HIROSHIMA MATH. J.
27 (1996), 1–46

Robustness and constructions of some balanced block designs

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(Received July 18, 1995)

ABSTRACT. A statistical property of block designs with their construction problems is discussed in the light of practical analysis. Firstly, we discuss the robustness of some balanced block designs against the unavailability of some observations in terms of efficiency of the residual design. The block designs covered here are variance-balanced designs and augmented balanced incomplete block designs. The investigation shows that variance-balanced designs are fairly robust against the unavailability of any two observations. The bounds on the efficiency of the residual designs of a variance-balanced design are given. The robustness of augmented balanced incomplete block designs against the unavailability of any two blocks is also investigated. Secondly, some block designs with missing observations are characterized as partially efficiency-balanced designs which provide a simple statistical analysis for the residual designs. Some constructions of equireplicate, proper partially efficiency-balanced designs are also given. Thirdly, as a by-product, some partially balanced incomplete block designs are provided.

0. Introduction

The basic principles of experimental designs as we know them today were formulated by R. A. Fisher in his famous book "Statistical Methods for Research Workers" (1925) and in his paper "The arrangement of field experiments" (1926). The design of such statistical experiments often used combinatorial structures that yielded a simple calculation of estimates and/or a symmetric structure of their variance and covariance. Typical examples are block designs with some balancing. However, when some observations are missing or become unavailable in a designed experiment for some reason, the combinatorial structures of the block designs have been also destroyed. This causes the following two interesting problems in two different directions. One is to see an insensitive or robust property against the unavailability of observations. The robustness of block designs against the unavailability of data has been investigated in abundance, for example, see Hedayat and John (1974), Ghosh (1979, 1982a, b, c), Dey and Dhall (1988), Srivastava, Gupta

1991 *Mathematics Subject Classitication.* 62K05, 62K10; 05B05.
Key words and phrases. block design, VB, PEB, BIB, PBIB, GD, efficiency, M-matrix, C-matrix, connected, binary.

and Dey (1990), Kageyama (1990), Mukerjee and Kageyama (1990), Bhaumik and Whittinghill (1991), Ghosh, Kageyama and Mukerjee (1992), Das and Kageyama (1992), Dey (1993), and Duan and Kageyama (1995b, 1996). For an excellent review of the subject refer to Kageyama (1993). Most of the robustness criteria of block designs against the unavailability of data are (1) to get the connectedness of the residual design obtained after the unavailability of data (in which all elementary treatment contrasts can be estimated under the usual linear model); (2) to have the variance-balance (VB) of the residual design (in which every normalized treatment contrast can be estimated with the same variance); (3) to consider the efficiency of the residual design. Criterion 1 of the robustness was introduced by Ghosh (1982a) who has shown that balanced incomplete block (BIB) designs are robust according to Criterion 1 against the loss of any $r-1$ observations or blocks in BIB designs with the replication number r of each treatment. Similar results on certain partially balanced incomplete block (PBIB) designs were obtained by Ghosh et al. (1983) and Kageyama (1986). See also Baksalary and Tabis (1987) who presented some sufficient conditions for arbitrary blocks to be robust under Criterion 1. Criterion 2 of the robustness was introduced and studied by Hedayat and John (1974). It is well known that all BIB designs are VB. Unfortunately, this desirable feature of BIB designs can easily be lost if, due to some unforeseen circumstances, some or all of the data related to experimental units assigned to one or more treatments are lost in actual experimentation. In some cases nothing can be done to prevent such an undesirable outcome, but fortunately in many cases there are ways which we can apply to preserve the variance-balance of the remaining design if we are careful in our selection of the design to begin with. The papers by Hedayat and John (1974), Most (1975), Shah and Gujarathi (1977, 1983), Chandak (1980) and Kageyama (1987) are in this category. Criterion 3 that has received attention by various authors is in terms of efficiency of the residual design. This criterion will be adopted in the present paper. The papers by Kageyama (1980), Dey and Dhall (1988), Mukerjee and Kageyama (1990), Ghosh et al. (1992), Dey (1993), and Duan and Kageyama (1995b, 1996) are in this category. Another problem is that when some observations are missing or become unavailable in a designed experiment, it is interesting to see what kind of residual designs appear, and how to take statistical analysis for the designs under the usual linear homoscedastic additive model. It is known that block designs can be also regarded as partially efficiency-balanced (PEB) designs with some number of efficiency classes (Puri and Nigam (1977)). For a PEB design, we can easily obtain the pseudo variance-covariance matrix Ω. Once Ω is known, the estimate of the vector of treatment effects is given by ΩQ and the adjusted sum of squares attributed to treatments is $Q'\Omega Q$ under the usual linear homoscedastic additive model, where

Q is the column vector of adjusted treatment totals (cf. Puri and Nigam (1983), Puri (1984), Puri and Kageyama (1985)).

The present paper concerns the two problems mentioned above and consists of two parts. Part I is devoted to the investigation of the robustness of some balanced block designs against the unavailability of data in the designs in terms of efficiency of the residual designs. The block designs covered here are VB designs and augmented BIB designs. In Part II, some block designs with missing observations are characterized as PEB designs which provide a simple statistical analysis for the residual designs. Furthermore, some constructions of equireplicate, proper PEB designs are also given. As a by-product, some PBIB designs are provided, which cover some results of Duan and Kageyama (1993, 1995a) as special cases.

1. Preliminaries

Consider a block design $d(v, b, \mathbf{r}, \mathbf{k})$ with v treatments arranged in b blocks of sizes $k_1, \dots, k_b$ such that the i-th treatment is replicated r_i times and occurs n_{ij} times in the j-th block $(i = 1, \dots, v;\ j = 1, \dots, b)$. Here $\mathbf{r} = (r_1, \dots, r_v)'$, $\mathbf{k} = (k_1, \dots, k_b)'$ and n_{ij} can take non-negative integers. Let $\mathbf{r} = (r_1, \dots, r_v)'$ and $\mathbf{k} = (k_1, \dots, k_b)'$, and R and K stand for diagonal matrices with diagonal elements $r_1, \dots, r_v$ and $k_1, \dots, k_b$, respectively, and $n = \sum_{i=1}^{v} r_i = \sum_{j=1}^{b} k_j$. Further let $N = (n_{ij}), i = 1, \dots, v;\ j = 1, \dots, b$, be the $v \times b$ incidence matrix of the design. If $n_{ij} = 1$ or 0 for all i and j, the design is called a binary design. When all r_i $(i = 1, \dots, v)$ are equal, the design is said to be equireplicate. If all k_j $(j = 1, \dots, b)$ are equal, the design is said to be proper.

The intra-block linear model of n observations obtained through the design $d(v, b, \mathbf{r}, \mathbf{k})$ can be written as

$$y_{ijl} = \mu + \tau_i + \beta_j + e_{ijl}, \tag{1.1}$$

where y_{ijl}, $i = 1, \dots, v; j = 1, \dots, b;\ l = 1, \dots, n_{ij}$, is the l-th observation from the i-th treatment in the j-th block, μ is the general mean, τ_i is the i-th treatment effect, β_j is the j-th block effect and e_{ijl} are residuals distributed identically and independently such that $E(e_{ijl}) = 0, V(e_{ijl}) = \sigma^2$ and $\mathrm{Cov}(e_{ijl}, e_{i'j'l'}) = 0$ for all $(i, j, l) \neq (i', j', l')$. We refer to (1.1) as a fixed-effects linear model. It can be shown that the reduced normal equations for estimating the treatment effects are $C\tau = Q$, where $C = R - NK^{-1}N'$ is the information matrix which plays an important role in statistical analysis, $\tau = (\tau_1, \dots, \tau_v)'$ and $Q = T - NK^{-1}B'$ is the vector of adjusted treatment totals. Here, T and B are the vectors of treatment totals and block totals, respectively. Note that the rank of C is at most $v - 1$. A block design is said to be connected if all elementary contrasts are estimable. It is well known that a block design is connected if and only if the rank of its C-matrix is exactly $v-1$. In this paper only connected binary block designs are considered.

A $\mathrm{BIB}(v,b,r,k,\lambda)$ design is an arrangement of v treatments in b blocks of size k such that (i) every treatment occurs in exactly r blocks and (ii) every pair of distinct treatments occurs together in λ blocks. An augmented BIB design d is obtained by augmenting each block of a $\mathrm{BIB}(v,b,r,k,\lambda)$ design, having the usual $v \times b$ incidence matrix N for v test treatments, with a new treatment (control treatment). This design d has incidence matrix $N_d = [N' : \mathbf{1}_b]'$ with parameters $v_0 = v+1$, $b_0 = b$, $\mathbf{k}_0 = (k+1)\mathbf{1}_b$ and $\mathbf{r}_0 = (r\mathbf{1}'_v, b)'$. Here $\mathbf{1}_b$ denotes a b-dimensional column vector with all components being unity. A block design with parameters $v, b, \mathbf{r} = (r_1, \dots, r_v)'$ and $\mathbf{k} = (k_1, \dots, k_b)'$ is said to be VB if and only if all the non-zero eigenvalues of C are equal, which means that all elementary contrasts are estimated with the same precision. In this case we have $C = \theta(I_v - v^{-1}J_v)$ with $\theta = (n-b)/(v-1)$, where I_v is the identity matrix of order v and $J_v = \mathbf{1}_v\mathbf{1}_v'$. It is known that in the class of all connected binary incomplete block designs the balanced design is the most efficient design. Among equiblock-sized and equireplicate designs the BIB design is the only balanced, and hence the most efficient, design. Unfortunately, however, BIB designs exist in a limited number of cases, and hence it is necessary to introduce new designs. Bose and Nair (1939) defined PBIB designs in which all elementary contracts were not estimated with the same variance. Before defining a PBIB design, we need the concept of an association scheme of v treatments as given below.

DEFINITION 1.1. *Given v treatments $1, 2, \dots, v$, a relation satisfying the following conditions is said to be an association scheme with m classes:*

1. *Any two treatments are either 1st, 2nd, …, or m-th associates, the relation of association being symmetrical; that is, if the treatment α is the i-th associate of the treatment β then β is the i-th associate of α.*
2. *Each treatment α_i has n_i i-th associates, the number n_i being independent of α_i.*
3. *If any two treatments α and β are i-th associates, then the number of treatments that are j-th associates of α, and k-th associates of β, is p^i_{jk} and is independent of the pair of i-th associates α and β.*

Given an association scheme for a set of the v treatments, we define a PBIB design as follows:

DEFINITION 1.2. *If we have an association scheme with m classes and given parameters, we get a* PBIB *design with m associate classes if the v treatments are arranged into b blocks of size k $(<v)$ such that*

1. *Every treatment occurs in exactly r blocks;*

2. *If two treatments α and β are i-th associates, then they occur together in λ_i blocks, the number λ_i being independent of the particular pair of i-th associates α and β.*

The known two-associate PBIB designs were classified by Bose and Shimamoto (1952) into the following five types depending on the association scheme: Group divisible (GD), Simple, Triangular, Latin-square type (L_i) and Cyclic. A PBIB design with two associate classes is said to be group divisible if there are $v = mn$ treatments and the treatments can be divided into m groups of n treatments each, such that any two treatments of the same group are first associates and two treatments from different groups are second associates. A PBIB design with two associate classes is called a Latin-square type design if there are $v = s^2$ treatments that are arranged in an $s \times s$ square array, such that two treatments are first associates if and only if they occur in the same row or column of the array, otherwise second associates. A PBIB design with three associate classes is said to be rectangular if there are $v = mn$ treatments arranged in a rectangle of m rows and n columns, and with respect to each treatment, the first associates are the other $n - 1$ treatments of the same row, the second associates are the other $m - 1$ treatments of the same column, and the remaining $(m - 1)(n - 1)$ treatments are the third associates. These PBIB designs will be discussed in Section 5.

When some observations become unavailable in a designed experiment for some reason, it is of interest to examine the loss of information, defined suitably, that is incurred due to missing data. Designs for which this loss is "small" may be termed *robust.* Let d^* be the design obtained by removing some observations in the original design d. Let C and C^* be the usual C-matrices of d and d^*, respectively. In this case, a criterion of the robustness against the unavailability of such observations is the efficiency of the residual design d^*, given by

$$e(*) = \frac{\text{sum of reciprocals of non-zero eigenvalues of } C}{\text{sum of reciprocals of non-zero eigenvalues of } C^*} \left(= \frac{\phi_2}{\phi_1(*)}, \text{ say} \right) \tag{1.2}$$

(see Das and Kageyama (1992)), which is equivalent to the ratio of the average variances of all elementary treatment contrasts in the original and the residual design. Assume d^* to be connected [this assumption is made only for the convenience of general presentation of the eigenvalues of C^* in a closed form, because the calculation of $\phi_1(*)$ in (1.2) can be done also for a disconnected design d^*].

For the evaluation of eigenvalues, the following lemma (Mukerjee and Kageyama (1990)) is useful.

LEMMA 1.1. *Let $u, s_1, \dots, s_u$ be positive integers, and consider the $s \times s$ matrix*

$$A = \begin{bmatrix} a_1 I_{s_1} + b_{11} J_{s_1 s_1} & b_{12} J_{s_1 s_2} & \cdots & b_{1u} J_{s_1 s_u} \\ b_{21} J_{s_2 s_1} & a_2 I_{s_2} + b_{22} J_{s_2 s_2} & \cdots & b_{2u} J_{s_2 s_u} \\ \vdots & \vdots & \ddots & \vdots \\ b_{u1} J_{s_u s_1} & b_{u2} J_{s_u s_2} & \cdots & a_u I_{s_u} + b_{uu} J_{s_u s_u} \end{bmatrix},$$

where $s = s_1 + s_2 + \cdots + s_u$ and the $u \times u$ matrix $B = (b_{ij})$ is symmetric. Then the eigenvalues of A are a_i with multiplicity $s_i - 1$ $(1 \le i \le u)$ and $\mu_1^*, \dots, \mu_u^*$, where $\mu_1^*, \dots, \mu_u^*$ are the eigenvalues of $\Delta = D_a + D_s^{1/2} B D_s^{1/2}$ with $D_a =$ diag $\{a_1, \dots, a_u\}$, $D_s = \mathrm{diag}\{s_1, \dots, s_u\}$ and $D_s^{1/2} = \mathrm{diag}\{s_1^{1/2}, \dots, s_u^{1/2}\}$.

The following notations are used throughout the paper: I_s is the identity matrix of order s, $J_{s\times t} = \mathbf{1}_s \mathbf{1}_t'$ is an $s \times t$ matrix with all elements unity, $O_{s\times t}$ denotes an $s \times t$ matrix with all elements zero, $J_{s\times s}$ $(O_{s\times s})$ is specially denoted by $J_s(O_s)$ (hereafter, J denotes such matrix of appropriate size), and $A \otimes B$ denotes the Kronecker product of two matrices A and B.

Part I. Robustness of some block designs

The robustness of VB and augmented BIB designs against the unavailability of some observations has been investigated in terms of efficiency of the residual design by Gupta and Srivastava (1992), Duan and Kageyama (1995b, 1996). For a VB design, Gupta and Srivastava (1992) investigated the robustness of the design against the unavailability of some disjoint blocks. Duan and Kageyama (1996) investigated the robustness of VB designs against the unavailability of any number of observations in a block or any two blocks which are not necessarily disjoint. For an augmented BIB design, Gupta and Srivastava (1992) investigated the robustness of the design against the unavailability of all observations in a block. Duan and Kageyama (1995b) investigated the robustness of the design against the unavailability of any two observations.

In Part I, we pay our attention to the cases that any two observations in the VB design or any two blocks in an augmented BIB design are missing. These are feasible situations in practice.

2. Robustness of variance balanced block designs

A binary connected VB design with parameters $v, b, \mathbf{r} = (r_1, \dots, r_v)'$, $\mathbf{k} = (k_1, \dots, k_b)'$, $n = \sum_{i=1}^{v} r_i = \sum_{j=1}^{b} k_j$ and $\theta = (n - b)/(v - 1)$ in which $C =$

$\theta(I_v - v^{-1}J_v)$, is considered here to show the robustness against the unavailability of data. Further assume that the present VB design does not have the incidence matrix of type $J_{v\times b}$ and also satisfies $k_j \geq 2$ for all $j = 1, \dots, b$. This is the usual assumption in the area of design of experiments to avoid trivial designs. On account of Corollary 6 of Kageyama and Tsuji (1980) and Proposition 3.1 of Kageyama (1984), we can show that $\theta = (n-b)/(v-1) > 1$. This property will be used later.

2.1. Unavailability of any two observations in a design

Suppose that two observations in any one block or two different blocks of a VB design d with incidence matrix $N_d = (n_{ij})$, $i = 1, \dots, v; j = 1, \dots, b$, are lost. Without loss of generality, the situation can be treated by separating it into the following five patterns.

Case 1: Two observations corresponding to positions n_{11} and n_{21} in a block are lost.

Note that Duan and Kageyama (1996) investigated the robustness of the design against the unavailability of any number s $(1 \leq s \leq k_j - 1)$ of observations in a block by showing an expression of the efficiency of the residual design as

$$e(s) = \frac{(v-1)(\theta-1)}{(v-1)(\theta-1)+s}. \tag{2.1}$$

Hence Case 1 can be treated as a special case of (2.1) when $s = 2$, which is also rewritten as

$$e_1(x) = \frac{(v-1)\{(\theta-1)^2 - x^2\}}{(v-3)\{(\theta-1)^2 - x^2\} + 2\theta(\theta-1)} \tag{2.2}$$

with $x = 0$. The formal expression (2.2) will be utilized in Section 2.2.

Case 2: Two observations n_{11} and n_{12} in different blocks but for the same treatment are lost.

