, *then*

$$T(A, A^c) \leqq C_4(|A|^{2-p} + (n-2)|A|),$$

where $C_4 = C_4(n,p) = 2C_3(n,p)$.

PROOF. In addition to open sets $Q_i^{\pm}$ $(i = 2, \ldots, n-1)$ in the identity 40, we consider two more open subsets $Q_1^+ = s((a,\infty;\infty))$ and $Q_1^- = s((-\infty,b;\infty))$. Since

$$(\bar{A})^c \subset \bigcup_{i=1}^{n-1} (Q_i^+ \cup Q_i^-),$$

the identities 39 and 40 imply that

$$T(A, A^c) \leqq T\left(A, \bigcup_{i=1}^{n-1} (Q_i^+ \cup Q_i^-)\right) \leqq \sum_{i=1}^{n-1} (T(A, Q_i^+) + T(A, Q_i^-))$$

$$= (T(A, Q_1^+) + T(A, Q_1^-)) + \sum_{i=2}^{n-1} (T(A, Q_i^+) + T(A, Q_i^-))$$

$$= 2C_3|A|^{2-p} + (n-2)\cdot 2C_3|A| = C_4(|A|^{2-p} + (n-2)|A|). \qquad \square$$

If $|A| \leqq 1$, then, since $1 < p < 2$, we see that $|A| \leqq |A|^{2-p}$. Hence the estimate 41 trivially yields the following result.

42. ESTIMATE. *If* $A = s((b,a;1/2))$ *with* $0 < b < a < 1$, *then*

$$T(A, A^c) \leqq C_5|A|^{2-p},$$

where $C_5 = C_5(n,p) = (n-1)C_4(n,p)$.

Recall that ω in (35) is the spherical cap in S^{n-1} with center $S = (0, \ldots, 0, -1)$, the south pole of S^{n-1}, such that $\omega^\wedge = \{x \in \boldsymbol{R}^{n-1} \times \{0\} : |x| <$

$\sqrt{n}\}$; Ω in (35) is the spherical cap in S^{n-1} with center at the north pole N such that $\Omega^{\wedge} = \{x \in \boldsymbol{R}^{n-1} \times \{0\} : |x| > 2\sqrt{n}\}$. Observe that, if $A = s((b,a;1/2))$ with $0 < b < a < 1$, then $\bar{A} \subset \omega$.

43. ESTIMATE. *If* $A = s((b,a;1/2))$ *with* $0 < b < a < 1$, *then*

$$T(A,\Omega) \leqq C_6|A|$$

where $C_6 = C_6(n,p) = (p-1)^{-1}n^{(1-p)/2}\sigma_{n-2}, \sigma_{n-2}$ *being the area of* S^{n-2}.

PROOF. For any (x,y) in $A^{\wedge} \times \Omega^{\wedge}$, we have $|x-y| \geqq \sqrt{n}$. Since $\|x-y\| \geqq |x-y|$, we see that $\|x-y\|^{-p-n+2} \leqq |x-y|^{-p-n+2}$. Using the surface element $d\sigma_{n-2}$ on S^{n-2} we deduce

$$\begin{aligned} T(A,\Omega) &= \iint_{A^{\wedge}\times\Omega^{\wedge}} \|x-y\|^{-p-n+2} d\lambda(x)d\lambda(y) \\ &\leqq \int_{A^{\wedge}} d\lambda(x) \int_{|x-y|\geqq\sqrt{n}} |x-y|^{-p-n+2} d\lambda(y) \\ &= \int_{A^{\wedge}} d\lambda(x) \left(\int_{S^{n-2}} d\sigma_{n-2} \int_{\sqrt{n}}^{\infty} r^{-p-n+2} r^{n-2} dr \right) \\ &= (p-1)^{-1} n^{(1-p)/2} \sigma_{n-2}|A|. \end{aligned}$$

