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On products of some β-elements in the homotopy of the mod 3 Moore spectrum

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Abstract. By β-elements, we mean the v_2-periodic maps on the sphere spectrum S^0 or on the mod 3 Moore spectrum M. For the prime number $p > 3$, we can tell many examples of non-trivial products of β-elements, since $\pi_*(L_2S^0)$ is determined in [23], where L_2 denotes the Bousfield-Ravenel localization functor. On the other hand we have no idea about $\pi_*(L_2S^0)$ at the prime 3, and so the situation is different from the case $p > 3$. Here we study products related to β-elements in the homotopy groups $\pi_*(M)$ of the mod 3 Moore spectrum M, using our results [1] on $H^1M_1^1$ which relates to the E_2-term of the Adams-Novikov spectral sequence for computing $\pi_*(M)$.

1. Introduction

Let BP denote the Brown-Peterson spectrum at a prime number p. Then the coefficient ring BP_* of the BP_*-homology theory is the polynomial algebra $\boldsymbol{Z}_{(p)}[v_1, v_2, \dots]$. The Morava K-theory $K(n)_*(-)$ is characterized by the coefficient ring $K(n)_* = \boldsymbol{Z}/p[v_n, v_n^{-1}]$. A spectrum X is of type n if $K(n)_*(X) \neq 0$ and $K(n-1)_*(X) = 0$, and a self-map $f : X \to X$ of a type n spectrum X is said to be a v_n-map if $K(n)_*(f) \neq 0$. By the name β-element, we mean an element of the homotopy groups of the mod p Moore spectrum M or the sphere spectrum defined by using a v_2-map on a type 2 spectrum V. For $p > 3$, we take V to be Toda-Smith spectrum $V(1)$ and v_2-map is β with $K(2)_*(\beta) = v_2$ constructed by [24] (*cf.* [25]). The homotopy β-elements are given by [5], [6], [7], [8], [9], [15], [26]. The non-triviality of β-elements itself is shown by Miller, Ravenel and Wilson [4]. The non-triviality of products of β-elements at the prime > 3 is studied in [2], [3] and [22] for the mod p Moore spectrum and [10], [11], [16], [17], [18], [19], [20] for the sphere spectrum. In [23], the homotopy groups $\pi_*(L_2S^0)$ are determined and render the non-triviality of the products of two β-elements for $p > 3$.

At the prime 3, S. Pemmaraju [12] shows the existence of $\beta_s \in \pi_{4(3s+s-1)-2}(S^0)$ for $s \equiv 0, 1, 2, 5, 6 \bmod 9$, while $\beta_s \in \pi_{(sp+s-1)q-2}(S^0)$ at the prime $p > 3$ exists for any $s > 0$. In this paper, we assume his results

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including that $\beta'_s \in \pi_{4(3s+s-1)-1}(M)$ also exists if $s \equiv 0,1,2,5,6 \bmod 9$. Here M denotes the mod 3 Moore spectrum. Note that the second author shows that β_s does not exist in $\pi_*(S^0)$ if $s \equiv 4,7,8 \bmod 9$, and further shows the existence of $\beta_t \in \pi_*(L_2S^0)$ and $\beta'_t \in \pi_*(L_2M)$ if $t \equiv 0,1,5 \bmod 9$ [21], where L_2 denotes the Bousfield-Ravenel localization functor with respect to $v_2^{-1}BP$. As is seen in his book [14], Ravenel shows the existence of another type of β-elements $\beta_{6/3} \in \pi_{82}(S^0)$. This result also indicates the existence of another β-element $\beta'_{6/3} \in \pi_{83}(M)$. As v_1-periodic maps, we have the α-elements. The α-elements α_s's are all seen to be non-trivial elements of $\pi_{4s-1}(S^0)$ by the existence of the Adams map $\alpha : \Sigma^4 M \to M$ such that $BP_*(\alpha) = v_1$. In this paper we show the following theorems under the assumption of Pemmariaju's results. The first one is on the products with α-elements:

THEOREM A. *In the homotopy groups $\pi_*(M)$ of the* mod 3 *Moore spectrum M,*

$$\alpha_s\beta'_t \neq 0 \qquad \textit{if } 3 \nmid st.$$

We next consider products of β'-elements:

THEOREM B. *In the homotopy groups $\pi_*(M)$ of the* mod 3 *Moore spectrum M,*

$$\beta'_s\beta'_t \neq 0 \qquad \textit{if } 3 \nmid st,$$

and

$$\beta'_s\beta'_{6/3} \neq 0 \qquad \textit{if } 3 \nmid s.$$

Since β'_t exists unless $t \equiv 3,4,7,8 \bmod 9$, these theorems are restated as:

THEOREM A′. *The element $\alpha_s\beta'_t$ for $s,t > 0$ is essential in $\pi_*(M)$ if $s \not\equiv 0$* mod 3 *and $t \equiv 1,2,5$* mod 9.