Let w be the number of treatments common to two such blocks. Then $1 \leq w \leq \min\{k_1, k_2\}$. In this case, the C-matrix of the residual design d^* can be given by

$$C^* = \begin{bmatrix} c_{11} & c_{12}J & c_{13}J & c_{14}J & c_{15}J \\ & C_{22} & c_{23}J & c_{24}J & c_{25}J \\ & & C_{33} & c_{34}J & c_{35}J \\ & \text{Sym} & & C_{44} & c_{45}J \\ & & & & C_{55} \end{bmatrix}$$

with

$$\begin{aligned}
&c_{11} = \theta - 2 - v^{-1}\theta + k_1^{-1} + k_2^{-1},\\
&C_{22} = \theta I_{w-1} - \{v^{-1}\theta + k_1^{-1}(k_1-1)^{-1} + k_2^{-1}(k_2-1)^{-1}\}J_{w-1},\\
&C_{33} = \theta I_{k_1-w} - \{v^{-1}\theta + k_1^{-1}(k_1-1)^{-1}\}J_{k_1-w},\\
&C_{44} = \theta I_{k_2-w} - \{v^{-1}\theta + k_2^{-1}(k_2-1)^{-1}\}J_{k_2-w},\\
&C_{55} = \theta I_{v-k_1-k_2+w} - v^{-1}\theta J_{v-k_1-k_2+w},\\
&c_{12} = -v^{-1}\theta + k_1^{-1} + k_2^{-1}, \qquad c_{13} = -v^{-1}\theta + k_1^{-1},\\
&c_{14} = -v^{-1}\theta + k_2^{-1}, \qquad c_{23} = -v^{-1}\theta - k_1^{-1}(k_1-1)^{-1},\\
&c_{24} = -v^{-1}\theta - k_2^{-1}(k_2-1)^{-1}, \qquad c_{15} = c_{25} = c_{34} = c_{35} = c_{45} = -v^{-1}\theta.
\end{aligned}$$

Hence we can obtain the following after some calculation through Lemma 1.1.

LEMMA 2.1. *The $v-1$ non-zero eigenvalues of C^*, $1 \le w \le \min\{k_1, k_2\}$, are given by*

$$\begin{array}{ll}
\theta & \textit{with multiplicity (w.m.) } v-3,\\
\theta - 1 - x & \textit{w.m. } 1,\\
\theta - 1 + x & \textit{w.m. } 1,
\end{array}$$

where $x = (k_1k_2 - k_1 - k_2 + w)\{k_1k_2(k_1-1)(k_2-1)\}^{-1/2}$.

REMARK 2.1. When $w = 1$, the C-matrix of the residual design d^* can be given by removing the second partitioned row (submatrix) and column (submatrix) in the original C^* of the residual design. Then by Lemma 1.1, the same result as Lemma 2.1 with $w = 1$ can be given. Similarly we can get Lemma 2.1 also for each case of $w = k_1$ or $w = k_2$, or $v - k_1 - k_2 + w = 0$. Later such special cases will be discussed similarly.

Recall that $\phi_1(k_1, k_2, w)$ and ϕ_2 are the sum of reciprocals of non-zero eigenvalues of C^* and C, respectively. Hence, in (1.2),

$$\phi_1(k_1, k_2, w) = \frac{v-3}{\theta} + \frac{2(\theta-1)}{(\theta-1)^2 - x^2}, \qquad \phi_2 = \frac{v-1}{\theta},$$

which yield the efficiency of the residual design d^* as

$$e(k_1, k_2; w) = \frac{\phi_2}{\phi_1(k_1, k_2, w)} = \frac{(v-1)\{(\theta-1)^2 - x^2\}}{(v-3)\{(\theta-1)^2 - x^2\} + 2\theta(\theta-1)} \tag{2.3}$$

for $1 \le w \le \min\{k_1, k_2\}$ with $x = (k_1k_2 - k_1 - k_2 + w)\{k_1k_2(k_1-1)(k_2-1)\}^{-1/2}$.

For a given VB design, we here consider how the efficiency of the residual designs changes in terms of w like (2.3). As for behaviour of the values of efficiencies in the case above, we have the following.

THEOREM 2.1. *In a* VB *design with parameters* $v, b, \mathbf{r} = (r_1, \ldots, r_v)'$, $\mathbf{k} =$

$(k_1,\dots,k_b)'$, $n=\sum_{i=1}^{v} r_i=\sum_{j=1}^{b} k_j$ *and* $\theta=(n-b)/(v-1)$, *for Case* 2

$$e(k_1,k_2;1)>e(k_1,k_2;2)>\cdots>e(k_1,k_2;w_{\max}),$$

where $w_{\max}\le\min\{k_1,k_2\}$.

Proof. Note that x in (2.3) is an increasing function of w. This implies that $\phi_1(k_1,k_2,w)$ is also an increasing function of w, since $\phi_1(k_1,k_2,w)$ is an increasing function of x. For any integers w', w such that $1\le w'<w\le\min\{k_1,k_2\}$, it follows that $\phi_1(k_1,k_2,w')<\phi_1(k_1,k_2,w)$, which through (2.3) implies that $e(k_1,k_2;w')>e(k_1,k_2;w)$ for $1\le w'<w\le\min\{k_1,k_2\}$. This completes the proof. □

Theorem 2.1 implies that the behaviour of $e(k_j,k_{j'};w_{\max})$, $j\ne j', j,j'=1,\dots,b$, is important to judge whether the design is robust or not for Case 2. All VB designs listed in Kageyama (1976), Gupta and Jones (1983), Jones, Sinha and Kageyama (1987), and Gupta and Kageyama (1992) have been worked out. The evaluation reveals that except for a few, all the VB designs have high values of $e(k_j,k_{j'};w_{\max})$. In fact, 318 designs satisfy $e\ge 0.90$, 5 designs of series numbers 2, 7 in Kageyama (1976), 6, 7 in Gupta and Jones (1983) and 2 in Gupta and Kageyama (1992) get $0.90>e\ge 0.80$, and only one design of series number 1 in Kageyama (1976) has smaller values of efficiency as $e(2,2;2)=e(4,4;4)=0.50$ and $e(2,4;2)=0.59$ show. It appears that all VB designs are fairly robust against the unavailability of any two observations pertaining to the same treatment in any two different blocks.

Case 3: Two observations n_{11} and n_{22} in different blocks and corresponding to different treatments are lost. This case can also be treated by separating it into following three types:

Case 3.1: $n_{21}=0$ and $n_{12}=0$ with $0\le w\le\min\{k_1,k_2\}-1$.
In this case, the C-matrix of the residual design d^* can be given by

$$C^*=\begin{bmatrix} c_{11} & c_{12}J & c_{13}J & c_{14}J & c_{15}J & c_{16}J \\ & c_{22} & c_{23}J & c_{24}J & c_{25}J & c_{26}J \\ & & C_{33} & c_{34}J & c_{35}J & c_{36}J \\ & & & C_{44} & c_{45} & c_{46}J \\ & \text{Sym} & & & C_{55} & c_{56}J \\ & & & & & C_{66} \end{bmatrix} \tag{2.4}$$

with

$$c_{11}=\theta-1-v^{-1}\theta+k_1^{-1},\qquad c_{22}=\theta^{-1}-1-v^{-1}\theta+k_2-1,$$
$$C_{33}=\theta I_w-\{v^{-1}\theta+k_1^{-1}(k_1-1)^{-1}+k_2^{-1}(k_2-1)^{-1}\}J_w,$$
$$C_{44}=\theta I_{k_1-w-1}-\{v^{-1}\theta+k_1^{-1}(k_1-1)^{-1}\}J_{k_1-w-1},$$

$$\begin{aligned}
&C_{55} = \theta I_{k_2-w-1} - \{v^{-1}\theta + k_2^{-1}(k_2-1)^{-1}\}J_{k_2-w-1},\\
&C_{66} = \theta I_{v-k_1-k_2+w} - v^{-1}\theta J_{v-k_1-k_2+w},\\
&c_{12} = c_{15} = c_{16} = c_{24} = c_{26} = c_{36} = c_{45} = c_{46} = c_{56} = -v^{-1}\theta,\\
&c_{13} = c_{14} = -v^{-1}\theta + k_1^{-1}, \qquad c_{23} = c_{25} = -v^{-1}\theta + k_2^{-1},\\
&c_{34} = -v^{-1}\theta - k_1^{-1}(k_1-1)^{-1}, \qquad c_{35} = -v^{-1}\theta - k_2^{-1}(k_2-1)^{-1}.
\end{aligned}$$

Hence we can obtain the following after some calculation through Lemma 1.1.

LEMMA 2.2. *The $v-1$ non-zero eigenvalues of C^*, $0 \le w \le \min\{k_1, k_2\} - 1$, are given by*

$$\begin{array}{ll}
\theta & w.m.\ v-3,\\
\theta - 1 - x & w.m.\ 1,\\
\theta - 1 + x & w.m.\ 1,
\end{array}$$

where $x = w\{k_1k_2(k_1-1)(k_2-1)\}^{-1/2}$.

By Lemma 2.2, in (1.2),

$$\phi_1(k_1, k_2, w) = \frac{v-3}{\theta} + \frac{2(\theta-1)}{(\theta-1)^2 - x^2}.$$

Hence the efficiency of the residual design d^* is given by

$$e(k_1, k_2; w) = \frac{\phi_2}{\phi_1(k_1, k_2, w)} = \frac{(v-1)\{(\theta-1)^2 - x^2\}}{(v-3)\{(\theta-1)^2 - x^2\} + 2\theta(\theta-1)} \tag{2.5}$$

for $0 \le w \le \min\{k_1, k_2\} - 1$ with $x = w\{k_1k_2(k_1-1)(k_2-1)\}^{-1/2}$.

As for behaviour of the values of efficiencies in Case 3.1, we have the following.

THEOREM 2.2. *In a* VB *design with parameters v, b, $\mathbf{r} = (r_1, \dots, r_v)'$, $\mathbf{k} = (k_1, \dots, k_b)'$, $n = \sum_{i=1}^{v} r_i = \sum_{j=1}^{b} k_j$ and $\theta = (n-b)/(v-1)$, for Case* 3.1

$$e(k_1, k_2; 0) > e(k_1, k_2; 1) > \cdots > e(k_1, k_2; w_{\max}),$$

where $w_{\max} \le \min\{k_1, k_2\} - 1$.

PROOF. Note that x in (2.5) is an increasing function of w. This implies that $\phi_1(k_1, k_2, w)$ is also an increasing function of w. For any integers w', w such that $0 \le w' < w \le \min\{k_1, k_2\} - 1$, it follows that $\phi_1(k_1, k_2, w') < \phi_1(k_1, k_2, w)$, which through (2.4) implies that $e(k_1, k_2; w') > e(k_1, k_2; w)$ for $0 \le w' < w \le \min\{k_1, k_2\} - 1$. This completes the proof. □

THEOREM 2.3. *In a* VB *design with parameters v, b, $\mathbf{r} = (r_1, \dots, r_v)'$, $\mathbf{k} =$

$(k_1, \ldots, k_b)'$, $n = \sum_{i=1}^{v} r_i = \sum_{j=1}^{b} k_j$ *and* $\theta = (n-b)/(v-1)$, *for Case* 3.1

$$e(k_{\min}, k_{\min}; w) \leq e(k_1, k_2; w) \leq e(k_{\max}, k_{\max}; w)$$

for a fixed positive integer w *such that* $1 \leq w \leq \min\{k_1, k_2\} - 1$.

Proof. First note that x in (2.5) is a symmetric function of k_1 and k_2. The partial derivatives of x with respect to k_1 and k_2 can be given by

$$\frac{\partial x}{\partial k_1} = -\frac{k_2(2k_1-1)(k_2-1)w}{2\{k_1k_2(k_1-1)(k_2-1)\}^{3/2}}, \qquad \frac{\partial x}{\partial k_2} = -\frac{k_1(2k_2-1)(k_1-1)w}{2\{k_1k_2(k_1-1)(k_2-1)\}^{3/2}},$$

respectively, which are negative for $1 \leq w \leq \min\{k_1, k_2\} - 1$. Thus, x is a decreasing function of k_1 and k_2. This implies that $\phi_1(k_1, k_2, w)$ is also a decreasing function of k_1 and k_2. It follows that $\phi_1(k_{\min}, k_{\min}, w) \geq \phi_1(k_1, k_2, w) \geq \phi_1(k_{\max}, k_{\max}, w)$, which through (2.5) implies that $e(k_{\min}, k_{\min}; w) \leq e(k_1, k_2; w) \leq e(k_{\max}, k_{\max}; w)$. This completes the proof. □

Remark 2.2. Note that $x = 0$ in (2.5) when $w = 0$. Thus, the efficiency of the residual design, $e(k_1, k_2; w = 0)$, is independent of block sizes k_1 and k_2.

We now compare values of $e(k_1, k_2; w)$ for Cases 2 and 3.1. The corresponding efficiencies are denoted by $e_2(k_1, k_2; w)$ and $e_{3.1}(k_1, k_2; w)$, respectively.

Theorem 2.4. *In a* VB *design with parameters* v, b, $\mathbf{r} = (r_1, \ldots, r_v)'$, $\mathbf{k} = (k_1, \ldots, k_b)'$,

$$e_2(k_1, k_2; w) \leq e_{3.1}(k_1, k_2; w)$$

for a fixed positive integer w *such that* $1 \leq w \leq \min\{k_1, k_2\} - 1$.

Proof. Let x_i stand for x in Case i. Recall that $x_{3.1} = w\{k_1k_2(k_1-1) \times (k_2-1)\}^{-1/2}$ and $x_2 = (k_1k_2 - k_1 - k_2 + w)\{k_1k_2(k_1-1)(k_2-1)\}^{-1/2}$. It can be shown that $x_{3.1} \leq x_2$. Since $e(k_1, k_2; w)$ is a decreasing function of x_i as shown before, it follows that $e_2(k_1, k_2; w) \leq e_{3.1}(k_1, k_2; w)$. This completes the proof. □

Theorems 2.2 and 2.3 imply that the behaviour of $e(k_{\min}, k_{\min}; w_{\max})$ is important to judge whether the design is robust or not for Case 3.1. However, Theorem 2.4 shows that the efficiency of the residual design with pattern of missing observations in Case 3.1 is not smaller than the one with pattern of missing observations in Case 2 for any VB design. Hence it is enough to evaluate the values of $e(k_j, k_{j'}; w)$ for Case 2 to show the robustness for Case 3.1 pattern. Thus, from the previous evaluation for Case 2, it follows that VB designs are fairly robust against the unavailability of the two observations in Case 3.1.

Case 3.2: $n_{21} = 0$ and $n_{12} = 1$ with $1 \le w \le \min\{k_1, k_2\} - 1$.

In this case, the C-matrix of the residual design d^* can be given by C^* in (2.4) with

$$c_{11} = \theta - 1 - v^{-1}\theta + k_1^{-1} - k_2^{-1}(k_2 - 1)^{-1}, \qquad c_{22} = \theta - 1 - v^{-1}\theta + k_2^{-1},$$
$$C_{33} = \theta I_{w-1} - \{v^{-1}\theta + k_1^{-1}(k_1 - 1)^{-1} + k_2^{-1}(k_2 - 1)^{-1}\}J_{w-1},$$
$$C_{44} = \theta I_{k_1 - w} - \{v^{-1}\theta + k_1^{-1}(k_1 - 1)^{-1}\}J_{k_1 - w},$$
$$C_{55} = \theta I_{k_2 - w - 1} - \{v^{-1}\theta + k_2^{-1}(k_2 - 1)^{-1}\}J_{k_2 - w - 1},$$
$$C_{66} = \theta I_{v - k_1 - k_2 + w} - v^{-1}\theta J_{v - k_1 - k_2 + w},$$
$$c_{12} = -v^{-1}\theta + k_2^{-1}, \qquad c_{13} = -v^{-1}\theta + k_1^{-1} - k_2^{-1}(k_2 - 1)^{-1},$$
$$c_{14} = -v^{-1}\theta + k_1^{-1}, \qquad c_{15} = -v^{-1}\theta - k_2^{-1}(k_2 - 1)^{-1},$$
$$c_{16} = c_{24} = c_{26} = c_{36} = c_{45} = c_{46} = c_{56} = -v^{-1}\theta, \qquad c_{23} = c_{25} = -v^{-1}\theta + k_2^{-1},$$
$$c_{34} = -v^{-1}\theta - k_1^{-1}(k_1 - 1)^{-1}, \qquad c_{35} = -v^{-1}\theta - k_2^{-1}(k_2 - 1)^{-1}.$$

Hence we can obtain the following after some calculation through Lemma 1.1.

LEMMA 2.3. *The $v - 1$ non-zero eigenvalues of C^*, $1 \le w \le \min\{k_1, k_2\} - 1$, are given by*

θ	*w.m.* $v - 3$,
$\theta - 1 - x$	*w.m.* 1,
$\theta - 1 + x$	*w.m.* 1,

where $x = (k_1 - w)\{k_1 k_2 (k_1 - 1)(k_2 - 1)\}^{-1/2}$.

By Lemma 2.3, in (1.2),

$$\phi_1(k_1, k_2, w) = \frac{v - 3}{\theta} + \frac{2(\theta - 1)}{(\theta - 1)^2 - x^2}.$$

Hence the efficiency of the residual design d^* is given by

$$e(k_1, k_2; w) = \frac{\phi_2}{\phi_1(k_1, k_2, w)} = \frac{(v - 1)\{(\theta - 1)^2 - x^2\}}{(v - 3)\{(\theta - 1)^2 - x^2\} + 2\theta(\theta - 1)} \tag{2.6}$$

for $1 \le w \le \min\{k_1, k_2\} - 1$ with $x = (k_1 - w)\{k_1 k_2 (k_1 - 1)(k_2 - 1)\}^{-1/2}$.

As for behaviour of the values of efficiencies in Case 3.2, we have the following.

THEOREM 2.5. *In a* VB *design with parameters v, b, $\mathbf{r} = (r_1, \ldots, r_v)'$, $\mathbf{k} = (k_1, \ldots, k_b)'$, $n = \sum_{i=1}^{v} r_i = \sum_{j=1}^{b} k_j$ and $\theta = (n - b)/(v - 1)$, for Case* 3.2

$$e(k_1, k_2; 1) < e(k_1, k_2; 2) < \cdots < e(k_1, k_2; w_{\max}),$$

where $w_{\max} \le \min\{k_1, k_2\} - 1$.

