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Immersions and embeddings of orientable manifolds up to unoriented cobordism

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ABSTRACT. We investigate the existence of immersions and embeddings of orientable manifolds in the Euclidean space up to unoriented cobordism, and we get the best estimates in some cases. Our study is an orientable version of the work investigated by R. L. Brown.

1. Introduction

In this paper, we investigate immersions and embeddings of orientable closed manifolds in the Euclidean space $\mathbf{R}^m$ up to unoriented cobordism. Manifolds are always assumed to be C^∞ differentiable, and two closed n-dimensional manifolds M_1^n and M_2^n are *cobordant* if there exists a compact manifold N^{n+1} whose boundary ∂N is the disjoint union of M_1 and M_2. We refer to a closed manifold simply as a manifold.

The source of our study is the next theorem by Brown [1]:

THEOREM 1.1 (Brown). *Let $\alpha(n)$ denote the number of* 1 *in the dyadic expansion of n.*

(1) *For $n \geq 2$, any manifold M^n is cobordant to a manifold which immerses in $\mathbf{R}^{2n-\alpha(n)}$ and embeds in $\mathbf{R}^{2n-\alpha(n)+1}$.*

(2) *For each $n \geq 2$ with $n \neq 3$, there is an n-dimensional manifold such that any manifold cobordant to it does not immerse in $\mathbf{R}^{2n-\alpha(n)-1}$ and does not embed in $\mathbf{R}^{2n-\alpha(n)}$.*

Our main results are stated as follows:

THEOREM A. *Let $\beta(n) = 2n - \alpha(n) - \min\{\alpha(n), \nu(n)\}$, where $\nu(n)$ is the integer determined by $n = (2m+1)2^{\nu(n)}$.*

(1) *Any orientable manifold M^n is cobordant to a manifold which immerses in $\mathbf{R}^{\beta(n)}$ and embeds in $\mathbf{R}^{\beta(n)+1}$ for $n \geq 4$.*

(2) *If n satisfies one of the following conditions* (i)–(iii), *then there is an*

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n-dimensional orientable manifold such that any manifold cobordant to it does not immerse in $\mathbf{R}^{\beta(n)-1}$ *and does not embed in* $\mathbf{R}^{\beta(n)}$:

(i) $n \geq 4$ *and* $\alpha(n) \leq \nu(n)$;

(ii) $\alpha(n) = 3$ *and* $n \equiv 2 \pmod 4$;

(iii) $\alpha(n) = 2$ *and* $n \equiv 1 \pmod 4$.

In this theorem, (2) implies that (1) is the best estimate in the cases (i)–(iii). On the other hand, we will have a better estimate in the following cases, where $\min\{\alpha(n), \nu(n)\}$ is always equal to $\nu(n)$.

THEOREM B. *If n satisfies one of the following conditions* (i)–(iv), *then any orientable manifold* M^n *is cobordant to a manifold which immerses in* $\mathbf{R}^{\gamma(n)}$ *and embeds in* $\mathbf{R}^{\gamma(n)+1}$ *for* $\gamma(n) = 2n - \alpha(n) - \nu(n) - 1$, *and there is an n-dimensional orientable manifold such that any manifold cobordant to it does not immerse in* $\mathbf{R}^{\gamma(n)-1}$ *and does not embed in* $\mathbf{R}^{\gamma(n)}$:

(i) $\alpha(n) = 2$ *and* $n \equiv 2 \pmod 8$;

(ii) $\alpha(n) = 3$ *and* $n \equiv 4 \pmod 8$;

(iii) $\alpha(n) = 4$ *and* $n \equiv 2 \pmod 4$;

(iv) $n \neq 7$ *is odd and* $\alpha(n) = 3$.

We note that, for all $n \leq 26$ but $n = 15$ and 23, Theorems A and B give the best estimate.

The paper is organized as follows: In §2 we recall some results concerning the cobordism theory. In §3 we prove Theorem A(1) and a part of Theorem B. In §4 we complete the proof of Theorems A and B.