□

44. Dirichlet integrals of harmonic measures

In this section we study the p-Dirichlet finiteness of the $\mathscr{A}$-harmonic measure $\omega(A,B^n;\mathscr{A})$ of the boundary set A given in (12). Observe that the $(n\text{-}p)$-Hausdorff measures of boundary components of $s^{-1}(A)$ relative to $\boldsymbol{R}^{n-1} \times \{0\}$ and hence of boundary components of A relative to S^{n-1} are finite for $1 < p \leqq 2$ when $n > 2$. Hence the variational p-capacity of $\bar{A}\backslash A$ is zero (cf. e.g. [2, 2.27 and 2.8] for $n > 2$; [2, 2.12 and 2.8] for $n = 2$):

$$\operatorname{cap}_p(\bar{A}\backslash A) = 0 \qquad (1 < p \leqq 2, n \geqq 2).$$

We also use the following form of *comparison principle* which is a special case of Lemma 7.37 in [2]: suppose that G is a bounded subregion of $\boldsymbol{R}^n$ and $E \subset \partial G$ satisfies $\operatorname{cap}_p E = 0$; assume that u and v are bounded $\mathscr{A}$-harmonic functions on G, where $\mathscr{A} \in \mathscr{A}_p(\boldsymbol{R}^n)$ with $1 < p \leqq n$, such that

$$\limsup_{y\to x} u(y) \leqq \liminf_{y\to x} v(y)$$

for all $x \in \partial G\backslash E$; if at least one of u and v is p-Dirichlet finite on G, then $u \leqq v$

everywhere on G. Using these two facts mentioned above we first prove the following result announced in the introduction.

45. PROPOSITION. *Suppose* $1 < p \leqq 2$ *and* $A \in \mathscr{A}_p(\boldsymbol{R}^n)$. *The function* $\omega = \omega(A, B^n; \mathscr{A})$ *is an* $\mathscr{A}$*-harmonic measure in the sense of Heins:* $\omega \wedge (1 - \omega) = 0$ *on* B^n.

PROOF. In view of (14) we see that $\lim_{y \to x} (\omega \wedge (1 - \omega))(y) = 0$ for all x in $S^{n-1} = \partial B^n$ except for the set $\bar{A} \backslash A$ of p-capacity zero. By the above comparison principle we deduce that $\omega \wedge (1 - \omega) = 0$ on B^n, as desired. □

Recall that 1_A is the characteristic function of the open set A considered on $S^{n-1} : 1_A = 1$ on A and $1_A = 0$ on $S^{n-1} \backslash A$ so that $1_A \in L_p(S^{n-1})$ $(1 \leqq p \leqq \infty)$.

46. PROPOSITION. *Suppose* $1 < p < 2$ *and* $\mathscr{A} \in \mathscr{A}_p(\boldsymbol{R}^n)$. *The* $\mathscr{A}$*-harmonic measure* $\omega(A, B^n; \mathscr{A})$ *is p-Dirichlet finite on* B^n *if and only if* $1_A \in \varLambda_p(B^n)$, *and in this case* $\omega(A, B^n; \mathscr{A}) = \tau_{\mathscr{A}} 1_A$ *on* B^n.

PROOF. First assume that $\omega = \omega(A, B^n; \mathscr{A})$ is p-Dirichlet finite on B^n. By (14) and (24) we see that $1_A = \gamma\omega \in \varLambda_p(S^{n-1})$. Hence, by the definition of $\tau = \tau_{\mathscr{A}} = \tau_{\mathscr{A}}^{B^n}$, we conclude that $\omega = \tau 1_A$. Conversely we suppose that $1_A \in \varLambda_p(S^{n-1})$. Proposition 32 assures that

$$\begin{cases} \lim_{y \to x} (\tau 1_A)(y) = 1 & (x \in A), \\ \lim_{y \to x} (\tau 1_A)(y) = 0 & (x \in S^{n-1} \backslash \bar{A}). \end{cases}$$