THEOREM B′. *The element $\beta'_s\beta'_t$ for $s,t > 0$ is essential in $\pi_*(M)$ if $s,t \equiv 1,2,5$* mod 9. *If $s \equiv 1,2,5$* mod 9, *then $\beta'_s\beta'_{6/3}$ is an essential element of $\pi_*(M)$.*

These follow from Theorem 3.3, which are the 2nd line phenomena. Now turn to the 3rd line phenomena.

THEOREM C. *In the homotopy groups $\pi_*(M)$ of the* mod 3 *Moore spectrum M,*

$$\beta'_s\beta_t \neq 0$$

if $3|(s+t)$ *and* $3\nmid s$, *or if* $3|(s-t)$ *and* $3|(s-1)$, *and*

$$\beta'_s\beta_{6/3} \neq 0 \qquad \text{if } 3\nmid s.$$

This is also restated as:

THEOREM C′. *The element* $\beta'_s\beta_t$ *is essential in* $\pi_*(M)$ *if* $s \equiv 1 \bmod 3$ *and* $t \equiv 1,2,5 \bmod 9$, *or if* $s \equiv 2 \bmod 3$ *and* $t \equiv 1 \bmod 9$. *If* $s \equiv 1,2,5 \bmod 9$, *then* $\beta'_s\beta_{6/3}$ *is an essential element of* $\pi_*(M)$.

This result follows from Theorem D below. In fact, Pemmaraju shows in [12] that β_i in the E_2-term of the Adams-Novikov spectral sequence is a permanent cycle if $i \equiv 0,1,2,5,6^* \bmod 9$. Therefore the result of the E_2-term implies the result of the homotopy, since nothing killes the products in the spectral sequence by degree reason.

THEOREM D. *In the* E_2*-term of the Adams-Novikov spectral sequence for computing* $\pi_*(M)$,

$$\beta'_s\beta_t \neq 0$$

if $3|(s+t)$ *and* $3\nmid s$, *or if* $3|(s-t)$ *and* $3|(s-1)$,

$$\beta'_s\beta_{3t/3} \neq 0$$

if $3\nmid st$, *and*

$$\beta'_s\beta_{3^nt/a_n} \neq 0$$

if $3|(m+2)$ *or* $27|(m-8)(m+1)$, *where* $s = 3^{k-1}m - 3^{n-1}(3t-1)$ *for* $k > 0$ *with* $3\nmid m$.

Here the integer a_k is $4\cdot 3^{k-1}-1$, and β-elements of the E_2-term are defined in the next section. This follows from the main result of [1] immediately.

2. β-elements in the E_2-term

Let BP denote the Brown-Peterson spectrum at the prime 3 whose homotopy groups $\pi_*(BP) = BP_*$ consist of a polynomial algebra $\boldsymbol{Z}_{(3)}[v_1, v_2, \ldots]$ over the Hazewinkel generators v_i with $|v_i| = 2(3^i-1)$. Then $BP_*(-) = \pi_*(BP\wedge -)$ is a homology theory over the category of spectra. Moreover, the

*There is a conjecture due to Ravenel that β_i exists if and only if $i \equiv 0,1,2,3,5,6 \bmod 9$. The 'only if' part is shown in [21]. In [12, Cor. 1.2], Pemmaraju claimed that the 'if' part is shown, but β_i with $i \equiv 3 \bmod 9$ stays still undetermined.