PROOF. Note that x in (2.6) is a decreasing function of w. This implies that $\phi_1(k_1, k_2, w)$ is also a decreasing function of w. For any integers w', w such that $1 \le w' < w \le \min\{k_1, k_2\} - 1$, it follows that $\phi_1(k_1, k_2, w') > \phi_1(k_1, k_2, w)$, which through (2.6) implies that $e(k_1, k_2; w') < e(k_1, k_2; w)$ for $1 \le w' < w \le \min\{k_1, k_2\} - 1$. This completes the proof. □

THEOREM 2.6. *In a* VB *design with parameters* $v, b, \mathbf{r} = (r_1, \dots, r_v)'$, $\mathbf{k} = (k_1, \dots, k_b)'$, $n = \sum_{i=1}^{v} r_i = \sum_{j=1}^{b} k_j$ *and* $\theta = (n - b)/(v - 1)$, *for Case* 3.2

$$e(k_{\max}, k_{\min}; w) \le e(k_1, k_2; w) \le e(k_{\min}, k_{\max}; w)$$

for a fixed positive integer w *such that* $1 \le w \le k_{\min} - 1$.

PROOF. The partial derivative of x in (2.6) with respect to k_1 can be given by

$$\frac{\partial x}{\partial k_1} = \frac{(2k_1 - 1)w - k_1}{2\{k_1(k_1 - 1)\}^{3/2}\{k_2(k_2 - 1)\}^{1/2}},$$

which is positive for $1 \le w \le k_{\min} - 1$. Thus, x is an increasing function of k_1. This implies that $\phi_1(k_1, k_2, w)$ is also an increasing function of k_1. On the other hand, the partial derivative of x in (2.6) with respect to k_2 can be given by

$$\frac{\partial x}{\partial k_2} = -\frac{k_1(k_1 - w)(k_1 - 1)(2k_2 - 1)}{2\{k_1 k_2(k_1 - 1)(k_2 - 1)\}^{3/2}},$$

which is negative for $1 \le w \le k_{\min} - 1$. Thus, x is a decreasing function of k_2. This implies that $\phi_1(k_1, k_2, w)$ is also a decreasing function of k_2. It follows that $\phi_1(k_{\max}, k_{\min}, w) \ge \phi_1(k_1, k_2, w) \ge \phi_1(k_{\min}, k_{\max}, w)$, which through (2.6) implies that $e(k_{\max}, k_{\min}; w) \le e(k_1, k_2; w) \le e(k_{\min}, k_{\max}; w)$. This completes the proof. □

When the sizes of two blocks in which the missing observations for Case 3.2 occur are equal, we can obtain the following which shows the sensitive behaviour of efficiency.

THEOREM 2.7. *In a* VB *design with parameters* $v, b, \mathbf{r} = (r_1, \dots, r_v)'$, $\mathbf{k} = (k_1, \dots, k_b)'$, $n = \sum_{i=1}^{v} r_i = \sum_{j=1}^{b} k_j$ *and* $\theta = (n - b)/(v - 1)$, *for Case* 3.2, *if* $k > k'$, *then*

$$e(k, k; w) \le e(k', k'; w)$$

for $kk'/(k + k' - 1) \le w \le k'$ *and*

$$e(k, k; w) > e(k', k'; w)$$

for $1 \le w < kk'/(k + k' - 1)$.

PROOF. Let $x(k) = (k - w)\{k(k-1)\}^{-1}$ stand for x in (2.6). For any two possible block sizes k, k' such that $k_{\min} \le k' < k \le k_{\max}$, it can be shown that

$$x(k) - x(k') = \frac{(k-k')\{w(k+k'-1) - kk'\}}{kk'(k-1)(k'-1)},$$

which means that $x(k) \ge x(k')$ if $kk'/(k+k'-1) \le w' \le k'$, and $x(k) < x(k')$ if $1 \le w < kk'/(k+k'-1)$. Since $e(k,k,w)$ in (2.6) is also a decreasing function of $x = x(k)$. It follows that $e(k,k;w) \le e(k',k';w)$ for $kk'/(k+k'-1) \le w \le k'$, and $e(k,k;w) > e(k',k';w)$ for $1 \le w < kk'/(k+k'-1)$. This completes the proof. □

We now compare values of $e(k_1,k_2;w)$ for Cases 2 and 3.2. A notation $e_{3.2}(k_1,k_2;w)$ is defined in the same way as Theorem 2.4.

THEOREM 2.8. *In a* VB *design with parameters* v, b, $\mathbf{r} = (r_1,\dots,r_v)'$, $\mathbf{k} = (k_1,\dots,k_b)'$,

$$e_2(k_1,k_2;w) \le e_{3.2}(k_1,k_2;w)$$

for a fixed positive integer w *such that* $1 \le w \le \min\{k_1,k_2\} - 1$.

PROOF. Let x_i stand for x in Case i. Recall that $x_{3.2} = (k_1 - w) \times \{k_1k_2(k_1-1)(k_2-1)\}^{-1/2}$ and $x_2 = (k_1k_2 - k_1 - k_2 + w)\{k_1k_2(k_1-1)(k_2-1)\}^{-1/2}$. It can be shown that $x_{3.2} \le x_2$. Since $e(k_1,k_2;w)$ is a decreasing function of x_i, it follows that $e_2(k_1,k_2;w) \le e_{3.1}(k_1,k_2;w)$. This completes the proof. □

Theorems 2.5 and 2.6 imply that the behaviour of $e(k_{\max},k_{\min};1)$ is important to judge whether the design is robust or not for Case 3.2. However, Theorem 2.8 shows that the efficiency of the residual design with pattern of missing observations in Case 3.2 is not smaller than the one with pattern of missing observations in Case 2 for any VB design. Hence it is enough to evaluate the values of $e(k_j,k_{j'};w)$ for Case 2 to show the robustness for Case 3.2 pattern. Thus, from the previous evaluation for Case 2, it follows that VB designs are fairly robust against the unavailability of the two observations in Case 3.2.

For $n_{21} = 1$ and $n_{12} = 0$, a case that two observations n_{11} and n_{22} are lost can be considered. However, this case will be omitted. Because this case is equivalent to Case 3.2 by exchanging the first and second treatments and simultaneously by exchanging the first and second blocks, respectively. Thus by exchanging k_1 and k_2 of (2.6), we obtain the same result in this case as Case 3.2.

Case 3.3: $n_{21} = 1$ and $n_{12} = 1$ with $2 \le w \le \min\{k_1,k_2\}$.

In this case, the C-matrix of the residual design d^* can be given by C^* in (2.4) with

$$
\begin{aligned}
&c_{11} = \theta - 1 - v^{-1}\theta + k_1^{-1} - k_2^{-1}(k_2-1)^{-1},\\
&c_{22} = \theta - 1 - v^{-1}\theta + k_2^{-1} - k_1^{-1}(k_1-1)^{-1},\\
&C_{33} = \theta I_{w-2} - \{v^{-1}\theta + k_1^{-1}(k_1-1)^{-1} + k_2^{-1}(k_2-1)^{-1}\}J_{w-2},\\
&C_{44} = \theta I_{k_1-w} - \{v^{-1}\theta + k_1^{-1}(k_1-1)^{-1}\}J_{k_1-w},\\
&C_{55} = \theta I_{k_2-w} - \{v^{-1}\theta + k_2^{-1}(k_2-1)^{-1}\}J_{k_2-w},\\
&C_{66} = \theta I_{v-k_1-k_2+w} - v^{-1}\theta J_{v-k_1-k_2+w},\\
&c_{12} = -v^{-1}\theta + k_1^{-1} + k_2^{-1}, \qquad c_{13} = -v^{-1}\theta + k_1^{-1} - k_2^{-1}(k_2-1)^{-1},\\
&c_{14} = -v_1^{-1}\theta + k_1^{-1}, \qquad c_{15} = -v^{-1}\theta - k_2^{-1}(k_2-1)^{-1},\\
&c_{16} = c_{26} = c_{36} = c_{45} = c_{46} = c_{56} = -v^{-1}\theta,\\
&c_{23} = -v^{-1}\theta + k_2^{-1} - k_1^{-1}(k_1-1)^{-1}, \qquad c_{24} = -v^{-1}\theta - k_1^{-1}(k_1-1)^{-1},\\
&c_{25} = -v^{-1}\theta + k_2^{-1}, \qquad c_{34} = -v^{-1}\theta - k_1^{-1}(k_1-1)^{-1},\\
&c_{35} = -v^{-1}\theta - k_2^{-1}(k_2-1)^{-1}.
\end{aligned}
$$

Hence we can obtain the following after some calculation through Lemma 1.1.

LEMMA 2.4. *The $v-1$ non-zero eigenvalues of C^*, $2 \le w \le \min\{k_1, k_2\}$, are given by*

$$
\begin{array}{ll}
\theta & w.m.\ v-3,\\
\theta - 1 - x & w.m.\ 1,\\
\theta - 1 + x & w.m.\ 1,
\end{array}
$$

where $x = (k_1 + k_2 - w)\{k_1k_2(k_1-1)(k_2-1)\}^{-1/2}$.

By Lemma 2.4, in (1.2),

$$
\phi_1(k_1, k_2, w) = \frac{v-3}{\theta} + \frac{2(\theta-1)}{(\theta-1)^2 - x^2}.
$$

Hence the efficiency of the residual design d^* is given by

$$
e(k_1, k_2; w) = \frac{\phi_2}{\phi_1(k_1, k_2, w)} = \frac{(v-1)\{(\theta-1)^2 - x^2\}}{(v-3)\{(\theta-1)^2 - x^2\} + 2\theta(\theta-1)} \tag{2.7}
$$

for $2 \le w \le \min\{k_1, k_2\}$ with $x = (k_1 + k_2 - w)\{k_1k_2(k_1-1)(k_2-1)\}^{-1/2}$.

As for behaviour of the values of efficiencies in Case 3.3, we have the following.

THEOREM 2.9. *In a VB design with parameters $v, b, \mathbf{r} = (r_1, \dots, r_v)'$, $\mathbf{k} = (k_1, \dots, k_b)'$, $n = \sum_{i=1}^{v} r_i = \sum_{j=1}^{b} k_j$ and $\theta = (n-b)/(v-1)$, for Case 3.3*

$$
e(k_1, k_2; 2) < e(k_1, k_2; 3) < \cdots < e(k_1, k_2; w_{\max}),
$$

where $w_{\max} \le \min\{k_1, k_2\}$.

Proof. Note that x in (2.7) is a decreasing function of w. This implies that $\phi_1(k_1, k_2, w)$ is also a decreasing function of w. For any integers w', w such that $2 \leq w' < w \leq \min\{k_1, k_2\}$, it follows that $\phi_1(k_1, k_2, w') > \phi_1(k_1, k_2, w)$, which through (2.7) implies that $e(k_1, k_2; w') < e(k_1, k_2; w)$ for $2 \leq w' < w \leq \min\{k_1, k_2\}$. This completes the proof. □

Theorem 2.10. *In a* VB *design with parameters* v, b, $\mathbf{r} = (r_1, \ldots, r_v)'$, $\mathbf{k} = (k_1, \ldots, k_b)'$, $n = \sum_{i=1}^{v} r_i = \sum_{j=1}^{b} k_j$ *and* $\theta = (n - b)/(v - 1)$, *for Case* 3.3

$$e(k_{\min}, k_{\min}; w) \leq e(k_1, k_2; w) \leq e(k_{\max}, k_{\max}; w)$$

for a fixed positive integer w *such that* $2 \leq w \leq \min\{k_1, k_2\}$.

Proof. First note that x in (2.7) is a symmetric function of k_1 and k_2. The partial derivatives of x with respect to k_1 and k_2 can be given by

$$\frac{\partial x}{\partial k_1} = \frac{-k_1(2k_2 + 1) + (2k_1 - 1)w + k_2}{2\{k_1(k_1 - 1)\}^{3/2}\{k_2(k_2 - 1)\}^{1/2}},$$

$$\frac{\partial x}{\partial k_2} = \frac{-k_2(2k_1 + 1) + (2k_2 - 1)w + k_1}{2\{k_1(k_1 - 1)\}^{1/2}\{k_2(k_2 - 1)\}^{3/2}},$$

respectively. Without loss of generality, we assume that $k_1 \leq k_2$. Then, for any $2 \leq w \leq \min\{k_1, k_2\} = k_1$, we have

$$\begin{aligned} -k_1(2k_2 + 1) + (2k_1 - 1)w + k_2 &\leq -k_1(2k_2 + 1) + (2k_1 - 1)k_1 + k_2 \\ &= (2k_1 - 1)(k_1 - k_2) - k_1 < 0, \\ -k_2(2k_1 + 1) + (2k_2 - 1)w + k_1 &\leq -k_2(2k_1 + 1) + (2k_2 - 1)k_1 + k_2 \\ &= -k_2 < 0. \end{aligned}$$

Thus, x is a decreasing function of k_1 and k_2. This implies that $\phi_1(k_1, k_2, w)$ is also a decreasing function of k_1 and k_2. It follows that $\phi_1(k_{\min}, k_{\min}, w) \geq \phi_1(k_1, k_2, w) \geq \phi_1(k_{\max}, k_{\max}, w)$, which through (2.7) implies that $e(k_{\min}, k_{\min}; w) \leq e(k_1, k_2; w) \leq e(k_{\max}, k_{\max}; w)$. This completes the proof. □

We now compare values of $e(k_1, k_2; w)$ for Cases 2 and 3.3.

Theorem 2.11. *In a* VB *design with parameters* v, b, $\mathbf{r} = (r_1, \ldots, r_v)'$, $\mathbf{k} = (k_1, \ldots, k_b)'$,

$$e_2(k_1, k_2; w) \leq e_{3.3}(k_1, k_2; w)$$

for a fixed positive integer w *such that* $2 \leq w \leq \min\{k_1, k_2\}$.

PROOF. Let x_i stand for x in Case i. Recall that $x_{3.3} = (k_1 + k_2 - w) \times \{k_1 k_2 (k_1 - 1)(k_2 - 1)\}^{-1/2}$ and $x_2 = (k_1 k_2 - k_1 - k_2 + w)\{k_1 k_2 (k_1 - 1)(k_2 - 1)\}^{-1/2}$. It can be shown that $x_{3.3} \le x_2$. Since $e(k_1, k_2; w)$ is a decreasing function of x_i, it follows that $e_2(k_1, k_2; w) \le e_{3.3}(k_1, k_2; w)$. This completes the proof. □

Theorems 2.9 and 2.10 imply that the behaviour of $e(k_{\min}, k_{\min}; 2)$ is important to judge whether the design is robust or not for Case 3.3. However, Theorem 2.11 shows that the efficiency of the residual design with pattern of missing observations in Case 3.3 is not smaller than the one with pattern of missing observations in Case 2 for any VB design. Hence it is enough to evaluate the values of $e(k_j, k_{j'}; w)$ for Case 2 to show the robustness for Case 3.3 pattern. Thus, from the previous evaluation for Case 2, it follows that VB designs are fairly robust against the unavailability of two observations in Case 3.3.

Thus, from the previous discussions, we can conclude that VB designs are fairly robust against the unavailability of any two observations in the design.

2.2. Bound on the efficiency of the residual designs

For a given VB design, we consider how the efficiency of residual designs changes according to positions of missing observations. A bound on the efficiency of the residual designs of VB designs is given in this section. It can be shown that the best design (which has the smallest loss of information) in a class of residual designs is derived by removing two observations in the same block or these corresponding to two different treatments in two different blocks which have disjoint sets of treatments. The worst design happens when two observations corresponding to the same treatment in two different blocks are removed.

Let $e_i(k_1, k_2; w)$ stand for the efficiency of the residual designs in Case i with $i = 2, 3.1, \ldots, 3.3$. For Case 1 see (2.2) as $e_1(x = 0)$ there. We can obtain the following.

THEOREM 2.12. *In a* VB *design with parameters* $v, b, \mathbf{r} = (r_1, \ldots, r_v)'$, $\mathbf{k} = (k_1, \ldots, k_b)'$, $n = \sum_{i=1}^{v} r_i = \sum_{j=1}^{b} k_j$ *and* $\theta = (n - b)/(v - 1)$,

$$e_2(k_1, k_2; w) \le e_i(k_1, k_2; w) < e_1(x = 0) = e_{3.1}(k_1, k_2; w = 0)$$

for all possible Cases i except for Cases 1, 2 *and* 3.1 *with* $w = 0$.

PROOF. First note that the expressions of efficiency of the residual designs for the five cases mentioned above are the same as a function on x, which is given by

$$\frac{(v-1)\{(\theta-1)^2-x^2\}}{(v-3)\{(\theta-1)^2-x^2\}+2\theta(\theta-1)} \quad (=e(x),\ \text{say}).$$

It can be shown that $e(x)$ is a decreasing function of x. Recall that x_i stands for x in the efficiency factor of Case i. Note that $0 \le x \le x_2$ in the five cases mentioned above. Thus it follows that the maximum and minimum of efficiency of the residual designs are given by $e(x=0)$ and $e(x=x_2)$, which correspond to Cases 1 and 3.1 with $w=0$, and Case 2, respectively. This completes the proof. □

An example is illustrated. A VB design with 7 treatments, 14 blocks of size $k_1=3$ and $k_2=4$, respectively, having 7 replicates is considered, whose blocks are given by

$$\begin{bmatrix} 1 & 3 & 2 & 1 & 3 & 1 & 4 & 1 & 1 & 2 & 2 & 1 & 1 & 2 \\ 2 & 5 & 3 & 4 & 4 & 2 & 5 & 2 & 5 & 3 & 6 & 3 & 3 & 4 \\ 4 & 6 & 5 & 6 & 6 & 5 & 7 & 3 & 6 & 4 & 7 & 4 & 7 & 5 \\ & 7 & & 7 & & 7 & & 6 & & 7 & & 5 & & 6 \end{bmatrix},$$

where columns show blocks of three or four treatments. Let d be two copies of this design which yields another VB design. All values of efficiency for the residual design d^* are shown in the following tables. Here "—" denotes that such case does not exist. The data with h and l denote the highest and lowest values of efficiencies of the residual designs in that tables, respectively.