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2. Preliminaries

Let $\mathcal{N}_*$ be the unoriented cobordism ring, Ω_* the oriented cobordism ring, and $I: \Omega_* \to \mathcal{N}_*$ the natural map obtained by ignoring orientation. We call $\omega = (a_1, \ldots, a_k)$ a *partition* of n if it consists of positive integers with $\sum_{j=1}^{k} a_j = n$, and put $|\omega| = n$. We regard that two partitions which consist of the same integers are the same, for example $(2, 2, 5, 6) = (6, 2, 5, 2)$. For the partitions $\omega_j = (a_{j1}, \ldots, a_{jm_j})$ $(1 \leq j \leq k)$, we denote $(\omega_1, \ldots, \omega_k) = (a_{11}, \ldots, a_{1m_1}, \ldots, a_{k1}, \ldots, a_{km_k})$. Let $P = \{\omega = (a_1, \ldots, a_k)\}$ be the set of all partitions. Then, we consider the following subsets of P:

$P_0 = \{(a_1, \ldots, a_k) \in P \mid \text{for } 1 \leq j \leq k,\ a_j \neq 2^i - 1 \text{ for any } i \geq 1\}$;

$P_1 = \{(a_1, \ldots, a_k) \in P_0 \mid \text{the number of } j\text{'s with } a_j = 2^i \text{ is even for any } i\}$;

$P_2 = \{(2a_1, \ldots, 2a_k) \in P_1 \mid a_i \neq a_j \text{ for any } i \neq j\}$;

$P_3 = \{(2a, 2a) \in P_1\}$;

$P_4 = \{(2a_1, \ldots, 2a_j - 1, \ldots, 2a_k) \in P_1 \mid 1 \le j \le k,\ (2a_1, \ldots, 2a_k) \in P_2\}$;
$P_5 = \{\omega = (\omega_1, \ldots, \omega_k) \in P_1 \mid \omega_j \in P_3\ (1 \le j \le k)\}$;
$P_6 = \{\omega = (\omega_1, \ldots, \omega_k) \in P_1 \mid \omega_j \in P_3 \cup P_4\ (1 \le j \le k)\} \supset P_5$.
The following result of Wall [4] is crucial in our study.

THEOREM 2.1 (Wall). *There are elements* $x_q \in \mathcal{N}_q$ $(q \neq 2^i - 1)$, $h_{4q} \in \Omega_{4q}$ $(q \ge 1)$ *and* $g_\omega \in \Omega_{|\omega|-1}$ $(\omega = (2a_1, \ldots, 2a_k) \in P_2)$ *which satisfy the following* (i)–(iii):

(i) $\mathcal{N}_* = \mathbf{Z}_2[x_q \mid q \neq 2^i - 1]$;

(ii) h_{4q} $(q \ge 1)$ *and* g_ω $(\omega \in P_2)$ *form a set of generators for* Ω_*;

(iii) $I(h_{4q}) = x_{2q}^2$ *and* $I(g_\omega) = \sum_{j=1}^k x_{2a_1} \cdots x_{2a_j - 1} \cdots x_{2a_k}$.

We put $x_\omega = \prod_{j=1}^k x_{a_j}$ for a partition $\omega = (a_1, \ldots, a_k)$. Then, we have the following corollary.

COROLLARY 2.2. *Concerning the image of* $I\colon \Omega_* \to \mathcal{N}_*$, *we have that* $I(\Omega_*)$ *is generated by* $I(h_{4q})$ $(q \ge 1)$ *and* $I(g_\omega)$ $(\omega \in P_2)$. *Hence, for any* $y \in \Omega_*$, $I(y)$ *is a finite sum of* x_ω *with* $\omega \in P_6$.

Remark that the cobordism class $[M^n]$ is indecomposable in $\mathcal{N}_*$ if and only if it equals x_n up to decomposable elements by Theorem 2.1(i). Brown [1; §6] has shown some manifolds representing indecomposable elements in $\mathcal{N}_*$ as follows:

THEOREM 2.3 (Brown). *For every positive integer* $n \neq 2^i - 1$ $(i \ge 1)$, *there are manifolds* $\{W^n\}$ *which satisfy the following* (i)–(iii):

(i) $[W^n]$ *is indecomposable in* $\mathcal{N}_*$, *and hence* $\mathcal{N}_* = \mathbf{Z}_2[[W^n] \mid n \neq 2^i - 1\ (i \ge 1)]$;

(ii) W^n *immerses in* $\mathbf{R}^{2n-\alpha(n)}$ *and embeds in* $\mathbf{R}^{2n-\alpha(n)+1}$;

(iii) W^n *immerses in* $\mathbf{R}^{2n-\alpha(n)-1}$ *if* $n \neq 2^i,\ 2^i + 1$ $(i \ge 1)$, *and embeds in* $\mathbf{R}^{2n-\alpha(n)}$ *if* $n \neq 6,\ 2^i,\ 2^i + 1$ $(i \ge 1)$.