pair $(BP_*, BP_*(BP))$ is a Hopf algebroid with structure maps $\eta_R = (i \wedge BP)_*$, $\eta_L = (BP \wedge i)_*$ and $\Delta = (BP \wedge i \wedge BP)_*$, where $i : S^0 \to BP$ denotes the unit map of the ring spectrum BP. The unit map also defines the exact couple which yields the Adams-Novikov spectral sequence

$$E_2^s(X) = \mathrm{Ext}^s_{BP_*(BP)}(BP_*, BP_*(X)) \Longrightarrow \pi_*(X \wedge S\boldsymbol{Z}_{(3)})$$

for a spectrum X and the Moore spectrum $S\boldsymbol{Z}_{(3)}$ with $\pi_0(S\boldsymbol{Z}_{(3)}) = \boldsymbol{Z}_{(3)}$. Here the E_2-term is defined as a cohomology of the cobar complex $(\Omega^s(X), d_s) = (\Omega^s_{BP_*(BP)} BP_*(X), d_s)$, which is defined by

$$\Omega^s(X) = BP_*(X) \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} \cdots \otimes_{BP_*} BP_*(BP),$$
$$(s \text{ copies of } BP_*(BP))$$

$$\begin{aligned} d_s(x \otimes \gamma_1 \otimes \cdots \otimes \gamma_s) &= \eta_R(x) \otimes \gamma_1 \otimes \cdots \otimes \gamma_s \\ &\quad + \sum_{k=1}^{s} (-1)^k x \otimes \gamma_1 \otimes \cdots \otimes \gamma_{k-1} \otimes \Delta(\gamma_k) \otimes \gamma_{k+1} \otimes \cdots \otimes \gamma_s \\ &\quad + (-1)^{s+1} x \otimes \gamma_1 \otimes \cdots \otimes \gamma_s \otimes 1, \end{aligned}$$

for $x \in BP_*(X)$ and $\gamma_i \in BP_*(BP)$.

First we define the β-elements in the E_2-terms $E_2^1(M)$ and $E_2^2(S^0)$ at the prime 3 in the same way as those at the prime $p > 3$. Here M denotes the mod 3 Moore spectrum. Recall [4] the elements x_i of $v_2^{-1}BP_*$:

$$x_0 = v_2, \qquad x_1 = v_2^3 - v_1^3 v_2^{-1} v_3, \qquad x_2 = x_1^3 - v_1^8 v_2^7 - v_1^{11} v_2^3 v_3 \qquad \text{and}$$
$$x_n = x_{n-1}^3 + v_1^{a_n - 3} v_2^{3^n - 3^{n-1} + 1} \qquad \text{for } n > 2,$$

for the integer a_n with $a_0 = 1$ and

$$a_n = 4 \cdot 3^{n-1} - 1.$$

Now consider the differential $d_0 = \eta_R - \eta_L : v_2^{-1}BP_* \to v_2^{-1}BP_*(BP)$, and it is shown [4] that

$$(2.1) \qquad \begin{aligned} d_0(x_n) &\equiv v_1 t_1^3 && n = 0, \\ &\equiv v_1^3 v_2^2 (t_1 + v_1(v_2^{-1}(t_2 - t_1^4) - \zeta_2)) && n = 1, \\ &\equiv -v_1^{a_n} v_2^{2 \cdot 3^{n-1}} (t_1 + v_1 \zeta_2^{3^{n-1}}) && n > 1. \end{aligned}$$

Here

(2.2) ([4]) *ζ_2 represents a cocycle $v_2^{-1} t_2 + v_2^{-3}(t_2^3 - t_1^{12}) - v_2^{-4} v_3 t_1^3$, which is homologous to $\zeta_2^{3^i}$ for $i \geq 0$ in $\Omega^1_\Gamma E(2)_*/(3, v_1)$.*

Consider the comodules

$$\begin{aligned} N_0^0 &= BP_*, \\ N_1^0 &= BP_*/(3), \\ N_0^1 &= BP_*/(3^\infty), \\ N_2^0 &= BP_*/(3, v_1), \\ N_1^1 &= BP_*/(3, v_1^\infty), \\ N_0^2 &= BP_*/(3^\infty, v_1^\infty), \end{aligned}$$

and $M_l^k = v_{k+l}^{-1} N_l^k$, whose comodule structures are induced from the right unit η_R. Then we have the short exact sequences

$$\begin{gathered} 0 \longrightarrow N_0^0 \xrightarrow{\subset} M_0^0 \longrightarrow N_0^1 \longrightarrow 0, \\ 0 \longrightarrow N_0^1 \xrightarrow{\subset} M_0^1 \longrightarrow N_0^2 \longrightarrow 0, \qquad \text{and} \\ 0 \longrightarrow N_1^0 \xrightarrow{\subset} M_1^0 \longrightarrow N_1^1 \longrightarrow 0, \end{gathered}$$

with the associated connecting homomorphisms

$$\begin{aligned} \delta &: H^s N_0^1 \to H^{s+1} N_0^0, \\ \delta' &: H^s N_0^2 \to H^{s+1} N_0^1, \\ \delta_1 &: H^s N_1^1 \to H^{s+1} N_1^0. \end{aligned}$$