Through Tables 2.1 to 2.4, it follows that the maximum of efficiency of the residual designs is $e_{\max} = e_1(x=0) = e_{3.1}(k_1, k_2; w=0) = 0.96969$. The residual designs corresponding to Cases 1 and 3.1 with $w=0$ are the designs with the highest efficiency in the class of the residual designs. Note that $e_{\max}$ is independent of k_1 and k_2. The minimum of efficiency of the residual designs is given by $e_{\min} = e_2(3,3;3) = e_2(4,4;4) = 0.96666$.

3. Robustness of augmented BIB designs

Consider an augmented BIB design d with parameters $v_0 = v+1$, $b_0 = b$, $\mathbf{k}_0 = (k+1)\mathbf{1}_b$ and $\mathbf{r}_0 = (r\mathbf{1}'_v, b)'$. It is easily shown that the non-zero eigenvalues of the C-matrix of d are $(rk+\lambda)/(k+1)$ with multiplicity $v-1$ and $(vr+r)/(k+1)$ with multiplicity one.

Suppose that any two blocks in an augmented BIB design d are lost. Without loss of generality, assume that the first two blocks are lost. Let w be the number of test treatments common to two such blocks in the original BIB

Table 2.1 $e(k_1, k_1; w)$ for $0 \le w \le k_1$

Case i	$w = 0$	$w = 1$	$w = 2$	$w = 3$	$w = 4$
Case 1	0.96969^h	—	—	—	—
Case 2	—	0.96835	0.96759	0.96666^l	—
Case 3.1	0.96969^h	0.96961	0.96936	—	—
Case 3.2	—	0.96936	0.96961	—	—
Case 3.3	—	—	0.96835	0.96894	—

Table 2.2 $e(k_2, k_2; w)$ for $0 \le w \le k_2$

Case i	$w = 0$	$w = 1$	$w = 2$	$w = 3$	$w = 4$
Case 1	0.96969^h	—	—	—	—
Case 2	—	0.96799	0.96759	0.96715	0.96666^l
Case 3.1	0.96969^h	0.96967	0.96961	0.96950	—
Case 3.2	—	0.96950	0.96961	0.96967	—
Case 3.3	—	—	0.96894	0.96917	0.96936

Table 2.3 $e(k_1, k_2; w)$ for $0 \le w \le \min\{k_1, k_2\}$

Case i	$w = 0$	$w = 1$	$w = 2$	$w = 3$	$w = 4$
Case 1	0.96969^h	—	—	—	—
Case 2	—	0.96818	0.96763	0.96700	—
Case 3.1	0.96969^h	0.96965	0.96952	—	—
Case 3.2	—	0.96952	0.96965	—	—
Case 3.3	—	—	0.96864	0.96902	—

Table 2.4 $e(k_2, k_1; w)$ for $0 \le w \le \min\{k_1, k_2\}$

Case i	$w = 0$	$w = 1$	$w = 2$	$w = 3$	$w = 4$
Case 1	0.96969^h	—	—	—	—
Case 2	—	0.96818	0.96763	0.96700	—
Case 3.1	0.96969^h	0.96965	0.96952	—	—
Case 3.2	—	0.96932	0.96952	—	—
Case 3.3	—	—	0.96864	0.96902	—

design. It is known (Connor (1952)) that the number w of treatments common to any two blocks satisfies $-(r-\lambda-k) \leq w \leq 2\lambda k/r + (r-\lambda-k)$ $(= w_{\max}, \text{say})$.

In this case, the C-matrix of the residual design d^* can be given by $C^*(w)$, where

$$(k+1)C^*(w) = \begin{bmatrix} C_{11} & c_{12}J & c_{13}J & c_{14}J & c_{15}J \\ & C_{22} & c_{23}J & c_{24}J & c_{25}J \\ & & C_{33} & c_{34}J & c_{35}J \\ & \text{Sym} & & C_{44} & c_{45}J \\ & & & & c_{55} \end{bmatrix}$$

with

$C_{11} = (q-2k-2)I_w - (\lambda-2)J_w$,
$C_{22} = C_{33} = (q-k-1)I_{k-w} - (\lambda-1)J_{k-w}$,
$C_{44} = qI_{v-2k+w} - \lambda J_{v-2k+w}$, $c_{55} = vr - 2k$, $c_{12} = c_{13} = -(\lambda-1)$,
$c_{14} = c_{23} = c_{24} = c_{34} = -\lambda$, $c_{15} = -(r-2)$, $c_{25} = c_{35} = -(r-1)$,
$c_{45} = -r$, $q = rk + \lambda$.

The calculation of eigenvalues of $C^*(w)$ can be done by separating it into three cases of $w = 0$, $w = k$ (for the same blocks) and $1 \leq w \leq k-1$.

Case 1: $w = 0$. In this case, we can obtain the following after some tedious calculation through Lemma 1.1.

LEMMA 3.1. *The v non-zero eigenvalues of $C^*(w)$ with $w = 0$ are given by*

$q/(k+1)$	*w.m.* $v-2k-1$,
$q/(k+1)-1$	*w.m.* $2k-2$,
$(q-1)/(k+1)$	*w.m.* 1,
$(\alpha \pm (\alpha^2-4\beta)^{1/2})/\{2(k+1)\}$	*w.m.* 1 *each*,

where

$\alpha = 2(q-k) + v(r-\lambda) - 1$, $\beta = (v+1)\{r(q-1) - 2k\lambda\}$, $q = rk + \lambda$.

Recall that $\phi_1(w)$ and ϕ_2 are the sum of reciprocals of non-zero eigenvalues of C^* and C, respectively. By Lemma 3.1, in (1.2),

$$\begin{aligned} \phi_1(0) &= (k+1)\left(\frac{v-2k-1}{q} + \frac{2k-2}{q-(k+1)} + \frac{1}{q-1} + \frac{\alpha}{\beta}\right), \\ \phi_2 &= (k+1)\left(\frac{v-1}{rk+\lambda} + \frac{1}{vr+r}\right), \end{aligned} \tag{3.1}$$

which yield the efficiency of the residual design d^* in $e(0) = \phi_2/\phi_1(0)$.

Case 2: $w = k$. In this case, we can obtain the following through Lemma 1.1.

LEMMA 3.2. *The v non-zero eigenvalues of $C^*(w)$ with $w = k$ are given by*

$q/(k+1)$	*w.m.* $v-k-1$,
$q/(k+1)-2$	*w.m.* $k-1$,
$(\alpha \pm (\alpha^2-4\beta)^{1/2})/\{2(k+1)\}$	*w.m.* 1 *each*,

where

$$\alpha = (v+1)r + q - 2(k+1),\ \beta = (v+1)\{rq - 2(k\lambda + r)\},\ q = rk + \lambda.$$

By Lemma 3.2, in (1.2),

$$\phi_1(k) = (k+1)\left(\frac{v-k-1}{q} + \frac{k-1}{q-2(k+1)} + \frac{\alpha}{\beta}\right).$$

Hence the effciency of the residual design d^* is, in (1.2), $e(k) = \phi_2/\phi_1(k)$ with (3.1).

Case 3: $1 \le w \le k-1$. It follows from Lemma 1.1 that the $v-4$ non-zero the eigenvalues of C^* are $q/(k+1)-2$, $q/(k+1)-1$ and $q/(k+1)$ with respective multiplicities $w-1$, $2(k-w-1)$ and $v-2k+w-1$. Other four non-zero eigenvalues of C^* are $\mu_1^*/(k+1), \dots, \mu_4^*/(k+1)$, where $\mu_1^*, \dots, \mu_4^*$ are non-zero eigenvalues of $\Delta = D_a + D_s^{1/2} B D_s^{1/2}$ with

$$D_a = \mathrm{diag}\{q-2k-2, q-k-1, q-k-1, q, vr-2k\},$$

$$D_s = \mathrm{diag}\{w, k-w, k-w, v-2k+w, 1\},$$

$$B = \begin{bmatrix} -(\lambda-2) & -(\lambda-1) & -(\lambda-1) & -\lambda & -(r-2) \\ & -(\lambda-1) & -\lambda & -\lambda & -(r-1) \\ & & -(\lambda-1) & -\lambda & -(r-1) \\ \text{Sym} & & & -\lambda & -r \\ & & & & 0 \end{bmatrix}.$$

It is easy to see that $q-w-1$ is an eigenvalue of Δ with multiplicity one. Thus $(q-w-1)/(k+1)$ is the eigenvalue of $C^*(w)$ with multiplicity one. The remaining three non-zero eigenvalues of Δ have to satisfy a polynomial equation of degree 3, say $\mu^3 + a_2\mu^2 + a_1\mu - a_0 = 0$. It is seen that the remaining three eigenvalues, say μ_1^*, μ_2^* and μ_3^*, of Δ satisfy

$$\frac{1}{\mu_1^*} + \frac{1}{\mu_2^*} + \frac{1}{\mu_3^*} = \frac{a_1}{a_0}$$

with $a_0 = \beta\gamma + (v+1)\{rq - 2(k+1)\lambda\}w$, $a_1 = \beta + \alpha\gamma + \{(v+1)r + \gamma\}w$, $a_2 = -(\alpha + \gamma + w)$, where $\alpha = 2(q-k) + v(r-\lambda) - 1$, $\beta = (v+1)\{r(q-1) - 2k\lambda\}$, $\gamma = q - 2(k+1)$, $q = rk + \lambda$.

In (1.2)

$$\phi_1(w) = (k+1)\left(\frac{w-1}{q-2(k+1)} + \frac{2(k-w-1)}{q-(k+1)} + \frac{v-2k+w-1}{q} + \frac{1}{q-(w+1)} + \frac{a_1}{a_0}\right).$$

Hence the efficiency of the residual design d^* is, in (1.2), given by $e(w) = \phi_2/\phi_1(w)$ with (3.1) for $1 \leq w \leq k-1$.

REMARK 3.1. Note that the expression, $e(w)$, of efficiency mentioned above also holds for $w = 0$ and $w = k$.

An example is presented as an illustration. For an augmented BIB design derivable from a BIB(9, 12, 4, 3, 1) design, when $w = 1$, the characteristic equation of degree 5 is given by

$$\det(\Delta - \mu I) = -\mu(\mu - 11)(\mu^3 - 52\mu^2 + 695\mu - 2540) = 0$$

with $a_0 = 2540$, $a_1 = 695$ and $a_2 = -52$. Hence $e(1) = 0.783$.

The augmented BIB designs to be used here are derived from existing BIB designs listed in Hall (1986) and Raghavarao (1971). The values of $e(w)$, $0 \leq w \leq w_{\max}$, for 168 augmented BIB designs derivable from existing BIB designs listed in Hall (1986) and Raghavarao (1971) were worked out. From the evaluation it follows that for an augmented BIB design, $e(w)$ is decreasing as w is increasing. Thus, it is sufficient to investigate the values of $e(w_{\max})$ for all augmented BIB designs to show the robustness. The evaluation reveals that, except for some cases, all the designs have high values of $e(w_{\max})$. In fact, 128 augmented BIB designs satisfy $e(w_{\max}) \geq 0.90$, 24 augmented BIB designs get $0.90 > e(w_{\max}) \geq 0.80$, 10 augmented BIB designs satisfy $0.80 > e(w_{\max}) \geq 0.70$ and 6 augmented BIB designs derivable from BIB designs of series numbers 1, 2, 4, 8 and 11 in Raghavarao (1971) and of series number 1 in Hall (1986) have $e(1) = 0.54$, $e(2) = 0.43$, $e(3) = 0.57$, $e(4) = 0.65$, $e(2) = 0.67$ and $e(1) = 0.62$, respectively. Thus, we can conclude that augmented BIB designs are fairly robust against the unavailability of any two blocks in a design.

REMARK 3.2. The worst design in the sense of efficiency is an augmented BIB design derivable from a BIB$(4, 4, 3, 3, 2)$ design.

Part II. Constructions and analysis of some PEB designs

A block design $d(v, b, \mathbf{r}, \mathbf{k})$ is called a partially efficiency-balanced (PEB) design with m efficiency classes (Puri and Nigam (1977)), if there exists a set of $v-1$ linearly independent treatment contrasts $\{s_{ij}\}$ which can be partitioned into $m(\leq v-1)$ disjoint classes such that all the ρ_i contrasts of i-th class are estimated with the same relative loss of information μ_i, i.e., they satisfy the equations

$$M_0 s_{ij} = \mu_i s_{ij}, \qquad j = 1, \ldots, \rho_i \qquad \text{for } i = 1, \ldots, m,$$

where M_0 is defined by

$$M_0 = M - \frac{1}{n}\mathbf{1}_v \mathbf{r}', \qquad M = R^{-1} N K^{-1} N'.$$

The parameters of a PEB design with m efficiency classes can be written as $v, b, \mathbf{r}, \mathbf{k}, \mu_i, \rho_i, L_i$, $i = 1, \ldots, m$, where L_i's are mutually orthogonal idempotent matrices of ranks ρ_i and given by

$$L_i = \sum_{j=1}^{\rho_i} \{1/(s'_{ij} R s_{ij})\} s_{ij} s'_{ij} R, \qquad i = 1, \ldots, m$$

(see Puri and Nigam (1976)).

A particular class of PEB designs, in which μ_i takes only two distinct values $\mu_1 \neq 0$ and $\mu_2 = 0$ (no relative loss of information) with respective multiplicities ρ_1 and $\rho_2 = v - \rho_1 - 1$, is called a simple PEB design (Puri and Nigam (1977)). If $\mu_i = \mu$ for all i, the design is called an efficiency-balanced (EB) design (Puri and Nigam (1975)).

For a PEB design with m efficiency classes, M- and M_0-matrices have the spectral decomposition $M = \sum_{i=0}^{m} \mu_i L_i$ and $M_0 = \sum_{i=1}^{m} \mu_i L_i$, respectively, such that $\sum_{i=0}^{m} L_i = I_v$, where $\mu_0 = 1$, $L_0 = (1/n)\mathbf{1}_v \mathbf{r}'$ and $0 \leq \mu_i < 1$, $i = 1, \ldots, m$. In this case, the pseudo variance-covariance matrix Ω of adjusted treatment means under model (1.1) is shown by Caliński (1971) as

$$\Omega = \left\{ I_v + \sum_{i=1}^{m} [\mu_i/(1-\mu_i)] L_i \right\} R^{-1} \tag{4.0}$$

which plays an important role in statistical analysis.

Most works on PEB designs in the literature have been devoted to discussion of the construction and statistical analysis of the designs, for example, see Caliński (1971), Puri and Nigam (1976, 1977, 1978, 1983), Puri et al. (1977), Nigam and Puri (1982), Kageyama and Puri (1983), Puri (1984), Puri and Kageyama (1985, 1988), and Kageyama and Saha (1988). Further statistical justification on PEB designs can be found in Caliński and Kageyama (1996).

In Part II, we discuss the structural patterns and statistical analysis of equireplicate non-proper balanced designs, BIB designs and augmented BIB designs with some missing blocks. Several methods of constructing equireplicate, proper PEB designs are also given together with some PBIB designs.

4. Structural patterns of some block design with missing blocks

If some observations are lost in balanced designs or PEB designs, the residual designs will become new PEB designs. In this section, we discuss the following three cases of missing patterns: (i) one block is lost in an equireplicate balanced design; (ii) some disjoint blocks are lost in a BIB design; (iii) one block is lost in an augmented BIB design. The PEB property of these designs are clarified.

4.1. Some partially efficiency-balanced designs

(i) Equireplicate balanced design with one missing block

Consider a binary connected equireplicate VB design d with parameters v, b, r, $k_j, j = 1, \dots, b$, $n = \sum_{j=1}^{b} k_j$ and $\theta = (n-b)/(v-1)$, where θ is a non-zero eigenvalue of matrix $C = \theta(I_v - v^{-1}J_v)$ with multiplicity $v-1$. Since a proper binary VB design is a BIB design, we mainly consider non-proper binary VB designs. It is well known (cf. Dey et al. (1981)) that an equireplicate VB design is also an EB design. Hence, the present design is simply called an equireplicate balanced design here. Let d^* be the residual design by removing one block, for example, the first block of size k_1, in the equireplicate balanced design d. Without loss of generality, suppose that all observations occur in the first k_1 positions of the missing block. It is easily shown that the M-matix of the residual design d^* is given by

$$M^* = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

with

$$M_{11} = \frac{r-\theta}{r-1} I_{k_1} + \frac{1}{r-1}\left(\frac{\theta}{v} - \frac{1}{k_1}\right) J_{k_1}, \qquad M_{12} = \frac{\theta}{v(r-1)} J_{k_1 \times (v-k_1)},$$

$$M_{21} = \frac{\theta}{vr} J_{(v-k_1)\times k_1}, \qquad M_{22} = \frac{r-\theta}{r} I_{v-k_1} + \frac{\theta}{vr} J_{v-k_1}.$$

Hence we can obtain the following after some calculation through Lemma 1.1.

LEMMA 4.1. *The v eigenvalues of M^* are given by*

$$
\begin{array}{ll}
\mu_0^* = 1 & w.m.\ 1, \\
\mu_1^* = (r-\theta)/(r-1) & w.m.\ k_1 - 1, \\
\mu_2^* = (r-\theta)/r & w.m.\ v - k_1 - 1, \\
\mu_3^* = \{vr(r-\theta-1) + k_1\theta\}/\{vr(r-1)\} & w.m.\ 1.
\end{array}
$$

It follows from Lemma 4.1 that the design d^* is a PEB design with three efficiency classes. Hence we can get the following.