We need the precise construction of W^n, and we recall it below:

First, let n be even. When $\alpha(n) \le 2$, we put $W^n = \mathbf{R}P^n$ the real projective space. When $\alpha(n) \ge 3$, for the dyadic expansion $n = \sum_{j=1}^{k+1} r_j$ of n, where $k + 1 = \alpha(n)$, $2 \le r_1 < \cdots < r_{k+1}$ and each r_j is a power of 2, we put

$$K^{n+1} = \prod_{j=1}^{k} \mathbf{R}P^{s_j} \quad \text{for } s_j = r_j \quad (j \le k-2), \quad s_{k-1} = r_{k-1} + 1 \text{ and } s_k = r_k + r_{k+1}.$$

Then, $H^*(K^{n+1}) = \mathbf{Z}_2[\alpha_1, \ldots, \alpha_k]/(\alpha_1^{s_1+1}, \ldots, \alpha_k^{s_k+1})$ with $\deg(\alpha_j) = 1$, and we define W^n as a submanifold of K^{n+1} which represents the Poincaré dual of $\sum_{j=1}^k \alpha_j \in H^1(K^{n+1})$ (see [3; pp. 78–81]), where the cohomology is always assumed to be with the coefficient group $\mathbf{Z}_2$.

Next, consider the definition of W^n for odd $n \neq 2^i - 1$. Since $n \neq 2^i - 1$, we can write n uniquely as $n = a + 2b$, $a = 2^r - 1$ and $b = 2^r s$ $(r, s \geq 1)$. By using W^b for even b, we define W^n to be

$$W^n = P(a, W^b) = S^a \times W^b \times W^b/(u, x, y) \sim (-u, y, x). \tag{2.4}$$

We notice that, if M^{m_j} immerses in $\mathbf{R}^{s_j}$ $(1 \leq j \leq k)$, then $\prod_{j=1}^k M^{m_j}$ immerses in $\mathbf{R}^s$ where $s = \sum_{j=1}^k s_j$. We need the following result to prove the possibilities of embeddings.

THEOREM 2.5 (Cf. [1]). *If M^m immerses in $\mathbf{R}^s$ and N^n embeds in $\mathbf{R}^t$ for $s + t \geq 2m + 1$, then $M^m \times N^n$ embeds in $\mathbf{R}^{s+t}$.*

COROLLARY 2.6. (i) *If M^m immerses in $\mathbf{R}^s$ $(m \leq s)$ and N^n embeds in $\mathbf{R}^t$ $(n < t)$ for $m \leq n$, then $M^m \times N^n$ embeds in $\mathbf{R}^{s+t}$.*

(ii) *Let $m_1 \leq \cdots \leq m_k$. If M^{m_j} immerses in $\mathbf{R}^{s_j}$ $(1 \leq j \leq k-1, m_j \leq s_j)$ and M^{m_k} embeds in $\mathbf{R}^{s_k}$ $(m_k < s_k)$, then $\prod_{j=1}^k M^{m_j}$ embeds in $\mathbf{R}^s$ where $s = \sum_{j=1}^k s_j$.*

3. Possibilities of immersions and embeddings

In this section, we verify the possibilities of immersions and embeddings of manifolds stated in Theorems A and B. Let $\omega = (a_1, \dots, a_k)$ be a partition, and put $l(\omega) = k$ and $\alpha(\omega) = \sum_{j=1}^k \alpha(a_j)$, where $\alpha(a_j)$ is the number of 1 in the dyadic expansion of a_j. For the manifolds $\{W^n\}$ of Theorem 2.3, we define $W^\omega = \prod_{j=1}^k W^{a_j}$. We represent x_ω $(\omega \in P_6)$ by using the manifolds $\{W^n\}$. Since x_a equals the cobordism class $[W^a]$ up to decomposable elements, x_a is represented by W^a plus a finite union of W^{ω_a} with $|\omega_a| = a$ and $l(\omega_a) > 1$ and $\omega_a \in P_0$. Thus, x_ω is represented by W^ω plus a finite union of $W^{\omega'}$ with $l(\omega') > l(\omega)$ and $\omega' = (\omega'_1, \dots, \omega'_k) \in P_0$ where ω'_j is a partition of a_j $(1 \leq j \leq k)$. We denote by X_ω this manifold which represents x_ω. Here, we remark that $\alpha(\omega') \geq \alpha(\omega)$, since $\alpha(a) + \alpha(b) \geq \alpha(a + b)$ for any integers a and b.