This with (14) implies $\lim_{y \to x} \omega(y) = \lim_{y \to x} (\tau 1_A)(y)$ for all x in $S^{n-1} \backslash (\bar{A} \backslash A)$ with $\mathrm{cap}_p(\bar{A} \backslash A) = 0$. The above comparison principle yields $\omega = \tau 1_A$, which is p-Dirichlet finite on B^n. □

47. COROLLARY. *Suppose* $1 < p < 2$ *and* $\mathscr{A} \in \mathscr{A}_p(\boldsymbol{R}^n)$. *If* $K < \infty$, *then* $\omega(A, B^n; \mathscr{A})$ *is p-Dirichlet finite.*

PROOF. In view of (34), (36) and the estimate 42 we see that

$$\begin{aligned} \|1_A; \varLambda_p(S^{n-1})\| &\leqq \sum_{k=1}^{K} \|1_{A_k}; \varLambda_p(S^{n-1})\| \\ &\leqq \sum_{k=1}^{K} (\sigma(A_k)^{1/p} + (2C_2C_5)^{1/p} |A_k|^{(2-p)/p}) < \infty \end{aligned}$$

so that $1_A \in \varLambda_p(S^{n-1})$ and therefore $\omega(A, B^n; \mathscr{A}) = \tau 1_A$ is p-Dirichlet finite. □

48. Proof of Main theorem 15

If $K < \infty$, then, by Corollary 47, $\omega = \omega(A, B^n; \mathscr{A})$ is p-Dirichlet finite on B^n. Hence, hereafter in this proof, we assume that $K = \infty$ so that $A = A((a_k), (b_k)) = \bigcup_{k=1}^{\infty} A_k$.

We now start the essential part of this proof by showing that (16) implies the p-Dirichlet finiteness of $\omega = \omega(A, B^n; \mathscr{A})$. Suppose first that $\sum_{k=1}^{\infty} |a_k - b_k|^{2-p} < \infty$ so that $\sum_{k=1}^{\infty} |A_k|^{2-p} < \infty$ since $A_k = s((b_k, a_k; 1/2))$ $(k = 1, 2, \ldots)$. By (34) we have

$$\|1_A; A_p(S^{n-1})\| = \sigma(A)^{1/p} + (2S(A, A^c))^{1/p}.$$

Using (36) we deduce

$$\begin{aligned} S(A, A^c) &= S(A, A^c \backslash \Omega) + S(A, A^c \cap \Omega) \leqq S(A, A^c \backslash \Omega) + S(\omega, \Omega) \\ &\leqq C_2 T(A, A^c \backslash \Omega) + S(\omega, \Omega) \leqq C_2 T(A, A^c) + S(\omega, \Omega), \end{aligned}$$

where $S(\omega, \Omega) < \infty$. Since $A^c \subset A_k^c$, by using Estimate 42, we see that

$$T(A, A^c) = \sum_{k=1}^{\infty} T(A_k, A^c) \leqq \sum_{k=1}^{\infty} T(A_k, A_k^c) \leqq \sum_{k=1}^{\infty} C_5 |A_k|^{2-p} < \infty.$$

We have thus shown that $\|1_A; A_p(S^{n-1})\| < \infty$ and a fortiori Proposition 46 yields that ω is p-Dirichlet finite on B^n.

We next consider the case $\sum_{k=1}^{\infty} |a_{k+1} - b_k|^{2-p} < \infty$. We wish to show that $\|1_A; A_p(S^{n-1})\| < \infty$ again. We set $B_k = s((a_{k+1}, b_k; 1/2))$ $(k = 1, 2, \cdots)$, $B = \bigcup_{k=1}^{\infty} B_k$ and $C = s((0, a_1; 1/2))$. Clearly $1_C = 1_A + 1_B$ a.e. on S^{n-1} and (34) assures that

$$\|1_C; A_p(S^{n-1})\| = \sigma(C)^{1/p} + (2S(C, C^c))^{1/p}.$$

As we estimated $S(A, A^c)$ above, we also see by Estimate 42 that

$$\begin{aligned} S(C, C^c) &\leqq C_2 T(C, C^c) + S(\omega, \Omega) \\ &\leqq C_2 C_5 |C|^{2-p} + S(\omega, \Omega) \\ &\leqq C_2 C_5 a_1^{2-p} + S(\omega, \Omega) < \infty. \end{aligned}$$