Here we use the abbreviation

$$H^s L = \operatorname{Ext}^s_{BP_*(BP)}(BP_*, L)$$

for a comodule L. Note that $H^s N_0^0 = E_2^s(S^0)$ and $H^s N_1^0 = E_2^s(M)$. Since we compute

(2.3) $$d_0(v_1^{3^n s}) \equiv 3^{n+1} s v_1^{3^n s - 1} t_1 \qquad \text{mod } (3^{n+2})$$

in $\Omega^1(S^0)$ by the formula $\eta_R(v_1) = v_1 + pt_1$ (*cf.* [14]), we see that

$$v_1^{3^n s}/3^k \in H^0 N_0^1$$

for $n \geq 0$, $s \geq 0$ and $0 < k \leq n+1$. Besides, we see by (2.1) that

$$x_n^s/v_1^j \in H^0 N_1^1 \quad \text{and} \quad x_n^s/3v_1^j \in H^0 N_0^2$$

for $n \geq 0$, $s > 0$ and $0 < j \leq a_n$. Now we can define the α- and β-elements:

$$\alpha_{3^n s/k} = \delta(v_1^{3^n s}/3^k) \in H^1 N_0^0 = E_2^1(S^0).$$
$$\beta'_{3^n s/j} = \delta_1(x_n^s/v_1^j) \in H^1 N_1^0 = E_2^1(M).$$
$$\beta_{3^n s/j} = \delta\delta'(x_n^s/3v_1^j) \in H^2 N_0^0 = E_2^2(S^0).$$

We abbreviate $\alpha_{s/1}$, $\beta_{s/1}$ and $\beta'_{s/1}$ by α_s, β_s and β'_s, respectively. Then the formula (2.3) yields immediately

$$\alpha_{3^n s/k} \equiv 3^{n-k+1} s v_1^{3^n s-1} h_{10} \qquad \mathrm{mod}\ (3^{n-k+2}) \tag{2.4}$$

in $E_2^1(S^0)$, since h_{10} is represented by t_1. Moreover, by definition together with (2.1), a β'-element is expressed by

$$\begin{aligned} \beta'_s &\equiv s v_2^{s-1} h_{11} && \mathrm{mod}\ (3, v_1), \\ \beta'_{3s/j} &\equiv s v_1^{3-j} v_2^{3s-1} h_{10} && \mathrm{mod}\ (3, v_1^{4-j}), \quad \text{and} \\ \beta'_{3^k s/j} &\equiv -s v_1^{a_k - j} v_2^{3^{k-1}(3s-1)} h_{10} && \mathrm{mod}\ (3, v_1^{a_k-j+1}) \quad \text{for } k > 1 \end{aligned} \tag{2.5}$$

in $E_2^1(M)$ by [4, Prop. 5.4] and β-elements are represented by the cocycles as follows (*cf.* [10]):

$$\begin{aligned} \beta_s &\equiv \binom{s}{2} v_2^{s-1} \zeta_2 h_{11} + \binom{s+1}{2} v_2^{s-1} b_0 && \mathrm{mod}\ (3, v_1), \\ \beta_{3s/3} &\equiv s v_2^{3s-3} b_1 && \mathrm{mod}\ (3, v_1), \quad \text{and} \\ \beta_{3^k s/a_k} &\equiv -s v_2^{3^{k-1}(3s-1)} h_{10} \zeta && \mathrm{mod}\ (3, v_1) \quad \text{for } k > 1 \end{aligned} \tag{2.6}$$

in $E_2^2(S^0)$. Here h_{11} and b_i are represented by t_1^3 and $-(t_1^{2\cdot 3^i} \otimes t_1^{3^i} + t_1^{3^i} \otimes t_1^{2\cdot 3^i})$, respectively. Moreover, ζ denotes the homology class which is represented by an element whose leading term is ζ_2.