LEMMA 3.1. *If $|\omega| = n$ and $\alpha(\omega) \geq \delta$, then X_ω immerses in $\mathbf{R}^{2n-\delta}$ and embeds in $\mathbf{R}^{2n-\delta+1}$.*

PROOF. Let $\omega = (a_1, \dots, a_k)$. Then, $2n - \alpha(\omega) = \sum_{j=1}^k \{2a_j - \alpha(a_j)\}$, and W^ω immerses in $\mathbf{R}^{2n-\alpha(\omega)}$ and embeds in $\mathbf{R}^{2n-\alpha(\omega)+1}$ by Theorem 2.3(ii) and Corollary 2.6(ii). Thus, W^ω immerses in $\mathbf{R}^{2n-\delta}$ and embeds in $\mathbf{R}^{2n-\delta+1}$ since $\alpha(\omega) \geq \delta$. For the other components $W^{\omega'}$ as above, each $W^{\omega'}$ also immerses in $\mathbf{R}^{2n-\delta}$ and embeds in $\mathbf{R}^{2n-\delta+1}$, because $\alpha(\omega') \geq \alpha(\omega) \geq \delta$. □

PROOF OF THEOREM A(1). Since any element of the subgroup $I(\Omega_*)$ of $\mathcal{N}_*$ is a finite sum of x_ω with $\omega \in P_6$ by Corollary 2.2, any orientable manifold M^n is cobordant to a finite union of X_ω with $\omega \in P_6$ and $|\omega| = n$. We show

the following for a partition ω with $|\omega| = n$:
(3.2) *If* $\omega \in P_5$, *then* $\alpha(\omega) \geq 2\alpha(n)$;
(3.3) *If* $\omega \in P_6 - P_5$, *then* $\alpha(\omega) \geq \alpha(n) + \nu(n)$.
Then, by Lemma 3.1, any manifold X_ω with $\omega \in P_6$ and $|\omega| = n$ immerses in $\mathbf{R}^{2n-2\alpha(n)}$ or $\mathbf{R}^{2n-\alpha(n)-\nu(n)}$ and embeds in $\mathbf{R}^{2n-2\alpha(n)+1}$ or $\mathbf{R}^{2n-\alpha(n)-\nu(n)+1}$. Thus, X_ω immerses in $\mathbf{R}^{\beta(n)}$ and embeds in $\mathbf{R}^{\beta(n)+1}$ for $\beta(n) = 2n - \alpha(n) - \min\{\alpha(n), \nu(n)\}$, and we have the required result for an orientable manifold M^n.

We show (3.2) and (3.3). Remark that $\alpha(2a) = \alpha(a)$, $\alpha(2a+1) = \alpha(2a) + 1$, $\alpha(a) + \alpha(b) \geq \alpha(a+b)$ and $\alpha(a-1) + 1 = \alpha(a) + \nu(a)$. Let $\omega = (2a_1, 2a_1, \ldots, 2a_k, 2a_k) \in P_5$. Then, we have $\alpha(\omega) = \sum_{j=1}^k 2\alpha(2a_j) \geq 2\alpha(\sum_{j=1}^k 2a_j) = 2\alpha(\sum_{j=1}^k 4a_j) = 2\alpha(n)$, and (3.2) is proved. Next, let $\omega = (a_1, \ldots, a_k) \in P_6 - P_5$, then ω has at least one odd number. So we assume a_k is odd, and put $\omega' = (a_1, \ldots, a_{k-1})$ and $a_k = 2l + 1$. Then, $n = |\omega'| + a_k$, and $\alpha(\omega) = \alpha(\omega') + \alpha(a_k) \geq \alpha(n - a_k) + \alpha(a_k) = \alpha(n - 2l - 1) + \alpha(2l) + 1 \geq \alpha(n-1) + 1 = \alpha(n) + \nu(n)$. Thus, we have (3.3). □

Now, we prepare two lemmas to prove the possibilities of immersions and embeddings in Theorem B.