THEOREM 4.1. *In an equireplicate balanced design with parameters v, b, r, k_j, $j = 1, \dots, b$, $n = \sum_{j=1}^{b} k_j$ and $\theta = (n-b)/(v-1)$, if one block is lost, the residual design is a* PEB *design having three efficiency classes with parameters*

$$v^* = v, \quad b^* = b-1, \quad \mathbf{r}^* = [(r-1)\mathbf{1}'_{k_1},\ r\mathbf{1}'_{v-k_1}]', \quad k_j^* = k_j, \quad j = 2, \dots, b,$$

$$\mu_0^* = 1, \quad \mu_1^* = \frac{r-\theta}{r-1}, \quad \mu_2^* = \frac{r-\theta}{r}, \quad \mu_3^* = \frac{vr(r-\theta-1)+k_1\theta}{vr(r-1)},$$

$$\rho_0^* = 1, \quad \rho_1^* = k_1 - 1, \quad \rho_2^* = v - k_1 - 1, \quad \rho_3^* = 1,$$

$$L_0^* = \frac{1}{vr-k_1}\mathbf{1}_v\mathbf{r}^{*\prime}, \qquad L_1^* = \operatorname{diag}\left\{I_{k_1} - \frac{1}{k_1}J_{k_1},\ O_{v-k_1}\right\},$$

$$L_2^* = \operatorname{diag}\left\{O_{k_1},\ I_{v-k_1} - \frac{1}{v-k_1}J_{v-k_1}\right\}, \qquad L_3^* = I_v - L_0^* - L_1^* - L_2^*.$$

PROOF. It is easily shown that the idempotent matrices corresponding to the eigenvalues μ_i^* of M^*, $i = 0, \dots, 3$, are given by $L_0^* = (vr-k_1)^{-1}\mathbf{1}_v\mathbf{r}^{*\prime}$, $L_1^* = \operatorname{diag}\{(I_{k_1} - k_1^{-1}J_{k_1}, O_{v-k_1}\}$, $L_2^* = \operatorname{diag}\{O_{k_1}, I_{v-k_1} - (v-k_1)^{-1}J_{v-k_1}\}$, $L_3^* = I_v - L_0^* - L_1^* - L_2^*$, respectively, which are mutually orthogonal such that $\sum_{i=0}^{3} L_i^* = I_v$. This completes the proof. □

Recall that an equireplicate proper balanced design is a BIB design. Thus, if the design d^* is obtained by deleting one block in a BIB(v, b, r, k, λ) design, Theorem 4.1 shows the following.

COROLLARY 4.1. *In a* BIB(v, b, r, k, λ) *design, if one block is lost, the residual design is a* PEB *design having three efficiency classes with parameters*

$$v^* = v, \quad b^* = b-1, \quad \mathbf{r}^* = [(r-1)\mathbf{1}'_{k},\ r\mathbf{1}'_{v-k}]', \quad k^* = k,$$

$$\mu_0^* = 1, \quad \mu_1^* = \frac{r-\lambda}{k(r-1)}, \quad \mu_2^* = \frac{r-\lambda}{rk}, \quad \mu_3^* = \frac{(r-\lambda)(r-k)}{rk(r-1)},$$

$$\rho_0^* = 1, \quad \rho_1^* = k - 1, \quad \rho_2^* = v - k - 1, \quad \rho_3^* = 1,$$

$$L_0^* = \frac{1}{vr-k}\mathbf{1}_v\mathbf{r}^{*\prime}, \qquad L_1^* = \mathrm{diag}\left\{I_k - \frac{1}{k}J_k,\ O_{v-k}\right\},$$
$$L_2^* = \mathrm{diag}\left\{O_k,\ I_{v-k} - \frac{1}{v-k}J_{v-k}\right\}, \qquad L_3^* = I_v - L_0^* - L_1^* - L_2^*.$$

(ii) BIB design with some disjoint missing blocks

Bhaumik and Whittinghill (1991) discussed the robustness of VB designs by showing that the best design is derived by removing blocks which have disjoint sets of treatments, and the worst design appears when identical blocks are removed. Here we consider an equireplicate proper balanced design, i.e. BIB design with parameters v, b, r, k and λ, in which some disjoint blocks are lost. Let d^* be a residual design by removing s disjoint blocks in a BIB design for $1 \le s \le v/k$. Note that $s = 1$ means the missing of one block.

Without loss of generality, suppose that the first s disjoint blocks are lost. It is easily shown that the M-matrix of the residual design d^* is given by

$$M^* = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{4.1}$$

with

$$M_{11} = \frac{r-\lambda}{k(r-1)}I_{sk} + \frac{\lambda}{k(r-1)}J_{sk} - \frac{1}{k(r-1)}I_s \otimes J_k,$$

$$M_{12} = \frac{\lambda}{k(r-1)}J_{sk\times(v-sk)},$$

$$M_{21} = \frac{\lambda}{rk}J_{(v-sk)\times sk}, \qquad M_{22} = \frac{r-\lambda}{rk}I_{v-sk} + \frac{\lambda}{rk}J_{v-sk}.$$

Hence we can obtain the following through Lemma 1.1.

LEMMA 4.2. *The v eigenvalues of M^* for $1 \le s < v/k$ are given by*

$\mu_0^* = 1$	*w.m.* 1,
$\mu_1^* = (r-\lambda)/\{k(r-1)\}$	*w.m.* $s(k-1)$,
$\mu_2^* = (r-\lambda)/(rk)$	*w.m.* $v-sk-1$,
$\mu_3^* = (r-k-\lambda)/\{k(r-1)\}$	*w.m.* $s-1$,
$\mu_4^* = \{r(r-\lambda)-(r-s\lambda)k\}/\{rk(r-1)\}$	*w.m.* 1.

It follows from Lemma 4.2 that the design d^* is a PEB design having at most four efficiency classes. Thus the following can be obtained.

THEOREM 4.2. *In a* BIB(v,b,r,k,λ) *design, if s disjoint blocks are lost for $1 \le s < v/k$, the residual design is a* PEB *design having at most four efficiency* *classes with parameters*

$$v^* = v, \qquad b^* = b - s, \qquad \mathbf{r}^* = [(r-1)\mathbf{1}'_{sk}, r\mathbf{1}'_{v-sk}]', \qquad k^* = k,$$

$$\mu_0^* = 1, \qquad \mu_1^* = \frac{r-\lambda}{k(r-1)}, \qquad \mu_2^* = \frac{r-\lambda}{rk}, \qquad \mu_3^* = \frac{r-k-\lambda}{k(r-1)},$$

$$\mu_4^* = \frac{r(r-\lambda)-(r-s\lambda)k}{rk(r-1)},$$

$$\rho_0^* = 1, \qquad \rho_1^* = s(k-1), \qquad \rho_2^* = v - sk - 1, \qquad \rho_3^* = s - 1, \qquad \rho_4^* = 1,$$

$$L_0^* = \frac{1}{vr - sk}\mathbf{1}_v\mathbf{r}^{*\prime}, \qquad L_1^* = \mathrm{diag}\left\{I_s \otimes \left(I_k - \frac{1}{k}J_k\right), O_{v-sk}\right\},$$

$$L_2^* = \mathrm{diag}\left\{O_{sk}, I_{v-sk} - \frac{1}{v-sk}J_{v-sk}\right\},$$

$$L_3^* = \mathrm{diag}\left\{\left(I_s - \frac{1}{s}J_s\right) \otimes \left(\frac{1}{k}J_k\right), O_{v-sk}\right\},$$

$$L_4^* = I_v - L_0^* - L_1^* - L_2^* - L_3^*.$$

Proof. It is easily shown that the idempotent matrices corresponding to the eigenvalues μ_i^* of M^*, $i = 0, \dots, 4$, are given by $L_0^* = (vr - sk)^{-1}\mathbf{1}_v\mathbf{r}^{*\prime}$, $L_1^* = \mathrm{diag}\{I_s \otimes (I_k - k^{-1}J_k), O_{v-sk}\}$, $L_2^* = \mathrm{diag}\{O_{sk}, I_{v-sk} - (v-sk)^{-1}J_{v-sk}\}$, $L_3^* = \mathrm{diag}\{(I_s - s^{-1}J_s) \otimes k^{-1}J_k, O_{v-sk}\}$, $L_4^* = I_v - L_0^* - L_1^* - L_2^* - L_3^*$, respectively, which are mutually orthogonal such that $\sum_{i=0}^4 L_i^* = I_{v^*}$. This completes the proof. □

When $s = v/k$, the M-matrix of the residual design d^* can be given by removing the second partitioned row (submatrix) and column (submatrix) of the original M^* of the residual design in (4.1). Then by Lemma 1.1, we can obtain the following.

Lemma 4.3. *The v eigenvalues of M^* for $s = v/k$ are given by*

$\mu_0^* = 1$	*w.m.* 1,
$\mu_1^* = (r-\lambda)/\{k(r-1)\}$	*w.m.* $v - s$,
$\mu_2^* = (r-k-\lambda)/\{k(r-1)\}$	*w.m.* $s - 1$.

It follows from Lemma 4.3 that the design d^* is a PEB design having two efficiency classes. Thus we can obtain the following.

Theorem 4.3. *In a* BIB(v, b, r, k, λ) *design, if $s = v/k$ disjoint blocks are lost, the residual design is a* PEB *design having two efficiency classes with* *parameters*

$$v^* = v, \quad b^* = b - s, \quad r^* = r - 1, \quad k^* = k,$$

$$\mu_0^* = 1, \quad \mu_1^* = \frac{r-\lambda}{k(r-1)}, \quad \mu_2^* = \frac{r-k-\lambda}{k(r-1)}, \quad \rho_0^* = 1, \quad \rho_1^* = v - s, \quad \rho_2^* = s - 1,$$

$$L_0^* = v^{-1}J_v, \quad L_1^* = I_s \otimes (I_k - k^{-1}J_k), \quad L_2^* = (I_s - s^{-1}J_s) \otimes k^{-1}J_k.$$

REMARK 4.1. The design given in Theorem 4.3 is actually a group divisible (GD) design with parameters $v^* = v$, $b^* = b - v/k$, $r^* = r - 1$, $k^* = k$, $\lambda_1^* = \lambda - 1$, $\lambda_2^* = \lambda$; $m^* = v/k$, $n^* = k$.

(iii) Augmented BIB design with a missing block

Consider an augmented BIB design d with parameters $v_0 = v + 1$, $b_0 = b$, $\mathbf{k}_0 = (k+1)\mathbf{1}_b$ and $\mathbf{r}_0 = (r\mathbf{1}_v{}', b)'$. Let d^* be a design by deleting one block in the augmented BIB design d. Without loss of generality, suppose that all observations in the missing block occur in the first k and the last positions respectively. It is easily shown that the M-matrix of the residual design d^* is given by

$$M^* = \frac{1}{k+1} \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & 1 \end{bmatrix}$$

with

$$M_{11} = \frac{r-\lambda}{r-1} I_k + \frac{\lambda - 1}{r-1} J_k, \qquad M_{12} = \frac{\lambda}{r-1} J_{k \times (v-k)}, \qquad M_{13} = J_{k \times 1},$$

$$M_{21} = \frac{\lambda}{r} J_{(v-k)\times k}, \qquad M_{22} = \frac{r-\lambda}{r} I_{v-k} + \frac{\lambda}{r} J_{v-k}, \qquad M_{23} = J_{(v-k)\times 1},$$

$$M_{31} = \frac{r-1}{b-1} J_{1\times k}, \qquad M_{32} = \frac{r}{b-1} J_{1\times(v-k)}.$$

Hence we can obtain the following through Lemma 1.1.

LEMMA 4.4. *The $v+1$ eigenvalues of M^* are given by*

$$\begin{array}{ll} \mu_0^* = 1 & w.m.\ 1, \\ \mu_1^* = (r-\lambda)/\{(r-1)(k+1)\} & w.m.\ k-1, \\ \mu_2^* = (r-\lambda)/\{r(k+1)\} & w.m.\ v-k-1, \\ \mu_3^* = \{rk(r-1) - (vr-k)\lambda\}/\{r(r-1)(k+1)\} & w.m.\ 1, \\ \mu_4^* = 0 & w.m.\ 1. \end{array}$$

It follows from Lemma 4.4 that the design d^* is a PEB design having at most four efficiency classes. Hence we can obtain the following.

THEOREM 4.4. *In an augmented* BIB *design with parameters* $v_0 = v + 1$, $b_0 = b$, $\mathbf{k}_0 = (k+1)\mathbf{1}_b$ *and* $\mathbf{r}_0 = (r\mathbf{1}_v', b)'$, *if one block is lost, the residual design is a* PEB *design having four efficiency classes with parameters*

$$v^* = v+1, \qquad b^* = b-1, \qquad \mathbf{r}^* = [(r-1)\mathbf{1}_k', r\mathbf{1}_{v-k}', b-1]', \qquad k^* = k+1,$$

$$\mu_0^* = 1, \qquad \mu_1^* = \frac{r-\lambda}{(r-1)(k+1)}, \qquad \mu_2^* = \frac{r-\lambda}{r(k+1)},$$

$$\mu_3^* = \frac{rk(r-1) - (vr-k)\lambda}{r(r-1)(k+1)}, \qquad \mu_4^* = 0,$$

$$\rho_0^* = 1, \qquad \rho_1^* = k-1, \qquad \rho_2^* = v-k-1, \qquad \rho_3^* = 1, \qquad \rho_4^* = 1,$$

$$L_0^* = \frac{1}{(b-1)(k+1)}\mathbf{1}_{v^*}\mathbf{r}^{*\prime}, \qquad L_1^* = \operatorname{diag}\left\{I_k - \frac{1}{k}J_k, O_{v-k}, 0\right\},$$

$$L_2^* = \operatorname{diag}\left\{O_k, I_{v-k} - \frac{1}{v-k}J_{v-k}, 0\right\}, \qquad L_3^* = \operatorname{diag}\{L, 0\},$$

$$L_4^* = I_{v^*} - L_0^* - L_1^* - L_2^* - L_3^*,$$

where

$$L = \frac{1}{vr-k}\begin{bmatrix} \{(v-k)r/k\}J_k & -rJ_{k\times(v-k)} \\ -(r-1)J_{(v-k)\times k} & \{(r-1)k/(v-k)\}J_{v-k} \end{bmatrix}.$$

PROOF. It is easily shown that the idempotent matrices corresponding to the eigenvalues μ_i^* of M^*, $i = 0, \dots, 4$, are given by $L_0^* = \{(b-1)(k+1)\}^{-1}\mathbf{1}_{v^*}\mathbf{r}^{*\prime}$, $L_1^* = \operatorname{diag}\{I_k - k^{-1}J_k, O_{v-k}, 0\}$, $L_2^* = \operatorname{diag}\{O_k, I_{v-k} - (v-k)^{-1}J_{v-k}, 0\}$, $L_3^* = \operatorname{diag}\{L, 0\}$, $L_4^* = I_{v^*} - L_0^* - L_1^* - L_2^* - L_3^*$, respectively, which are mutually orthogonal such that $\sum_{i=0}^4 L_i^* = I_{v^*}$. This completes the proof. □

If the original augmented BIB design d is derived from a symmetric BIB design, the residual design d^* is a PEB design with three efficiency classes since $\mu_3^* = 0$. Hence we have the following.

COROLLARY 4.2. *In an augmented* BIB *design with parameters* $v_0 = v + 1$, $b_0 = b$, $\mathbf{k}_0 = (k+1)\mathbf{1}_b$ *and* $\mathbf{r}_0 = (r\mathbf{1}_v', b)'$, *where* $v = b$ *and* $r = k$, *if one block is lost, the residual design is a* PEB *design having three efficiency classes with parameters*

$$v^* = v+1, \qquad b^* = b-1, \qquad \mathbf{r}^* = [(r-1)\mathbf{1}_k', r\mathbf{1}_{v-k}', b-1]', \qquad k^* = k+1,$$

$$\mu_0^* = 1, \qquad \mu_1^* = \frac{r-\lambda}{(r-1)(k+1)}, \qquad \mu_2^* = \frac{r-\lambda}{r(k+1)}, \qquad \mu_3^* = 0,$$

$$\rho_0^* = 1, \qquad \rho_1^* = k-1, \qquad \rho_2^* = v-k-1, \qquad \rho_3^* = 2,$$

$$L_0^* = \frac{1}{(b-1)(k+1)} \mathbf{1}_{v^*} \mathbf{r}^{*\prime}, \qquad L_1^* = \operatorname{diag}\left\{ I_k - \frac{1}{k} J_k, O_{v-k}, 0 \right\},$$

$$L_2^* = \operatorname{diag}\left\{ O_k, I_{v-k} - \frac{1}{v-k} J_{v-k}, 0 \right\}, \qquad L_3^* = I_{v^*} - L_0^* - L_1^* - L_2^*.$$

4.2. Statistical analysis

In this section, we shall present a basic formulae for analyzing the designs constructed in the previous section. Under model (1.1), we further assume that $e_{ijl}, i = 1, \dots, v; j = 1, \dots, b; l = 1, \dots, n_{ij}$, are independently distributed as a normal distribution with $E(e_{ijl}) = 0$ and $V(e_{ijl}) = \sigma^2$. For PEB designs, we can easily obtain the pseudo variance-covariance matrix Ω through idempotent matrices L_i and corresponding eigenvalues μ_i (see (4.0)). Once Ω is known, the estimate of the vector of treatment effects is given by $\Omega\mathbf{Q}$ and the adjusted sum of squares attributed to treatments is $\mathbf{Q}'\Omega\mathbf{Q}$, where $\mathbf{Q}$ is the column vector of adjusted treatment totals (cf. Puri and Nigam (1983), Puri (1984), Puri and Kageyama (1985)).