We end this section with explaining about the homotopy elements $\beta'_t \in \pi_{4(3t+3-1)-1}(L_2 M)$. In [21], the existence is shown of $B_j : S^{16j} \to L_2 V(1)$ for $j \equiv 0, 1, 5$ mod 9 such that $BP_*(B_j) = v_2^j$. Here $V(1)$ denotes the Toda-Smith spectrum, which is a cofiber of the Adams map $\alpha : \Sigma^4 M \to M$. Now define

$$\beta'_j = \pi(B_j) \in \pi_*(L_2 M),$$

where $\pi : V(1) \to \Sigma^5 M$ is the canonical projection. Then the Geometric Boundary Theorem (*cf.* [14]) shows that the β-elements of the E_2-term converge to the same named homotopy elements in the Adams-Novikov spectral sequence.

3. The non-triviality of products in $H^2BP_*/3$

We have the exact sequence

$$H^1M_1^0 \to H^1N_1^1 \xrightarrow{\delta_1} H^2BP_*/3,$$

in which $H^{1,t}M_1^0 = 0$ unless $t = 0$ by [4]. Therefore, δ_1 is monomorphic at $t > 0$. Moreover, (2.4) and (2.5) show the equations:

$$(3.1)\qquad \begin{aligned}\alpha_{3^ns/n+1}\beta'_{3^mt/j} &= \delta_1(x_m^t\alpha_{3^ns/n+1}/v_1^j)\\ &= s\delta_1(v_2^{3^mt}h_{10}/v_1^{j-3^ns+1})\\ \beta'_{3^ns/k}\beta'_{3^mt/j} &= \delta_1(x_m^t\beta'_{3^ns/k}/v_1^j)\\ &= \begin{cases} s\delta_1(v_2^{3^mt+s-1}h_{11}/v_1^j + \cdots) & n=0,\\ s\delta_1(v_2^{3^mt+3s-1}h_{10}/v_1^{j-3+k} + \cdots) & n=1,\\ -s\delta_1(v_2^{3^mt+3^{n-1}(3s-1)}h_{10}/v_1^{j-a_n+k} + \cdots) & n>1.\end{cases}\end{aligned}$$

Recall [1, Th. 6.1] the structure of $H^1M_1^1$:

(3.2) $H^1M_1^1 = A \oplus B$. *Here B is the direct sum of cyclic $k(1)_*$-modules generated by the elements represented by the cocycles whose leading terms are*:

$$v_2^{3^k(3t+1)}h_{10}/v_1^{a(k)},\quad v_2^{3^k(9t-1)}h_{10}/v_1^{a'(k)},\quad v_2^{3t-1}h_{11}/v_1^2 \quad \textit{and} \quad v_2^{3^ku}\zeta/v_1^{a_k}$$

for $k \geq 0$ and $t, u \in \mathbf{Z}$ with $3 \nmid u$. Here $k(1)_* = \mathbf{Z}/3[v_1]$ and the integers $a(k)$, $a'(k)$ and a_k are given by $a(0) = 2$, $a'(0) = 10$, $a_0 = 1$, $a(k) = 6\cdot 3^{k-1}+1$, $a'(k) = 28\cdot 3^{k-1}$ and $a_k = 4\cdot 3^{k-1} - 1$ for $k > 0$.

These facts show the following

THEOREM 3.3. *In the E_2-term of the Adams-Novikov spectral sequence for computing $\pi_*(M)$,*

$$\alpha_{3^ns/n+1}\beta'_{3^mt/j} \neq 0 \quad \textit{if} \quad 3\nmid st \quad \textit{and} \quad 3\nmid t+1 \quad \textit{or} \quad 9|t+1.$$

$$\beta'_s\beta'_{3^mt/j} \neq 0 \quad \textit{if} \quad 3\nmid s \quad \textit{for } m>0, \quad \textit{or if} \quad 3\nmid st \quad \textit{for } m=0.$$

Suppose that $m > 0$. Then,

$$\beta'_{3s/k}\beta'_{3^mt/j} \quad \textit{if } j+k>3,\quad 3\nmid su \quad \textit{and} \quad 3\nmid u+1 \quad \textit{or}$$
$$9|u+1 \quad \textit{for} \quad 3^lu = 3^mt+3s-1.$$