LEMMA 3.4. *Let* $\omega \in P_6 - P_5$. *Then, we have the following*:
(i) *If* $|\omega|$ *is even and* $l(\omega) \geq 3$ *(resp.* $l(\omega) = 2$*), then* $\alpha(\omega) \geq 6$ *(resp.* $\alpha(\omega) \geq 4$*);*
(ii) *If* $|\omega|$ *is odd and* $l(\omega) \geq 2$, *then* $\alpha(\omega) \geq 4$.

PROOF. (i) By assumptions, ω has at least two odd numbers $a_i \neq 1$ and $a_j \neq 1$. When $l(\omega) \geq 4$, ω has at least two other numbers, and so $\alpha(\omega) \geq \alpha(a_i) + \alpha(a_j) + 1 + 1 \geq 2 + 2 + 1 + 1 = 6$ as required. When $l(\omega) = 3$, ω has another even number a_k. Since $\omega \in P_1$, a_k is not a power of 2. Hence, $\alpha(\omega) = \alpha(a_i) + \alpha(a_j) + \alpha(a_k) \geq 2 + 2 + 2 = 6$ as required. When $l(\omega) = 2$, ω has just two odd numbers $a_i \neq 1$ and $a_j \neq 1$, and thus $\alpha(\omega) = \alpha(a_i) + \alpha(a_j) \geq 2 + 2 = 4$ as required.

(ii) By assumptions, ω has at least one odd number $a_i \neq 1$. When $l(\omega) \geq 3$, ω has at least two other numbers, and so $\alpha(\omega) \geq \alpha(a_i) + 1 + 1 \geq 2 + 1 + 1 = 4$ as required. When $l(\omega) = 2$, ω has another even number a_j. Since $\omega \in P_1$, a_j is not a power of 2. Hence, $\alpha(\omega) = \alpha(a_i) + \alpha(a_j) \geq 2 + 2 = 4$ as required. □

We remark that, if n is odd and $\alpha(n) \leq 3$, then W^n of (2.4) is cobordant to a Dold manifold which is orientable. In fact, in this case, $W^n = P(a, \mathbf{R}P^b)$ for $n = a + 2b$, $a = 2^r - 1$ and $b = 2^r s$ $(r, s \geq 1)$ by (2.4). Brown [1; Cor. 7.4] showed that $P(a, \mathbf{R}P^b)$ is cobordant to the Dold manifold $P(a, b) = S^a \times \mathbf{C}P^b/(u, z) \sim (-u, \bar{z})$, and Dold [2; Satz 1, p. 29 and Satz 2, p. 30] proved that the total Stiefel-Whitney class of $P(a, b)$ is

$$w(P(a, b)) = (1 + c)^a(1 + c + d)^{b+1}. \tag{3.5}$$

Here, $c \in H^1(P(a, b))$ and $d \in H^2(P(a, b))$, which satisfy $c^{a+1} = 0$ and $d^{b+1} = 0$. Thus, in this case, $P(a, b)$ is an orientable manifold, because $w_1(P(a, b)) = (a + b + 1)c = 0$.

LEMMA 3.6. *Let $\omega \in P_6$ and $|\omega| = n$. Then,*

(i) *when $\alpha(n) = 3$, $n \equiv 4 \pmod 8$ and $l(\omega) = 2$, X_ω is cobordant to a manifold which immerses in $\mathbf{R}^{2n-6}$ and embeds in $\mathbf{R}^{2n-5}$;*

(ii) *when $\alpha(n) = 4$, $n \equiv 2 \pmod 4$ and $l(\omega) = 2$, X_ω is cobordant to a manifold which immerses in $\mathbf{R}^{2n-6}$ and embeds in $\mathbf{R}^{2n-5}$;*

(iii) *when $n \neq 7$ is odd, $\alpha(n) = 3$ and $l(\omega) = 1$, X_ω is cobordant to a manifold which immerses in $\mathbf{R}^{2n-4}$ and embeds in $\mathbf{R}^{2n-3}$.*