(i) Analysis for equireplicate balanced designs when one block is lost

Let $\mathbf{Q} = (\mathbf{Q}_1{}', \mathbf{Q}_2{}')'$ be the vector of adjusted treatment totals, where $\mathbf{Q}_1 = (Q_1, \dots, Q_{k_1})'$, and $\mathbf{Q}_2 = (Q_{k_1+1}, \dots, Q_v)'$. Using Theorem 4.1, we get

$$\begin{aligned}
\mathbf{Q}'\Omega\mathbf{Q} = &\left\{ \sum_{i=1}^{k_1} Q_i^2 - \frac{1}{k_1}\left(\sum_{i=1}^{k_1} Q_i \right)^2 \right\} \Big/ \{(r-1)(1-\mu_1^*)\} \\
&+ \left\{ \sum_{i=k_1+1}^{v} Q_i^2 - \frac{1}{v-k_1}\left(\sum_{i=k_1+1}^{v} Q_i \right)^2 \right\} \Big/ \{r(1-\mu_2^*)\} \\
&+ \left\{ \frac{1}{k(r-1)}\left(\sum_{i=1}^{k_1} Q_i \right)^2 + \frac{1}{r(v-k_1)}\left(\sum_{i=k_1+1}^{v} Q_i \right)^2 \right\} \Big/ (1-\mu_3^*).
\end{aligned}$$

The following original data relate to an experiment on wheat crop using a BIB design with parameters $v = 9$, $b = 12$, $r = 4$, $k = 3$, $\lambda = 1$. The data used here are from Dey (1986) on page 98. The layout plan and yield figures are tabulated in Table 4.1.

Now suppose we are interested in testing a hypothesis involving the treatment effects. In comparative design of experiments, the interest is in comparing the various treatment effects and a hypothesis of common interest

Table 4.1

Block No.	Treatments and yield figures	Block totals (B_j)
1	(1) 77 (2) 65 (3) 65	217
2	(6) 54 (4) 60 (5) 65	179
3	(7) 47 (9) 61 (8) 60	168
4	(1) 70 (7) 62 (4) 62	194
5	(8) 72 (5) 55 (2) 55	182
6	(3) 50 (6) 40 (9) 60	150
7	(1) 63 (8) 67 (6) 54	184
8	(4) 62 (2) 53 (9) 57	172
9	(3) 68 (5) 67 (7) 66	201
10	(1) 69 (9) 62 (5) 52	183
11	(2) 61 (6) 63 (7) 79	203
12	(8) 65 (3) 65 (4) 38	168

is

$$H_0 : \tau_1 = \tau_2 = \cdots = \tau_v$$

against the alternative H_1 : at least one pair of τ_i's is different.

We reproduce the following tables from Dey (1986), which are useful for computing adjusted sum of squares attributed to treatments.

Table 4.2

Treatment No.	Treatment total T_i	Block Nos. in which the treatment i occurs	$\sum_{j(i)} B_j$	$kQ_i = kT_i - \sum_{j(i)} B_j$
1	279	1, 4, 7, 10	778	59
2	234	1, 5, 8, 11	774	−72
3	258	1, 6, 9, 12	736	38
4	222	2, 4, 8, 12	713	−47
5	239	2, 5, 9, 10	745	−28
6	211	2, 6, 7, 11	716	−83
7	254	3, 4, 9, 11	766	−4
8	264	3, 5, 7, 12	702	90
9	240	3, 6, 8, 10	673	47
Total	2201		6603	0

It follows from Table 4.2 that

$S_t = \mathbf{Q}'\Omega\mathbf{Q} = 1123, \qquad S_b = \sum_{j=1}^{b} B_j^2/k_j - G^2/(vr) = 1226,$

$S_T = \sum_{ijk} y_{ijk}^2 - G^2/(vr) = 2884, \qquad S_e = S_T - S_t - S_b = 535,$

which can yield the following analysis of variance table (Table 4.3).

Here we have values of $F = 4.2$ and $F_{8,16}(\alpha = 0.05) = 2.59$. Thus the hypothesis of equality of treatment effects is rejected at 5%.

Table 4.3

Source	d.f.	S.S.	M.S.	F
Treatment	8	1123	140.3	4.2
Block	11	1226		
Error	16	535	33.4	
Total	35	2884		

Table 4.4

Block No.	Treatments and yield figures	Block totals (B_j)
1	(1) * (2) * (3) *	*
2	(6) 54 (4) 60 (5) 65	179
3	(7) 47 (9) 61 (8) 60	168
4	(1) 70 (7) 62 (4) 62	194
5	(8) 72 (5) 55 (2) 55	182
6	(3) 50 (6) 40 (9) 60	150
7	(1) 63 (8) 67 (6) 54	184
8	(4) 62 (2) 53 (9) 57	172
9	(3) 68 (5) 67 (7) 66	201
10	(1) 69 (9) 62 (5) 52	183
11	(2) 61 (6) 63 (7) 79	203
12	(8) 65 (3) 65 (4) 38	168

When some disjoint blocks are lost in the design, the residual design becomes a PEB design. In the case, we can also take this hypothesis: $\tau_1 = \tau_2 = \cdots = \tau_v$. The following examples are utilized to illustrate how to treat this testing hypothesis.

EXAMPLE 1. When one block, for example, the first block, is lost, the data are tabulated in Table 4.4. Here "*" denotes that data in this position are lost.

We prepare the following auxiliary table (Table 4.5), which is useful for computing adjusted sum of squares attributed to treatments.

It follows from Table 4.5 that

$S_t = \mathbf{Q}'\Omega\mathbf{Q} = 1045.93, \qquad S_b = \sum_{j=2}^{b} B_j^2/k_j - G^2/(vr - k_1) = 815.52,$

$S_T = \sum_{ijk} y_{ijk}^2 - G^2/(vr - k_1) = 2391.51, \qquad S_e = S_T - S_t - S_b = 530.05,$

which can yield the following analysis of variance table (Table 4.6).

Here we have values of $F = 3.45$ and $F_{8,14}(\alpha = 0.05) = 2.70$. Thus the hypothesis of equality of treatment effects is rejected at 5%.

Table 4.5

Treatment No.	Treatment total T_i	Block Nos. in which the treatment i occurs	$\sum_{j(i)} B_j$	$kQ_i = kT_i - \sum_{j(i)} B_j$
1	202	*, 4, 7, 10	561	45
2	169	*, 5, 8, 11	557	−50
3	183	*, 6, 9, 12	519	30
4	222	2, 4, 8, 12	713	−47
5	239	2, 5, 9, 10	745	−28
6	211	2, 6, 7, 11	716	−83
7	254	3, 4, 9, 11	766	−4
8	264	3, 5, 7, 12	702	90
9	240	3, 6, 8, 10	673	47
Total	1984		5952	0

Table 4.6

Source	d.f.	S.S.	M.S.	F
Treatment	8	1045.93	130.74	3.45
Block	10	815.52		
Error	14	530.05	37.86	
Total	32	2391.51		

(ii) Analysis for BIB designs when s disjoint blocks are lost

Let $\mathbf{Q} = (\mathbf{Q}_1', \mathbf{Q}_2')'$ be the vector of adjusted treatment totals, where $\mathbf{Q}_1 = (Q_1, \ldots, Q_{sk})'$, and $\mathbf{Q}_2 = (Q_{sk+1}, \ldots, Q_v)'$. Using Theorem 4.2, we get

$$\begin{aligned}
\mathbf{Q}'\Omega\mathbf{Q} = {} & \left[\sum_{i=1}^{sk} Q_i^2 - \frac{1}{k}\left\{\left(\sum_{i=1}^{k} Q_i\right)^2 + \cdots + \left(\sum_{i=(s-1)k+1}^{sk} Q_i\right)^2\right\}\right] \Big/ \{(r-1)(1-\mu_1^*)\} \\
& + \left\{\sum_{i=sk+1}^{v} Q_i^2 - \frac{1}{v-sk}\left(\sum_{i=sk+1}^{v} Q_i\right)^2\right\} \Big/ \{r(1-\mu_2^*)\} \\
& + \frac{1}{k}\left\{\left(\sum_{i=1}^{k} Q_i\right)^2 + \cdots + \left(\sum_{i=(s-1)k+1}^{sk} Q_i\right)^2 - \frac{1}{s}\left(\sum_{i=1}^{sk} Q_i\right)^2\right\} \Big/ \{(r-1)(1-\mu_3^*)\} \\
& + \left\{\frac{1}{r(v-sk)}\left(\sum_{i=sk+1}^{v} Q_i\right)^2 + \frac{1}{sk(r-1)}\left(\sum_{i=1}^{sk} Q_i\right)^2\right\} \Big/ (1-\mu_4^*).
\end{aligned}$$

EXAMPLE 2. When two disjoint blocks, for example, the first two blocks, are lost in a BIB design with the data in Table 4.1, the resulting data are

Table 4.7

Block No.	Treatments and yield figures	Block totals (B_j)
1	(1) * (2) * (3) *	*
2	(6) * (4) * (5) *	*
3	(7) 47 (9) 61 (8) 60	168
4	(1) 70 (7) 62 (4) 62	194
5	(8) 72 (5) 55 (2) 55	182
6	(3) 50 (6) 40 (9) 60	150
7	(1) 63 (8) 67 (6) 54	184
8	(4) 62 (2) 53 (9) 57	172
9	(3) 68 (5) 67 (7) 66	201
10	(1) 69 (9) 62 (5) 52	183
11	(2) 61 (6) 63 (7) 79	203
12	(8) 65 (3) 65 (4) 38	168

Table 4.8

Treatment No.	Treatment total T_i	Block Nos. in which the treatment i occurs	$\sum_{j(i)} B_j$	$kQ_i = kT_i - \sum_{j(i)} B_j$
1	202	*, 4, 7, 10	561	45
2	169	*, 5, 8, 11	557	−50
3	183	*, 6, 9, 12	519	30
4	162	*, 4, 8, 12	713	−48
5	174	*, 5, 9, 10	745	−44
6	157	*, 6, 7, 11	716	−66
7	254	3, 4, 9, 11	766	−4
8	264	3, 5, 7, 12	702	90
9	240	3, 6, 8, 10	673	47
Total	1805		5415	0

tabulated in Table 4.7. Here "*" denotes that data in this position are lost.

We also prepare the following table (Table 4.8), which is useful for computing adjusted sum of squares attributed to treatments.

It follows from Table 4.8 that

$S_t = \mathbf{Q}'\Omega\mathbf{Q} = 1039.79, \qquad S_b = \sum_{j=3}^{b} B_j^2/k - G^2/(vr-2k) = 814.83,$

$S_T = \sum_{ijk} y_{ijk}^2 - G^2/(vr-2k) = 2330.17, \qquad S_e = S_T - S_t - S_b = 476.55,$

which yield the following analysis of variance table (Table 4.9).

Here we have values of $F = 3.27$ and $F_{8,12}(\alpha = 0.05) = 2.85$. Thus the hypothesis of equality of treatment effects is rejected at 5%.

Table 4.9

Source	d.f.	S.S.	M.S.	F
Treatment	8	1039.79	129.97	3.27
Block	9	814.83		
Error	12	476.55	39.71	
Total	29	2330.17		

(iii) Analysis for augmented BIB designs when one block is lost

Let $\mathbf{Q} = (\mathbf{Q}_1', \mathbf{Q}_2', Q_{v+1})'$ be the vector of adjusted treatment totals, where $\mathbf{Q}_1 = (Q_1, \ldots, Q_k)'$ and $\mathbf{Q}_2 = (Q_{k+1}, \ldots, Q_v)'$. Using Theorem 4.4, we get

$$\begin{aligned}
\mathbf{Q}'\Omega\mathbf{Q} = &\left\{\sum_{i=1}^{k} Q_i^2 - \frac{1}{k}\left(\sum_{i=1}^{k} Q_i\right)^2\right\}\Big/\{(r-1)(1-\mu_1^*)\} \\
&+\left\{\sum_{i=k+1}^{v} Q_i^2 - \frac{1}{v-k}\left(\sum_{i=k+1}^{v} Q_i\right)^2\right\}\Big/\{r(1-\mu_2^*)\} \\
&+\frac{1}{vr-k}\left\{\frac{r(v-k)}{k(r-1)}\left(\sum_{i=1}^{k} Q_i\right)^2 + \frac{k(r-1)}{r(v-k)}\left(\sum_{i=k+1}^{v} Q_i\right)^2\right. \\
&\left. - 2\sum_{i=1}^{k} Q_i \sum_{i=k+1}^{v} Q_i\right\}\Big/(1-\mu_3^*) \\
&+\frac{1}{vr-k}\left(\sum_{i=1}^{v} Q_i\right)^2 + \frac{k}{vr-k} Q_{v+1}^2.
\end{aligned}$$

Now suppose we are interested in testing a hypothesis involving the test and the control treatment effects. In augmented BIB designs, the common interest is in comparing the various test treatment effects and the control treatment effect, and a hypothesis of common interest is

$$H_0 : \tau_1 = \tau_2 = \cdots = \tau_v + \tau_{v+1}$$

against the alternative H_1: at least one pair of τ_i and τ_{v+1} is different for $i = 1, \ldots, v$.

Example 3. In an augmented BIB design with parameters $v_0 = 10$, $b_0 = 12$, $k_0 = 4$ and $\mathbf{r}_0 = (4\mathbf{1}_9', 12)'$, which is derived from the BIB(9, 12, 4, 3, 1) design as used before. When one block, for example, the first block, is lost, the data are tabulated in Table 4.10 (e.g. compare with Table 4.1). The data corresponding to the test treatments 1 to 9 are from Dey (1986), while the data corresponding to the control treatment 10 is artificially given by

Table 4.10

Block No.	Treatments and yield figures	Block totals (B_j)
1	(1) * (2) * (3) * (10) *	*
2	(6) 54 (4) 60 (5) 65 (10) 60	239
3	(7) 47 (9) 61 (8) 60 (10) 56	224
4	(1) 70 (7) 62 (4) 62 (10) 65	259
5	(8) 72 (5) 55 (2) 55 (10) 61	243
6	(3) 50 (6) 40 (9) 60 (10) 50	200
7	(1) 63 (8) 67 (6) 54 (10) 61	245
8	(4) 62 (2) 53 (9) 57 (10) 57	229
9	(3) 68 (5) 67 (7) 66 (10) 67	268
10	(1) 69 (9) 62 (5) 52 (10) 61	244
11	(2) 61 (6) 63 (7) 79 (10) 68	271
12	(8) 65 (3) 65 (4) 38 (10) 56	224

Table 4.11

Treatment No.	Treatment total T_i	Block Nos. in which the treatment i occurs	$\sum_{j(i)} B_j$	$k_0 Q_i = k_0 T_i - \sum_{j(i)} B_j$
1	202	*, 4, 7, 10	748	60
2	169	*, 5, 8, 11	743	−67
3	183	*, 6, 9, 12	692	40
4	222	2, 4, 8, 12	951	−63
5	239	2, 5, 9, 10	994	−38
6	211	2, 6, 7, 11	955	−111
7	254	3, 4, 9, 11	1022	−6
8	264	3, 5, 7, 12	936	120
9	240	3, 6, 8, 10	897	63
10	662	2, 3, ..., 12	2646	2
Total	2646		10584	0

rounding up an average of other data in the same block. Here "*" denotes that data in this position are lost.

We prepare the following auxiliary table (Table 4.11), which is useful for computing adjusted sum of squares attributed to treatments.

It follows from Table 4.11 that

$S_t = \mathbf{Q}'\Omega\mathbf{Q} = 908.56,\qquad S_b = \sum_{j=2}^{b} B_j^2/k_0 - G^2/(vr + b - k - 1) = 1096.68,$
$S_T = \sum_{ijk} y_{ijk}^2 - G^2/(vr + b - k - 1) = 2673.18,$
$S_e = S_T - S_t - S_b = 667.94,$

which can yield the following analysis of variance table (Table 4.12).

Here we have values of $F = 3.63$ and $F_{9,24}(\alpha = 0.05) = 2.30$. Thus the hypothesis of equality of treatment effects is rejected at 5%.

Table 4.12

Source	d.f.	S.S.	M.S.	F
Treatment	9	908.56	100.95	3.63
Block	10	1096.68		
Error	24	667.94	27.83	
Total	43	2673.18		

5. Constructions of equireplicate, proper PEB designs

The previous section also shows the usefulness of PEB designs for our present set-up. Such designs have been constructed in abundance as mentioned before. In this section, we provide further several construction methods for PEB designs.

A block design with parameters $v = 2k, b, r, k$ is said to be self-complementary. An Hadamard matrix H of order n is a matrix with elements $+1$'s and -1's such that $HH' = nI_n$. In this case, it is well known that $n = 2$ or $n \equiv 0 \pmod 4$ (see Bush (1979)). Let A denote an $(n-t) \times n$ matrix obtained by deleting any t rows of the matrix H for $0 \leq t \leq n-2$. Note that $t = 0$ implies no deletion of rows in matrix H. Let $\bar{N} = J_{v\times b} - N$ for the incidence matrix N.

Theorem 5.1. *The existence of an Hadamard matrix of order n and an equireplicate, proper self-complementary* PEB *design N with parameters $v = 2k, b, r, k, \mu_i, \rho_i, L_i$, $i = 1, \ldots, m$, implies the existence of a self-complementary* PEB *design N^*, having at most $m+1$ efficiency classes, with parameters $v^* = (n-t)v$, $b^* = nb$, $r^* = nr$, $k^* = (n-t)k$, $\mu_i^* = \mu_i/(n-t)$, $\mu_{m+1}^* = 0$, $\rho_i^* = (n-t)\rho_i$, $\rho_{m+1}^* = n-t-1$, $L_i^* = I_{n-t} \otimes L_i$, $i = 1, 2, \ldots, m$, $L_{m+1}^* = [I_{n-t} - \{1/(n-t)\}J_{n-t}] \otimes (1/v)J_v$:*

$$N^* = \frac{1}{2}\{(J_{(n-t)\times n} + A) \otimes N + (J_{(n-t)\times n} - A) \otimes \bar{N}\}.$$

Proof. Note that in N^* the elements $+1$ and -1 in the matrix A are replaced by N and $\bar{N}$, respectively. It follows that the M^*-matrix for the design N^* is given by

$$M^* = \frac{1}{(n-t)v} J_{(n-t)v} + \sum_{i=1}^{m} \frac{\mu_i}{n-t} I_{n-t} \otimes L_i.$$

It is easily shown that the eigenvalues of M^* are given by $\mu_0^* = 1$, $\mu_i^* = \mu_i/(n-t)$, $i = 1, \ldots, m$, and $\mu_{m+1}^* = 0$ with multiplicities $\rho_0^* = 1$, $\rho_i^* = (n-t)\rho_i$, $i = 1, \ldots, m$, and $\rho_{m+1}^* = n-t-1$, respectively. The idempotent matrices corresponding to the eigenvalues μ_i^*, $i=0,\dots,m+1$, are given by $L_0^* = \{(n-t)v\}^{-1}J_{(n-t)v}$, $L_i^* = I_{n-t}\otimes L_i$, $i=1,\dots,m$, $L_{m+1}^* = [I_{n-t}-(n-t)^{-1}J_{n-t}]\otimes v^{-1}J_v$, which are mutually orthogonal such that $\sum_{i=0}^{m+1} L_i^* = I_{(n-t)v}$. This completes the proof. □

REMARK 5.1. Note that if the original design N is a BIB or simple PEB design, the resulting design N^* is also a simple PEB design because of $\mu_{m+1}^* = 0$.