Therefore $\|1_C; A_p(S^{n-1})\| < \infty$. Hence we only have to show that $\|1_B; A_p(S^{n-1})\| < \infty$ in order to conclude $\|1_A; A_p(S^{n-1})\| < \infty$. Since $\sum_{k=1}^{\infty} |B_k|^{2-p} < \infty$, by repeating the above argument on replacing A_k by B_k, we deduce $\|1_B; A_p(S^{n-1})\| < \infty$.

We close this proof by showing that (17) implies that $\omega = \omega(A, B^n; \mathscr{A})$ is p-Dirichlet infinite. We prove this by contradiction. Suppose, contrary to the assertion, that ω is p-Dirichlet finite. By Proposition 46, we must have $1_A \in A_p(S^{n-1})$ so that, by (34), $S(A, A^c) < \infty$. Using (36) and the estimate 43, we deduce

$$\begin{aligned} S(A, A^c) &\geqq S(A, A^c \backslash \Omega) \geqq C_2^{-1} T(A, A^c \backslash \Omega) \\ &= C_2^{-1}(T(A, A^c) - T(A, A^c \cap \Omega)) \\ &\geqq C_2^{-1}(T(A, A^c) - T(A, \Omega)) \geqq C_2^{-1}(T(A, A^c) - C_6|A|) \\ &\geqq C_2^{-1} T(A, A^c) - C_2^{-1} C_6 \end{aligned}$$

since $|A| < 1$. We set $P_k = s((a_{k+1}, b_k; \infty))$ $(k = 1, 2, \cdots)$. Since $P_k \subset A^c$ for every $k = 1, 2, \ldots$, we see that

$$T(A, A^c) = \sum_{k=1}^{\infty} T(A_k, A^c) \geqq \sum_{k=1}^{\infty} T(A_k, P_k).$$

Hence, by using Identity 37, we deduce

$$T(A, A^c) \geqq \sum_{k=1}^{\infty} C_3(|A_k|^{2-p} + |P_k|^{2-p} - (|A_k| + |P_k|)^{2-p}).$$

Here recall the following well known elementary inequality valid for $1 < p < 2$ and also for real numbers x, y, a, and b:

$$x^{2-p} + y^{2-p} - (x+y)^{2-p} \geqq a^{2-p} + b^{2-p} - (a+b)^{2-p} \qquad (0 \leqq a \leqq x, 0 \leqq b \leqq y).$$

On setting $x = |A_k|$, $y = |P_k|$, and $a = b = \min(|A_k|, |P_k|)$ in the above inequality, we obtain

$$\begin{aligned} &|A_k|^{2-p} + |P_k|^{2-p} - (|A_k| + |P_k|)^{2-p} \\ &\qquad \geqq 2(\min(|A_k|, |P_k|))^{2-p} - (2\min(|A_k|, |P_k|))^{2-p}. \end{aligned}$$

Therefore we have

$$T(A, A^c) \geqq (2 - 2^{2-p}) C_3 \sum_{k=1}^{\infty} \min(|A_k|^{2-p}, |P_k|^{2-p}),$$

which implies the following contradiction:

$$\begin{aligned} \infty &> S(A, A^c) + C_2^{-1} C_6 \\ &\geqq C_2^{-1}(2 - 2^{2-p}) C_3 \sum_{k=1}^{\infty} \min(|a_k - b_k|^{2-p}, |a_{k+1} - b_k|^{2-p}) = \infty. \end{aligned}$$ □

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