PROOF. (i) When $\omega \in P_5$ or $\alpha(\omega) \geq 6$, X_ω itself immerses in $\mathbf{R}^{2n-6}$ and embeds in $\mathbf{R}^{2n-5}$ from Lemma 3.1 and (3.2). So we assume that $\omega \in P_6 - P_5$ and $\alpha(\omega) \leq 5$. Since $l(\omega) = 2$, $\omega = (a_i, a_j)$ for odd numbers $a_i \neq 1$ and $a_j \neq 1$. By (3.3), $\alpha(\omega) \geq 3 + 2 = 5$. Thus, we can assume that $\alpha(a_i) = 2$ and $\alpha(a_j) = 3$. Then, by the above remark, $W^\omega = W^{a_i}W^{a_j}$ is cobordant to an orientable manifold, and hence, W^ω is cobordant to X_ω plus a finite union of $X_{\omega'}$ with $\omega' \in P_6 - P_5$ and $l(\omega') \geq 3$. In other words, X_ω is cobordant to W^ω plus a finite union of $X_{\omega'}$ as above. By Theorem 2.3(ii), (iii) and Corollary 2.6(i), W^ω immerses in $\mathbf{R}^{2n-6}$ and embeds in $\mathbf{R}^{2n-5}$, and, by Lemmas 3.1 and 3.4(i), each $X_{\omega'}$ as above immerses in $\mathbf{R}^{2n-6}$ and embeds in $\mathbf{R}^{2n-5}$. Therefore, X_ω is cobordant to a manifold which immerses in $\mathbf{R}^{2n-6}$ and embeds in $\mathbf{R}^{2n-5}$ as required.

(ii) We remark that $\omega \notin P_5$. When $\alpha(\omega) \geq 6$, we have the required result by Lemma 3.1. So we assume that $\alpha(\omega) \leq 5$. Since $l(\omega) = 2$, $\omega = (a_i, a_j)$ for odd numbers $a_i \neq 1$ and $a_j \neq 1$. By (3.3), $\alpha(\omega) \geq 4 + 1 = 5$, and we can assume that $\alpha(a_i) = 2$ and $\alpha(a_j) = 3$. Then, we have the required result by doing just the same way as in (i).

(iii) In this case, $\omega = (n)$ since $l(\omega) = 1$. Then, similarly as in (i), W^n is cobordant to an orientable manifold, and hence, W^n is cobordant to $X_{(n)}$ plus a finite union of $X_{\omega'}$ with $\omega' \in P_6 - P_5$ and $l(\omega') \geq 2$. In other words, $X_{(n)}$ is cobordant to W^n plus a finite union of $X_{\omega'}$ as above. Then, W^n immerses in $\mathbf{R}^{2n-4}$ and embeds in $\mathbf{R}^{2n-3}$ by Theorem 2.3(iii), and each $X_{\omega'}$ as above immerses in $\mathbf{R}^{2n-4}$ and embeds in $\mathbf{R}^{2n-3}$ by Lemmas 3.1 and 3.4(ii). Hence, we have the required result. □

Now, the proof of the possibilities of immersions and embeddings stated in Theorem B is given as follows: Recall that any orientable manifold M^n

is cobordant to a finite union of X_ω with $\omega \in P_6$ and $|\omega| = n$. If n satisfies (i), the condition $n \equiv 2 \pmod 8$ implies $\omega \notin P_5$. Since n is even, each $\omega \in P_6 - P_5$ has at least two odd numbers, so $l(\omega) \geq 2$. Thus, by Lemmas 3.1 and 3.4(i), each manifold X_ω as above immerses in $\mathbf{R}^{2n-4}$ and embeds in $\mathbf{R}^{2n-3}$, and so we have that any orientable manifold M^n is cobordant to a manifold which immerses in $\mathbf{R}^{2n-4}$ and embeds in $\mathbf{R}^{2n-3}$ as required. Similarly, when n satisfies (ii) (resp. (iii); resp. (iv)), we have the required result, by Lemmas 3.1, 3.4(i) and 3.6(i) (resp. Lemmas 3.1, 3.4(i) and 3.6(ii); resp. Lemmas 3.1, 3.4(ii) and 3.6(iii)).