Since a PBIB design is a special case of PEB designs (cf. Puri and Nigam (1977)), if a self-complementary rectangular PBIB design with parameters $v = mn = 2k, b, r, k, \lambda_1, \lambda_2, \lambda_3$ is taken as a design N, we can get the following.

COROLLARY 5.1. *The existence of an Hadamard matrix of order n and a self-complementary rectangular* PBIB *design with parameters $v = mn = 2k, b, r, k, \lambda_1, \lambda_2, \lambda_3$ implies the existence of a self-complementary* PEB *design, having at most four efficiency classes, with parameters* $v^* = (n-t)v$, $b^* = nb$, $r^* = nr$, $k^* = (n-t)k$, $\mu_1^* = \{r-\lambda_1+(m-1)(\lambda_2-\lambda_3)\}/\{rk(n-t)\}$, $\mu_2^* = \{r-\lambda_2+(n-1)(\lambda_1-\lambda_3)\}/\{rk(n-t)\}$, $\mu_3^* = \{r-\lambda_1-\lambda_2+\lambda_3)\}/\{rk(n-t)\}$, $\mu_4^* = 0$, $\rho_1^* = (n-t)(n-1)$, $\rho_2^* = (n-t)(m-1)$, $\rho_3^* = (n-t)(m-1)(n-1)$, $\rho_4^* = n-t-1$, $L_1^* = I_{n-t}\otimes(1/m)J_m\otimes\{I_n-(1/n)J_n\}$, $L_2^* = I_{n-t}\otimes\{I_m-(1/m)J_m\}\otimes(1/n)J_n$, $L_3^* = I_{n-t}\otimes\{I_m-(1/m)J_m\}\otimes\{I_n-(1/n)J_n\}$, $L_4^* = [I_{n-t} - \{1/(n-t)\}J_{n-t}]\otimes(1/v)J_v$.

It is well known that if $\lambda_2 = \lambda_3$ or $\lambda_1 = \lambda_3$, a rectangular PBIB design is reducible to a GD design. Hence if a starting design is a GD design, we can easily present the parameters of the resulting design by letting $\lambda_2 = \lambda_3$ or $\lambda_1 = \lambda_3$ in Corollary 5.1, as the following shows.

COROLLARY 5.1.1. *The existence of an Hadamard matrix of order n and a self-complementary* GD *design with parameters $v = 2k, b, r, k, \lambda_1, \lambda_2; m, n$ implies the existence of a self-complementary* PEB *design, having at most three efficiency classes, with parameters* $v^* = (n-t)v$, $b^* = nb$, $r^* = nr$, $k^* = (n-t)k$, $\mu_i^* = (r-\lambda_1)/\{rk(n-t)\}$, $\mu_2^* = (rk-v\lambda_2)/\{rk(n-t)\}$, $\mu_3^* = 0$, $\rho_1^* = m(n-t)(n-1)$, $\rho_2^* = (n-t)(m-1)$, $\rho_3^* = n-t-1$, $L_1^* = I_{n-t}\otimes I_m\otimes\{I_n-(1/n)J_n\}$, $L_2^* = I_{n-t}\otimes\{I_m-(1/m)J_m\}\otimes(1/n)J_n$, $L_3^* = [I_{n-t}-\{1/(n-t)\}J_{n-t}]\otimes(1/v)J_v$.

REMARK 5.2. The resulting design N^* in Corollary 5.1.1 is a PBIB design based on a nested group divisible (NGD) association scheme (see Duan and Kageama (1993)). It is well known that NGD designs are useful as 3-factor experiments. The parameters λ_1^*, λ_2^* and λ_3^* of N^* are easily derived through the expressions of the eigenvalues of M^*, where $\lambda_1^* = n\lambda_1$, $\lambda_2^* = n\lambda_2$, $\lambda_3^* = nr/2$; $m^* = m$, $n^* = n$, $\rho^* = n-t$. Note that if $r = 2\lambda_1$ or $2\lambda_2$, or $\lambda_1 = \lambda_2$, the design

N^* is reducible to a semi-regular GD design. The related discussions are referred to Duan and Kageyama (1993, 1995a).

THEOREM 5.2. *The existence of an equireplicate, proper self-complementary* PEB *design* N *with parameters* $v = 2k, b, r, k, \mu_i, \rho_i, L_i$, $i = 1, \ldots, m$, *implies the existence of a self-complementary* PEB *design* N^*, *having at most* $m+1$ *efficiency classes, with parameters* $v^* = 2v$, $b^* = 2b$, $r^* = 2r$, $k^* = 2k$, $\mu_i^* = \mu_i$, $\mu_{m+1}^* = 0$, $\rho_i^* = \rho_i$, $\rho_{m+1}^* = v$, $L_i^* = \{I_2 - (1/2)J_2\} \otimes L_i$, $i = 1, \ldots, m$, $L_{m+1}^* = (I_2 - J_2) \otimes (1/v)J_v + (1/2)J_2 \otimes I_v$:

$$N^* = \begin{bmatrix} N & \bar{N} \\ \bar{N} & N \end{bmatrix}.$$

PROOF. It follows that the M^*-matrix for the design N^* is given by

$$M^* = \frac{1}{2v}J_{2v} + \sum_{i=1}^{m} \mu_i\left(I_2 - \frac{1}{2}J_2\right) \otimes L_i.$$

It is easily shown that the eigenvalues of M^* are given by $\mu_0^* = 1$, $\mu_i^* = \mu_i$, $i = 1, \ldots, m$, and $\mu_{m+1}^* = 0$ with multiplicities $\rho_0^* = 1$, $\rho_i^* = \rho_i$, $i = 1, \ldots, m$, and $\rho_{m+1} = v$, respectively. The idempotent matrices corresponding to eigenvalues μ_i^*, $i = 0, \ldots, m+1$, are given by $L_0^* = (2v)^{-1}J_{2v}$, $L_i^* = \{I_2 - 2^{-1}J_2)\} \otimes L_i$, $i = 1, \ldots, m$, $L_{m+1}^* = (I_2 - J_2) \otimes v^{-1}J_v + 2^{-1}J_2 \otimes I_v$, which are mutually orthogonal such that $\sum_{i=0}^{m+1} L_i^* = I_{2v}$. This yields the required result. □

REMARK 5.3. Note that if the original design is a BIB or simple PEB design, the resulting design is also a simple PEB design because of $\mu_{m+1}^* = 0$.

If we take a self-complementary rectangular PBIB design with parameters $v = mn = 2k, b, r, k, \lambda_1, \lambda_2, \lambda_3$ as a starting design, we can obtain the following.

COROLLARY 5.2. *The existence of a self-complementary rectangular* PBIB *design with parameters* $v = mn = 2k, b, r, k, \lambda_1, \lambda_2, \lambda_3$ *implies the existence of a self-complementary* PEB *design, having at most four efficiency classes, with parameters* $v^* = 2v$, $b^* = 2b$, $r^* = 2r$, $k^* = 2k$, $\mu_1^* = \{r - \lambda_1 + (m-1)(\lambda_2 - \lambda_3)\}/(rk)$, $\mu_2^* = \{r - \lambda_2 + (n-1)(\lambda_1 - \lambda_3)\}/(rk)$, $\mu_3^* = (r - \lambda_1 - \lambda_2 + \lambda_3)/(rk)$, $\mu_4^* = 0$, $\rho_1^* = (n-1)$, $\rho_2^* = (m-1)$, $\rho_3^* = (m-1)(n-1)$, $\rho_4^* = v$, $L_1^* = \{I_2 - (1/2)J_2\} \otimes (1/m)J_m \otimes \{I_n - (1/n)J_n\}$, $L_2^* = \{I_2 - (1/2)J_2\} \otimes \{I_m - (1/m)J_m\} \otimes (1/n)J_n$, $L_3^* = \{I_2 - (1/2)J_2\} \otimes \{I_m - (1/m)J_m\} \otimes \{I_n - (1/n)J_n\}$, $L_4^* = (I_2 - J_2) \otimes (1/v)J_v + (1/2)J_2 \otimes I_v$.

It is well known that a rectangular PBIB design is reducible to a BIB design when $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$. Hence if the starting design is a BIB design, the resulting design is simple PEB which is actually a rectangular PBIB design with parmeters $\lambda_1^* = 2\lambda$, $\lambda_2^* = 0$, $\lambda_3^* = 2(r - \lambda)$; $m = 2$, $n = v$.

THEOREM 5.3. *The existence of an equireplicate, proper self-complementary* PEB *design* N *with parameters* $v = 2k, b, r, k, \mu_i, \rho_i, L_i$, $i = 1, \dots, m$, *implies the existence of a* PEB *design* N^*, *having at most* $m+1$ *efficiency classes, with parameters* $v^* = 4v$, $b^* = 4b$, $r^* = 3r$, $k^* = 3k$, $\mu_i^* = \mu_i/3$, $\mu_{m+1}^* = 1/9$, $\rho_i^* = 4\rho_i$, $\rho_{m+1}^* = 3$, $L_i^* = I_4 \otimes L_i$, $i = 1, \dots, m$, $L_{m+1}^* = \{I_4 - (1/4)J_4\} \otimes (1/v)J_v$:

$$N^* = \begin{bmatrix} N & N & N & O_{v\times b} \\ N & \bar{N} & O_{v\times b} & N \\ N & O_{v\times b} & \bar{N} & \bar{N} \\ O_{v\times b} & N & \bar{N} & N \end{bmatrix}.$$

PROOF. It follows that the M^*-matrix for the design N^* is given by

$$\begin{aligned} M^* &= \frac{1}{4v} J_{4v} + \sum_{i=1}^{m} \frac{\mu_i}{3} I_4 \otimes L_i + \frac{1}{9}\left(I_4 - \frac{1}{4} J_4\right) \otimes \left(\frac{1}{v} J_v\right) \\ &= L_0^* + \sum_{i=1}^{m} \mu_i^* L_i^* + \mu_{m+1}^* L_{m+1}^* \end{aligned}$$

which yields the required result. □

If we take a self-complementary rectangular PBIB design with parameters $v = mn = 2k, b, r, k, \lambda_1, \lambda_2, \lambda_3$ as a starting design, we can get the following.

COROLLARY 5.3. *The existence of a self-complementary rectangular* PBIB *design with parameters* $v = mn = 2k, b, r, k, \lambda_1, \lambda_2, \lambda_3$ *implies the existence of a* PEB *design, having at most four efficiency classes, with parameters* $v^* = 4v$, $b^* = 4b$, $r^* = 3r$, $k^* = 3k$, $\mu_1^* = \{r - \lambda_1 + (m-1)(\lambda_2 - \lambda_3)\}/(3rk)$, $\mu_2^* = \{r - \lambda_2 + (n-1)(\lambda_1 - \lambda_3)\}/(3rk)$, $\mu_3^* = (r - \lambda_1 - \lambda_2 + \lambda_3)/(3rk)$, $\mu_4^* = 1/9$, $\rho_1^* = 4(n-1)$, $\rho_2^* = 4(m-1)$, $\rho_3^* = 4(m-1)(n-1)$, $\rho_4^* = 3$, $L_1^* = I_4 \otimes (1/m)J_m \otimes \{I_n - (1/n)J_n\}$, $L_2^* = I_4 \otimes \{I_m - (1/m)J_m\} \otimes (1/n)J_n$, $L_3^* = I_4 \otimes \{I_m - (1/m)J_m\} \otimes \{I_n - (1/n)J_n\}$, $L_4^* = \{I_4 - (1/4)J_4\} \otimes (1/v)J_v$.

As an application of Corollary 5.3, we have the following.

COROLLARY 5.3.1. *The existence of a self-complementary* GD *design with parameters* $v = 2k, b, r, k, \lambda_1, \lambda_2$; m, n *implies the existence of a* PEB *design, having at most three efficiency classes, with parameters* $v^* = 4v$, $b^* = 4b$, $r^* = 3r$, $k^* = 3k$, $\mu_1^* = (r - \lambda_1)/(3rk)$, $\mu_2^* = (rk - v\lambda_2)/(3rk)$, $\mu_3^* = 1/9$, $\rho_1^* = 4m(n-1)$, $\rho_2^* = 4(m-1)$, $\rho_3^* = 3$, $L_1^* = I_4 \otimes I_m \otimes \{I_n - (1/n)J_n\}$, $L_2^* = I_4 \otimes \{I_m - (1/m)J_m\} \otimes (1/n)J_n$, $L_3^* = \{I_4 - (1/4)J_4\} \otimes (1/v)J_v$.

REMARK 5.4. The resulting design in Corollary 5.3.1 is also regarded as an NGD design with parameters $\lambda_1^* = 3\lambda_1$, $\lambda_2^* = 3\lambda_2$, $\lambda_3^* = r$; $m^* = m$, $n^* = n$

and $p^* = 4$. Note that if $r = 3\lambda_1$ or $3\lambda_2$, or $\lambda_1 = \lambda_2$, the design is reducible to a regular GD design.

THEOREM 5.4. *The existence of an equireplicate, proper* PEB *design* N *with parameters* $v, b, r, k, \mu_i, \rho_i, L_i$, $i = 1, \ldots, m$, *implies the existence of a* PEB *design* N^*, *having at most* $m+1$ *efficiency classes, with parameters* $v^* = tv$, $b^* = tb$, $r^* = r + (t-1)b$, $k^* = k + (t-1)v$, $\mu_i^* = rk\mu_i/[\{r + (t-1)b\}\{k + (t-1)v\}]$, $\mu_{m+1}^* = (vb + rk - 2vr)/[\{r + (t-1)b\}\{k + (t-1)v\}]$, $\rho_i^* = t\rho_i$, $\rho_{m+1}^* = t - 1$, $L_i^* = I_t \otimes L_i$, $i = 1, \ldots, m$, $L_{m+1}^* = \{I_t - (1/t)J_t\} \otimes (1/v)J_v$ *for* $t \geq 2$:

$$N^* = I_t \otimes N + (J_t - I_t) \otimes J_{v\times b}.$$

PROOF. It follows that the M^*-matrix for the design N^* is given by

$$\begin{aligned} M^* =& \frac{1}{tv}J_{tv} + \sum_{i=1}^{m} \frac{rk\mu_i}{\{r + (t-1)b\}\{k + (t-1)v\}} I_t \otimes L_i \\ &+ \frac{vb + rk - 2vr}{\{r + (t-1)b\}\{k + (t-1)v\}} \left(I_t - \frac{1}{t}J_t\right) \otimes \left(\frac{1}{v}J_v\right) \\ =& L_0^* + \sum_{i=1}^{m} \mu_i^* L_i^* + \mu_{m+1}^* L_{m+1}^* \end{aligned}$$

which yields the required result. □

If we take a rectangular PBIB design with parameters $v = mn, b, r, k, \lambda_1, \lambda_2, \lambda_3$ as a starting design, we can obtain the following.

COROLLARY 5.4. *The existence of a rectangular* PBIB *design with parameters* $v = mn, b, r, k, \lambda_1, \lambda_2, \lambda_3$ *implies the existence of a* PEB *design, having at most four efficiency classes, with parameters* $v^* = tv$, $b^* = tb$, $r^* = r + (t-1)b$, $k^* = k + (t-1)v$, $\mu_1^* = rk\{r - \lambda_1 + (m-1)(\lambda_2 - \lambda_3)\}/[\{r + (t-1)b\}\{k + (t-1)v\}]$, $\mu_2^* = rk\{r - \lambda_2 + (n-1)(\lambda_1 - \lambda_3)\}/[\{r + (t-1)b\}\{k + (t-1)v\}]$, $\mu_3^* = rk(r - \lambda_1 - \lambda_2 + \lambda_3)/[\{r + (t-1)b\}\{k + (t-1)v\}]$, $\mu_4^* = (vb + rk - 2vr)/[\{r + (t-1)b\}\{k + (t-1)v\}]$, $\rho_1^* = t(n-1)$, $\rho_2^* = t(m-1)$, $\rho_3^* = t(m-1)(n-1)$, $\rho_4^* = t - 1$, $L_1^* = I_t \otimes (1/m)J_m \otimes \{I_n - (1/n)J_n\}$, $L_2^* = I_t \otimes \{I_m - (1/m)J_m\} \otimes (1/n)J_n$, $L_3^* = I_t \otimes \{I_m - (1/m)J_m\} \otimes \{I_n - (1/n)J_n\}$, $L_4^* = \{I_t - (1/t)J_t\} \otimes (1/v)J_v$ *for* $t \geq 2$.

It is well known that a rectangular PBIB design is reducible to a Latin square design, if $m = n$ and $\lambda_1 = \lambda_2$. Hence if a starting design is a PBIB design based on a Latin square association scheme, we can present the parameters of the resulting design by letting $m = n = s$ and $\lambda_1 = \lambda_2$ in the result above. As an application of Corollary 5.4, the following can be obtained.

