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On some equivalent properties of sub- and supersolutions in second order quasilinear elliptic equations*

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ABSTRACT. The paper is concerned with some equivalent properties of sub- and supersolutions of second order quasilinear elliptic equations. We answer (positively) a question arised from a paper by Kura ([4]). Namely, we show that the concepts of sub-(super)solutions and W-sub-(super)solutions defined in [4] are in fact equivalent.

1. Introduction

This paper is concerned with some properties of weak sub- and supersolutions of quasilinear elliptic equations of second order. Our investigation is motivated by a paper of Kura ([4]), in which were established several interesting results concerning weak sub- and supersolution methods for quasilinear elliptic equations. We study some issues raised in [4] about the equivalence of different concepts of sub- and supersolutions and answer certain questions asked in this paper. To have a more precise perspective of the issues discussed here, we first present the setting of the problem and its related assumptions and definitions. For the sake of convenience and clarity, we follow the notation and assumptions in [4].

Let $\Omega \subset \mathbf{R}^N$ be a region with smooth boundary (for example, $\partial\Omega$ is of class C^1; also, Ω may be unbounded). Let $A : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$, $B : \Omega \times \mathbf{R}^{N+1} \to \mathbf{R}$ be defined as in [4]. In particular, A and B are Carathéodory functions that satisfy the following conditions (in the sequel, p denotes a number in $(1, \infty)$ and $q = p'$ its conjugate exponent):

(H1) $$|A_i(x, \xi)| \leq |f_0(x)| + |c_0(x)|\,|\xi|^{p-1}, \qquad i = 1, \dots, N,$$

for a.e. $x \in \Omega$, $\forall \xi \in \mathbf{R}^N$, where $f_0 \in L^q(\Omega)$, $c_0 \in L^\infty_{\text{loc}}(\mathbf{R}^N)$;

(H2) $$(A(x, \xi) - A(x, \xi')) \cdot (\xi - \xi') > 0,$$

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for a.e. $x \in \Omega$, $\forall \xi, \xi' \in \mathbf{R}^N$, $\xi \neq \xi'$;

(H3) $$A(x,\xi) \cdot \xi \geq \alpha(x)|\xi|^p - |f_1(x)|\,|\xi|^{p-1} - |f_2(x)|,$$

for a.e. $x \in \Omega$, $\forall \xi \in \mathbf{R}^N$, where $\alpha : \mathbf{R}^N \to \mathbf{R}_+$ is a continuous function, and $f_1 \in L^p(\Omega)$, $f_2 \in L^1(\Omega)$.

We refer to [4] for more details about (H1)–(H3). Now, we recall the definitions of sub- and supersolutions and W-sub- and W-supersolutions as presented in [4].

Definition 1 *A function $u \in W^{1,p}(\Omega)$ is called a subsolution (respectively, supersolution) of the equation*

$$-\operatorname{div} A(x, \nabla u), + B(x,u,\nabla u) = 0 \, in \, \Omega \tag{1}$$

if

$$B(x,u,\nabla u) \in L^1_{\text{loc}}(\Omega), \tag{2}$$

and

$$\int_\Omega \{A(x,\nabla u) \cdot \nabla\phi + B(x,u,\nabla u)\phi\} \leq 0 (respectively, \geq 0), \tag{3}$$

for all $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$ in Ω.

Definition 2 *u is a W-subsolution (respectively, W-supersolution) of (1) if $u = \max\{x_i : i = 1,\ldots,m\}$ (respectively, $u = \min\{u_i : i = 1,\ldots,m\}$) for some $m \in \mathbf{N}$, where each u_i is a subsolution (respectively, supersolution) of (1).*

It is clear that if u is a subsolution (respectively, supersolution) or (1), then u is also a W-subsolution (respectively, W-supersolution) of (1). In [4] (page 8), the author raised the question about relationships between sub-(super)solutions and W-sub-(super)solutions in the other direction, namely, in what conditions are W-sub-(super)solutions also sub-(super)solutions. In Proposition 1 of [4], Kura also showed an interesting property that under some conditions on B, bounded W-subsolutions are subsolutions. In this paper, we show (without any additional conditions) that the concepts of sub-(super)solutions and W-sub-(super)solutions given in definitions 1 and 2 are, in fact, equivalent; hence answer the question in [4]. The arguments are given in the next section.

2. Main result

We prove the following result.