4. Best possibilities

In this section, we prove Theorem A(2) and the rest of Theorem B. Let $\bar{w}_i(M^n)$ for $i \geq 0$ be the dual Stiefel-Whitney classes of a manifold M^n. That is, they satisfy $(\sum_{i\geq 0} \bar{w}_i(M^n)) \times (\sum_{i\geq 0} w_i(M^n)) = 1$. Then, we have the following:

LEMMA 4.1. *Let L^n and $L_i^{n_i}$ be manifolds with $L^n = \prod_{i=1}^k L_i^{n_i}$. If each $L_i^{n_i}$ satisfies the following* (i) *and* (ii) *for some $\sigma_i < n_i$, then $\bar{w}_j(L^n) = 0$ for any $j > n - \sigma$ and $\bar{w}_\sigma \bar{w}_{n-\sigma}(L^n) \neq 0$, where $\sigma = \sum_{i=1}^k \sigma_i$:*

(i) *$\bar{w}_j(L_i^{n_i}) = 0$ for any $j > n_i - \sigma_i$;*

(ii) *$\bar{w}_{\sigma_i} \bar{w}_{n_i-\sigma_i}(L_i^{n_i}) \neq 0$.*

PROOF. Remark that the total dual Stiefel-Whitney classes satisfy $\bar{w}(L^n) = \prod_{i=1}^k \bar{w}(L_i^{n_i})$. Hence, by the hypothesis (i), we have $\bar{w}_j(L^n) = 0$ for any $j > n - \sigma$, and thus $\bar{w}_{n-\sigma}(L^n) = \prod_{i=1}^k \bar{w}_{n_i-\sigma_i}(L_i^{n_i})$. Also, $\bar{w}_\sigma(L^n)$ equals $\prod_{i=1}^k \bar{w}_{\sigma_i}(L_i^{n_i})$ up to terms which consist of $\bar{w}_j(L_i^{n_i})$ for some i and $j > \sigma_i$. Hence, $\bar{w}_\sigma \bar{w}_{n-\sigma}(L^n) = \prod_{i=1}^k \bar{w}_{\sigma_i} \bar{w}_{n_i-\sigma_i}(L_i^{n_i}) \neq 0$. □

Since the manifold L^n in the lemma satisfies $\bar{w}_\sigma \bar{w}_{n-\sigma}(L^n) \neq 0$, any manifold M^n cobordant to L^n satisfies $\bar{w}_\sigma \bar{w}_{n-\sigma}(M^n) \neq 0$. In particular, $\bar{w}_{n-\sigma}(M^n) \neq 0$. As a necessary condition for M^n to immerse in $\mathbf{R}^{2n-\sigma-1}$ or embed in $\mathbf{R}^{2n-\sigma}$ is that $\bar{w}_j(M^n) = 0$ for $j \geq n - \sigma$, we have the following:

COROLLARY 4.2. *Let L^n and σ be those given in* Lemma 4.1. *Then, any manifold cobordant to L^n does not immerse in $\mathbf{R}^{2n-\sigma-1}$ and does not embed in $\mathbf{R}^{2n-\sigma}$.*

LEMMA 4.3. (i) *Let $n = 2r$, where $r \geq 2$ and r is a power of* 2. *Then, $\bar{w}_j(\mathbf{C}P^r) = 0$ for any $j > n - 2$, and $\bar{w}_2 \bar{w}_{n-2}(\mathbf{C}P^r) \neq 0$.*

(ii) *Let $n = 2t + s - 1$, where $t \geq s \geq 2$ and t, s are both power of* 2. *Then, $\bar{w}_j(P(s-1, t)) = 0$ for any $j > n - s$, and $\bar{w}_s \bar{w}_{n-s}(P(s-1, t)) \neq 0$.*

PROOF. (i) Let $d \in H^2(\mathbf{C}P^r) \cong \mathbf{Z}_2$ be the generator. Then, it satisfies $d^{r+1} = 0$, and the total Stiefel-Whitney class of $\mathbf{C}P^r$ is given by $w(\mathbf{C}P^r) = (1+d)^{r+1}$. Then, $\bar{w}(\mathbf{C}P^r) = (1+d)^{-r-1}$, and thus $\bar{w}(\mathbf{C}P^r) = (1+d)^{r-1}$, since r is a power of 2 and $d^l = 0$ $(l > r)$. Hence, $\bar{w}_j(\mathbf{C}P^r) = 0$ for any $j > 2r - 2 = n - 2$, and $\bar{w}_{n-2}(\mathbf{C}P^r) = d^{r-1}$. Also, since $\bar{w}_2(\mathbf{C}P^r) = (r-1)d = d$, $\bar{w}_2\bar{w}_{n-2}(\mathbf{C}P^r) = d^r \neq 0$ as required.