Corollary 5.4.1. *The existence of a Latin square* PBIB *design with parameters* $v = s^2, b, r, k, \lambda_1, \lambda_2$ *implies the existence of a* PEB *design, having at most three efficiency classes, with parameters* $v^* = tv$, $b^* = tb$, $r^* = r + (t-1)b$, $k^* = k + (t-1)v$, $\mu_1^* = \{r - 2\lambda_1 + \lambda_2 + s(\lambda_1 - \lambda_2)\}/[\{r + (t-1)b\}\{k + (t-1)v\}]$, $\mu_2^* = (r - 2\lambda_1 + \lambda_2)/[\{r + (t-1)b\}\{k + (t-1)v\}]$, $\mu_3^* = (vb + rk - 2vr)/[\{r + (t-1)b\}\{k + (t-1)v\}]$, $\rho_1^* = 2t(s-1)$, $\rho_2^* = t(s-1)^2$, $\rho_3^* = t-1$, $L_1^* = I_t \otimes [\{I_s - (1/s)J_s\} \otimes (1/s)J_s + (1/s)J_s \otimes \{I_s - (1/s)J_s\}]$, $L_2^* = I_t \otimes \{I_s - (1/s)J_s\} \otimes \{I_s - (1/s)J_s\}$, $L_3^* = \{I_t - (1/t)J_t\} \otimes (1/v)J_v$ *for* $t \geq 2$.

Corollary 5.4.2. *The existence of a* GD *design with parameters* $v = mn, b, r, k, \lambda_1, \lambda_2$; m, n *implies the existence of a* PEB *design, having at most three efficiency classes, with parameters* $v^* = tv$, $b^* = tb$, $r^* = r + (t-1)b$, $k^* = k + (t-1)v$, $\mu_1^* = rk(r - \lambda_1)/[\{r + (t-1)b\}\{k + (t-1)v\}]$, $\mu_2^* = rk(rk - v\lambda_2)/[\{r + (t-1)b\}\{k + (t-1)v\}]$, $\mu_3^* = (vb + rk - 2vr)/[\{r + (t-1)b\}\{k + (t-1)v\}]$, $\rho_1^* = tm(n-1)$, $\rho_2^* = t(m-1)$, $\rho_3^* = t-1$, $L_1^* = I_t \otimes I_m \otimes \{I_n - (1/n)J_n\}$, $L_2^* = I_t \otimes \{I_m - (1/m)J_m\} \otimes (1/n)J_n$, $L_3^* = \{I_t - (1/t)J_t\} \otimes (1/v)J_v$ *for* $t \geq 2$.

Remark 5.5. The resulting design in Corollary 5.4.2 yields an NGD design with parameters $\lambda_1^* = \lambda_1 + (t-1)b$, $\lambda_2^* = \lambda_2 + (t-1)b$, $\lambda_3^* = 2r + (t-2)b$; $m^* = m$, $n^* = n$ and $p^* = t$. This result is also reported as Theorem 2.2 in Bhagwandas et al. (1992). Note that if $\lambda_1 = \lambda_2$ or $b = 2r - \lambda_1$, or $2r - \lambda_2$, the design is reducible to a regular GD design.

Another pattern will be presented to constuct more PEB designs. Though the pattern can be generalized, a case with small values of design parameters is chosen from a statistical point of view.

Theorem 5.5. *The existence of an equireplicate, proper* PEB *design* N *with parameters* $v, b, r, k, \mu_i, \rho_i, L_i$, $i = 1, \ldots, m$, *implies the existence of a* PEB *design* N^*, *having at most* $m+1$ *efficiency classes, with parameters* $v^* = 3v$, $b^* = 3b$, $r^* = r + b$, $k^* = k + v$, $\mu_i^* = rk\mu_i/\{(r+b)(k+v)\}$, $\mu_{m+1}^* = (vb + rk - vr)/\{(r+b)(k+v)\}$, $\rho_i^* = 3\rho i$, $\rho_{m+1}^* = 2$, $L_i^* = I_3 \otimes L_i$, $i = 1, \ldots, m$, $L_{m+1}^* = \{I_3 - (1/3)J_3\} \otimes (1/v)J_v$:

$$N^* = \begin{bmatrix} N & O_{v\times b} & J_{v\times b} \\ O_{v\times b} & J_{v\times b} & N \\ J_{v\times b} & N & O_{v\times b} \end{bmatrix}.$$

Proof. It follows that the M^*-matrix for the design N^* is given by

$$M^* = \frac{1}{3v} J_{3v} + \sum_{i=1}^{m} \frac{rk\mu_i}{(r+b)(k+v)} I_3 \otimes L_i$$

$$+\frac{vb+rk-vr}{(r+b)(k+v)}\left(I_3-\frac{1}{3}J_3\right)\otimes\left(\frac{1}{v}J_v\right)$$

$$=L_0^*+\sum_{i=1}^{m}\mu_i^*L_i^*+\mu_{m+1}^*L_{m+1}^*,$$

which yields the required result. □

If we take a rectangular PBIB design with parameters $v=mn, b, r, k, \lambda_1, \lambda_2, \lambda_3$ as a starting design, we have the following.

COROLLARY 5.5. *The existence of a rectangular* PBIB *design with parameters $v=mn, b, r, k, \lambda_1, \lambda_2, \lambda_3$ implies the existence of a* PEB *design, having at most four efficiency classes, with parameters $v^*=3v$, $b^*=3b$, $r^*=r+b$, $k^*=k+v$, $\mu_1^*=rk\{r-\lambda_1+(m-1)(\lambda_2-\lambda_3)\}/\{(r+b)(k+v)\}$, $\mu_2^*=rk\{r-\lambda_2+(n-1)(\lambda_1-\lambda_3)\}/\{(r+b)(k+v)\}$, $\mu_3^*=rk(r-\lambda_1-\lambda_2+\lambda_3)/\{(r+b)(k+v)\}$, $\mu_4^*=(vb+rk-vr)/\{(r+b)(k+v)\}$, $\rho_1^*=3(n-1)$, $\rho_2^*=3(m-1)$, $\rho_3^*=3(m-1)(n-1)$, $\rho_4^*=2$, $L_1^*=I_3\otimes(1/m)J_m\otimes\{I_n-(1/n)J_n\}$, $L_2^*=I_3\otimes\{I_m-(1/m)J_m\}\otimes(1/n)J_n$, $L_3^*=I_3\otimes\{I_m-(1/m)J_m\}\otimes\{I_n-(1/n)J_n\}$, $L_4^*=\{I_3-(1/3)J_3\}\otimes(1/v)J_v$.*

We also have some applications of Corollary 5.5.

COROLLARY 5.5.1. *The existence of a Latin square* PBIB *design with parameters $v=s^2, b, r, k, \lambda_1, \lambda_2$, implies the existence of a* PEB *design, having at most three efficiency classes, with parameters $v^*=3v$, $b^*=3b$, $r^*=r+b$, $k^*=k+v$, $\mu_1^*=\{r-2\lambda_1+\lambda_2+s(\lambda_1-\lambda_2)\}/\{(r+b)(k+v)\}$, $\mu_2^*=(r-2\lambda_1+\lambda_2)/\{(r+b)(k+v)\}$, $\mu_3^*=(vb+rk-vr)/\{(r+b)(k+v)\}$, $\rho_1^*=6(s-1)$, $\rho_2^*=3(s-1)^2$, $\rho_3^*=2$, $L_1^*=I_3\otimes[\{I_s-(1/s)J_s\}\otimes(1/s)J_s+(1/s)J_s\otimes\{I_s-(1/s)J_s\}]$, $L_2^*=I_3\otimes\{I_s-(1/s)J_s\}\otimes\{I_s-(1/s)J_s\}$, $L_3^*=\{I_3-(1/3)J_3\}\otimes(1/v)J_v$.*

COROLLARY 5.5.2. *The existence of a* GD *design with parameters $v, b, r, k, \lambda_1, \lambda_2; m, n$* implies the existence of a PEB *design with parameters $v^*=3v$, $b^*=3b$, $r^*=r+b$, $k^*=k+v$, $\mu_1^*=(r-\lambda_1)/\{(r+b)(k+v)\}$, $\mu_2^*=(rk-v\lambda_2)/\{(r+b)(k+v)\}$, $\mu_3^*=(vb+rk-vr)/\{(r+b)(k+v)\}$, $\rho_1^*=3m(n-1)$, $\rho_2^*=3(m-1)$, $\rho_3^*=2$, $L_1^*=I_3\otimes I_m\otimes\{I_n-(1/n)J_n\}$, $L_2^*=I_3\otimes\{I_m-(1/m)J_m\}\otimes(1/n)J_n$, $L_3^*=\{I_3-(1/3)J_3\}\otimes(1/v)J_v$.*

REMARK 5.6. The resulting design in Corollary 5.5.2 yields an NGD design with parameters $\lambda_1^*=\lambda_1+b$, $\lambda_2^*=\lambda_2+b$, $\lambda_3^*=r$; $m^*=m$, $n^*=n$ and $p^*=3$. Note that if $\lambda_1=\lambda_2$, the design is reducible to a regular GD design.

Since many families of PBIB designs based on other known association schemes are available in the literature, these PBIB designs can be utilized to construct more PEB designs through the methods mentioned in this section. The analysis of variance for theses designs can be easily made.

Acknowledgements

The author would like to express his sincere thank to Professor Sanpei Kageyama of Hiroshima University for his guidance and valuable comments and suggestions. The author also wishes to thank Professor Yasunori Fujikoshi of Hiroshima University for his helpful comments and encouragement.

References

[1] J. K. Baksalary and Z. Tabis, Conditions for the robustness of block designs against the unavailability of data, J. Statist. Plann. Inference **16** (1987), 49–54.

[2] K. Sinha Bhagwandas and S. Kageyama, Constructions of PBIB designs based on nested group divisible association scheme, Utilitas Math. **41** (1992), 169–174.

[3] D. K. Bhaumik and D. C. Whittinghill, Optimality and robustness to the unavailability of blocks in block designs, J. R. Statist. Soc. B **53(2)** (1991), 399–407.

[4] R. C. Bose and K. R. Nair, Partially incomplete block designs, Sankhyā B **4** (1939), 337–372.

[5] R. C. Bose and T. Shimamoto, Classification and analysis of partially incomplete block designs with two associate classes, J. Am. Stat. Assn. **47** (1952), 151–184.

[6] K. A. Bush, Families of Hadamard group divisible designs, J. Statist. Plan. Inference **3** (1979), 387–394.

[7] T. Caliński, On some desirable patterns in block designs (with discussions), Biometrics **27** (1971), 275–292.

[8] T. Caliński and S. Kageyama, Block designs: their combinatorial and statistical properties, Handbook of Statistics **14** (1996), to appear.

[9] M. L. Chandak On the theory of resistant block designs, Calcutta Statist. Assoc. Bull. **29** (1980), 27–34.

[10] W. S. J. Connor, On the structure of balanced incomplete block designs, Ann. Math. Statist. **23** (1952), 57–71.

[11] A. Das and S. Kageyama, Robustness of BIB and extended BIB designs, Computational Statistics & Data Analysis **14** (1992), 343–358.

[12] A. Dey, Theory of Bolck Designs, Wiley Eastern Limited, 1986.

[13] A. Dey, Robustness of block designs against missing data, Statistica Sincia **3** (1993), 219–231.

[14] A. Dey, M. Singh and G. M. Saha, Efficiency balanced block designs, Commun. Statist.-Theor. Meth. **A10** (1981), 237–247.

[15] A. Dey and S. P. Dhall, Robustness of augmented BIB designs, Sankhyā B **50** (1988), 376–381.

[16] X. Duan and S. Kageyama, Constructions of nested group divisible designs, Statist. & Prob. Lett. **18** (1993), 41–48.

[17] X. Duan and S. Kageyama, Constructions of group divisible designs, J. Japan Statist. Soc. Vol. **25(2)** (1995a), 121–128.

[18] X. Duan and S. Kageyama, Robustness of augmented BIB designs against unavailability of some observations, Sankhyā B Vol. **57(3)** (1995b), 405–419.

[19] X. Duan and S. Kageyama, Robustness of variance balanced designs against the unavailability of some observations, Hiroshima Mathematical Journal Vol. **26(2)** (1996), 351–362.

[20] R. A. Fisher, Statistical Methods for Research Workers, Oliver and Boya, Edinburgh, 1925.

[21] R. A. Fisher, The arrangement of field experiment, J. Ministry Agric. **23** (1926), 503–513.

[22] S. Ghosh, On robustness of design against the unavailability of data, Sankhyā B **40** (1979), 204–208.

[23] S. Ghosh, Robustness of BIBD against the unavailability of data, J. Statist. Plann. Inference **6** (1982a), 29–32.

[24] S. Ghosh, Robustness of designs against the unavailability of data, Sankhyā B **44** (1982b), 50–62.

[25] S. Ghosh, Information in an observation in robust designs, Commun. Statist.-Theor. Meth. **11** (1982c), 1173–1184.

[26] S. Ghosh, S. B. Rao and N. M. Singhi, On a robustness property of PBIB, J. Statist. Plann. Inference **8** (1983), 355–363.

[27] S. Ghosh, S. Kageyama and R. Mukerjee, Efficiency of connected binary block designs when a single observation is unavailable, Ann. Inst. Statist. Math. **44** (1992), 593–603.

[28] S. C. Gupta and B. Jones, Equireplicate balanced block designs with unequal block sizes, Biometrika **70** (1983), 433–439.

[29] S. Gupta and S. Kageyama, Variance balanced designs with unequal block sizes and unequal replications, Utilitas Math. **42** (1992), 15–24.

[30] V. K. Gupta and R. Srivastava, Investigations of robustness of block designs against missing observations, Sankhyā B **54** (1992), 100–105.

[31] M. Hall Jr., Combinatorial Theory, 2nd ed. (Wiley, New York), 1986.

[32] A. Hedayat and P. W. M. John, Resistant and susceptible BIB designs, Ann. Statist. **2** (1974), 148–158.

[33] B. Jones, K. Sinha and S. Kageyama, Furthere equireplicate variance balanced designs with unequal block sizes, Utilitas Math. **32** (1987), 5–10.

[34] S. Kageyama, Constructions of balanced block designs, Utilitas Math. **9** (1976), 209–229.

[35] S. Kageyama, Robustness of connected balanced block designs, Ann. Inst. Statist. Math. A **32** (1980), 255–261.

[36] S. Kageyama, Connected designs with the minimum number of experimental units, Lecture Notes in Statistics 35 (1984), 99–117, Springer-Verlag, New York.

[37] S. Kageyama, Robustness of 3-associate PBIB designs of two types, J. Japan Statist. Soc. **16** (1986), 21–24.

[38] S. Kageyama, Some characterization of locally resistant BIB designs of degree one, Ann. Inst. Statist. Math. A **39** (1987), 661–669.

[39] S. Kageyama, Robustness of block designs, Proceedings of the R. C. Bose Symposium on Probability, Statistics and Designs of Experiments, 425–438, Wiley Eastern, New Delhi, 1990.

[40] S. Kageyama, Robustness of some balanced block designs, Discrete Math. **116** (1993), 159–181.

[41] S. Kageyama and T. Tsuji, Some bounds on balanced block designs, J. Statist. Plann. Inference **4** (1980), 155–167.

[42] S. Kageyama and P. D. Puri, Properties of partially efficiency-balanced designs, Bull. Inform. Cyber. (formerly Bull. Math. Statist.) **21** No. 3–4 (1983), 19–28.

[43] S. Kageyama and G. M. Saha, D^{-1}-partially efficiency-balanced designs with at most two efficiency classes, Commun. Statist.-Theor. Meth. **17(5)** (1988), 1669–1683.

[44] B. M. Most, Resistance of balanced incomplete block designs, Ann. Statist. **3** (1975), 1149–1162.

[45] R. Mukerjee and S. Kageyama, Robustness of group divisible designs, Commun. Statist.-Theor. Meth. **19(9)** (1990), 3189–3203.

[46] A. K. Nigam and P. D. Puri, Partially efficiency-balanced designs II, Commun. Statist.-Theor. Meth. **A11** (1982), 2817–2830.

[47] P. D. Puri, Patterns and analysis of BIB and GD designs having a missing block, Sankhyā **B46** (1984), 44–53.

[48] P. D. Puri and S. Kageyama, Construction of partially efficiency-balanced designs and their analysis, Commun. Statist.-Theor. Meth. **14(6)** (1985), 1315–1342.

[49] P. D. Puri and S. Kageyama, Some structural properties of partially efficiency balanced block designs, Sankhyā **B50** (1988), 200–205.

[50] P. D. Puri and A. K. Nigam, On patterns of efficiency-balanced designs, J. Roy. Statist. Soc. **B37** (1975), 457–458.

[51] P. D. Puri and A. K. Nigam, Balanced factorial experiments, Commun. Statist.-Theor. Meth. **A5** (1976), 599–619.

[52] P. D. Puri and A. K. Nigam, Partially efficiency-balanced designs, Commun. Statist.-Theor. Meth. **A6** (1977), 753–771.

[53] P. D. Puri and A. K. Nigam, Balanced factorial experiments II, Commun. Statist.-Theor. Meth. **A7** (1978), 591–605.

[54] P. D. Puri, A. K. Nigam and P. Narain, Supplemented block designs, Sankhyā **B39** (1977), 189–195.

[55] P. D. Puri and A. K. Nigam, Merging of treatments in block designs, Sankhyā **B45** (1983), 50–59.

[56] D. Raghavarao, Constructions and Combinatorial Problems in Design of experiments, John Wiley, New York, 1971.

[57] S. M. Shah and C. C. Gujarathi, On a locally resistant BIB design of degree one, Sankhyā **B39** (1977), 46–48.

[58] S. M. Shah and C. C. Gujarathi, Resistant BIB designs, Sankhyā **B45** (1983), 225–232.

[59] R. Srivastava, V. K. Gupta and A. Dey, Robustness of some designs against missing observations, Commun. Statist.-Theor. Meth. **19(1)** (1990), 121–126.

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