Suppose that $m \geq n$. Then,

$$\beta'_{3^ns/k}\beta'_{3^mt/j} \quad \textit{if } j+k>a_n,\quad 3\nmid su \quad \textit{and} \quad 3\nmid u+1 \quad \textit{or}$$
$$9|u+1 \quad \textit{for} \quad 3^lu = 3^mt+3^{n-1}(3s-1).$$

PROOF. Consider the localization map $\lambda : H^1N_1^1 \to H^1M_1^1$ induced from the canonical localization map $N_1^1 \to M_1^1$. Let x denote the element found in $\delta_1(x)$ on the right hand side of (3.1). If we show that $\lambda(x) \neq 0$, then $x \neq 0$, and so is the product of the left hand side of (3.1). The non-triviality of $\lambda(x)$ follows from (3.2), immediately. q.e.d.

4. The non-triviality of products in $H^3BP_*/3$

Consider the short exact sequence

$$0 \longrightarrow M_2^0 \xrightarrow{\varphi} M_1^1 \xrightarrow{v_1} M_1^1 \longrightarrow 0$$

of comodules, and denote the connecting homomorphism by $\partial : H^sM_1^1 \to H^{s+1}M_2^0$. Here φ is defined by $\varphi(x) = x/v_1$.

LEMMA 4.1. *If* $v_2^s\beta_{t/j}$ *is not in* $\mathrm{Im}\{\partial : H^sM_1^1 \to H^{s+1}M_2^0\}$, *then*

$$\beta_s'\beta_{t/j} \neq 0 \in E_2^3(M).$$

PROOF. Consider the diagram

$$\begin{array}{ccccc} & & H^2M_1^0 \longrightarrow H^2N_1^1 & \xrightarrow{\delta_1} & H^3N_1^0 = E_2^3(M) \\ & & & & \big\downarrow \lambda \\ H^1M_1^1 & \xrightarrow{\partial} & H^2M_2^0 & \xrightarrow{\varphi_*} & H^2M_1^1 \end{array}$$

in which both sequences are exact, and λ denotes the localization map used in the proof of Proposition 3.3. It is shown that $H^2M_1^0 = 0$ in [13] (*cf.* [14]), and so the map δ_1 in the diagram is a monomorphism. Since $H^*N_0^0$ acts on H^*L for a comodule L naturally,

$$\beta_s'\beta_{t/j} = \delta_1(v_2^s/v_1)\beta_{t/j} = \delta_1(v_2^s\beta_{t/j}/v_1).$$

Therefore, the non-triviality of the element $v_2^s\beta_{t/j}/v_1$ implies the desired non-triviality of the product of the β-elements.

Note that $\lambda(v_2^s\beta_{t/j}/v_1) = v_2^s\beta_{t/j}/v_1$ in $H^2M_1^1$. Furthermore, $v_2^s\beta_{t/j}/v_1 = \varphi_*(v_2^s\beta_{t/j})$. Thus, if $v_2^s\beta_{t/j}$ is not in $\mathrm{Im}\,\partial$, then $\varphi_*(v_2^s\beta_{t/j}) \neq 0$ and so $v_2^s\beta_{t/j}/v_1 \neq 0$. q.e.d.

PROOF OF THEOREM D. By the result of [1], we see that Im ∂ is generated by the following elements:

(I) $v_2^{3t}b_0$ $(t \in \boldsymbol{Z})$,

$v_2^{9t-4}b_0 + v_2^{9t-4}h_{11}\zeta$ $(t \in \boldsymbol{Z})$,

(II) $v_2^{3^{n+1}t+i(n)}\xi$ $(n > 0, t \in \boldsymbol{Z})$,

(III) $v_2^{i'(t;n)}h_{10}\zeta$ $(n > 0, t \in \boldsymbol{Z})$,

$v_2^{3^{n-1}(3u-1)}h_{10}\zeta$ $(n > 0, u \in \boldsymbol{Z} - 3\boldsymbol{Z})$,

(IV) $v_2^{3t-3}b_1$ $(t \in \boldsymbol{Z})$,

(V) $v_2^u h_{11}\zeta$ $(u \in \boldsymbol{Z} - 3\boldsymbol{Z})$.

Here, integers $i(n) = \frac{1}{2}(3^n - 1)$ for $n \geq 0$, and $i'(t;0) = 9t - 4$ and $i'(t;n) = 3^{n-1}(9(3t-1)-1)$. Now Theorem D follows from (2.6) and Lemma 4.1. q.e.d.

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