(ii) By (3.5), $w(P(s-1, t)) = (1+c)^{s-1}(1+c+d)^{t+1}$ where $c \in H^1(P(s-1, t))$ and $d \in H^2(P(s-1, t))$, and they satisfy $c^s = 0$ and $d^{t+1} = 0$. Then, $\bar{w}(P(s-1, t)) = (1+c)^{-s+1}(1+c+d)^{-t-1} = (1+c)(1+c+d)^{t-1}$, because t and s are both powers of 2, $c^l = 0$ $(l \geq s)$ and $d^l = 0$ $(l > t)$. Hence, $\bar{w}_j(P(s-1, t)) = 0$ for any $j > 2t - 1 = n - s$, and $\bar{w}_{n-s}(P(s-1, t)) = cd^{t-1}$. Let $\{x, y, z\} = (x+y+z)!/(x!y!z!)$. Then, $\bar{w}_s(P(s-1, t)) = \{t-s, s-2, 1\}c^{s-2}d$ up to other terms consisting of c^l for some $l > s-2$ or d^l for some $l > 1$. Hence, $\bar{w}_s\bar{w}_{n-s}(P(s-1, t)) = \{t-s, s-2, 1\}c^{s-1}d^t = c^{s-1}d^t \neq 0$. □

Now, we can complete the proofs of Theorem A and Theorem B. By Corollary 4.2, the assertion of Theorem A(2) (resp. Theorem B) will be established if we find an orientable manifold L^n satisfying the conditions of Lemma 4.1 for $\sigma = \alpha(n) + \min\{\alpha(n), \nu(n)\}$ (resp. $\sigma = \alpha(n) + \nu(n) + 1$). In the below, we assume that each r or r_i is a power of 2. In each case, the following L^n satisfies the required conditions:

For Theorem A(2):

(i) Let $n = 2r_1 + \cdots + 2r_k$ where $2r_1 > \cdots > 2r_k \geq 2^k$. We put $L^n = \prod_{i=1}^k \mathbf{C}P^{r_i}$, then L^n satisfies the required conditions for $\sigma = 2k = 2\alpha(n)$ by Lemma 4.3;

(ii) Let $n = 2r_1 + 2r_2 + 2$ where $r_1 > r_2 \geq 2$. We put $L^n = P(1, r_1) \times P(1, r_2)$, then $\sigma = 2 + 2 = 4$;

(iii) Let $n = 2r + 1$ where $r \geq 2$. We put $L^n = P(1, r)$, then $\sigma = 2$.

For Theorem B:

(i) Let $n = 4r + 2$ where $r \geq 2$. We put $L^n = P(1, r) \times P(1, r)$, then $\sigma = 2 + 2 = 4$;

(ii) Let $n = 2r_1 + 2r_2 + 4$ where $r_1 > r_2 \geq 4$. We put $L^n = P(1, r_1) \times P(3, r_2)$, then $\sigma = 2 + 4 = 6$;

(iii) Let $n = 2r_1 + 2r_2 + 2r_3 + 2$ where $r_1 > r_2 > r_3 \geq 2$. We put $L^n = \mathbf{C}P^{r_1} \times P(1, r_2) \times P(1, r_3)$, then $\sigma = 2 + 2 + 2 = 6$;

(iv) If $n = 2r_1 + 2r_2 + 1$ where $r_1 > r_2 \geq 2$, then we put $L^n = \mathbf{C}P^{r_1} \times P(1, r_2)$, and we have $\sigma = 2 + 2 = 4$. If $n = 2r + 3$ where $r \geq 4$, then we put $L^n = P(3, r)$, and we have $\sigma = 4$.

REMARK. When $n = 6$ or 7, then there is no partition $\omega \in P_6$ such that $n = |\omega|$. It means that any closed orientable manifold of dimension 6 or 7 is cobordant to 0 $(\in \mathcal{N}_*)$. Hence, we omit the case $n = 6$ and 7 from Theorem B.

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