THEOREM 1 *u is a W-sub-(super) solution of (1) if and only if u is a sub-(super)solution of (1).*

PROOF. We prove the theorem for bounded domains Ω. The case of unbounded Ω immediately follows. Also, we establish the result for sub-solutions; the proof for supersolutions is similar. Without loss of generality, we can assume $m = 2$ in Definition 2; the general case follows easily by induction.

Assume $u_1, u_2 \in W^{1,p}(\Omega)$ satisfy (2) and (3). We show that

$$u = \max\{u_1, u_2\} \tag{4}$$

also satisfies (2) and (3). It is well-known (cf. e.g. Lemma 7.6, Theorem 7.8 of [2]) that $u \in W^{1,p}(\Omega)$ and

$$\nabla u = \begin{cases} \nabla u_1 & \text{on } \Omega_1 := \{x \in \Omega : u_1 > u_2\} \\ \nabla u_1 = \nabla u_2 & \text{on } \Omega_0 := \{x \in \Omega : u_1 = u_2\} \\ \nabla u_2 & \text{on } \Omega_2 := \{x \in \Omega : u_1 < u_2\}. \end{cases} \tag{5}$$

Hence,

$$B(x, u, \nabla u) = \begin{cases} B(x, u_1, \nabla u_1) & \text{on } \Omega_1 \\ B(x, u_2, \nabla u_2) & \text{on } \Omega \backslash \Omega_1 \end{cases}.$$

Since $B(x, u_i, \nabla u_i) \in L^1_{\text{loc}}(\Omega)$ for $i = 1, 2$, it follows that

$$B(x, u, \nabla u) = B(x, u_1, \nabla u_1)\chi_{\Omega_1} + B(x, u_2, \nabla u_2)\chi_{\Omega\backslash\Omega_1} \in L^1_{\text{loc}}(\Omega).$$

Now, we check that u satisfies (3). Let $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$. First, we define some auxiliary functions. We fix a function γ with the following properties (cf. [1], [3]):

$$\begin{cases} (i) \quad \gamma : \mathbf{R} \to \mathbf{R}, \quad \gamma \in C^\infty(\mathbf{R}) \\ (ii) \quad \gamma \text{ is nondecreasing on } \mathbf{R} \\ (iii) \quad 0 \leq \gamma \leq 1 \\ (iv) \quad \gamma(t) = 1 \text{ if } t \geq 1, \quad \gamma(t) = 0 \text{ if } t \leq 0. \end{cases} \tag{6}$$

We put, for $n \in \mathbf{N}$, $t \in \mathbf{R}$,

$$\gamma_n(t) = \gamma(nt). \tag{7}$$

Thus, γ_n satisfies (i)–(iii) and

$$\gamma_n(t) = 1 \text{ if } t \geq 1/n, \quad \gamma_n(t) = 0 \text{ if } t \leq 0. \tag{8}$$

Let $M = \max\{\gamma'(t) : t \in \mathbf{R}\} = \max\{\gamma'(t) : t \in [0,1]\} < \infty$. We have

$$|\gamma_n'(t)| = |n\gamma'(nt)| \leq Mn, \quad \forall n \in \mathbf{N}, \quad t \in \mathbf{R}. \tag{9}$$

Now, by classical density results for $W^{1,p}(\Omega)(\ni u_2 - u_1)$ (cf. [1]), there exists a sequence $\{\tilde{w}_n\} \subset C_0^\infty(\mathbf{R}^N)$ such that

$$w_n = \tilde{w}_n|_{\bar{\Omega}} \to w := u_2 - u_1 \text{ in } W^{1,p}(\Omega). \tag{10}$$

In particular, $w_n \in C^\infty(\bar{\Omega})$. By choosing a subsequence and relabeling the sequence $\{w_n\}$, if necessary, we can assume from (10) that

$$w_n \to w \quad \text{a.e. in } \Omega, \tag{11}$$

and

$$\|w_n - w\|_{W^{1,p}(\Omega)} \leq \frac{1}{n^2}, \quad \forall n. \tag{12}$$

Let $\phi_1 = (1 - \gamma_n \circ w_n)\phi$ and $\phi_2 = (\gamma_n \circ w_n)\phi$. Since $\gamma_n \circ w_n \in C^\infty(\Omega)$ and $\phi \in C_0^\infty(\Omega)$, both ϕ_1 and ϕ_2 belong to $C_0^\infty(\Omega)$. Since $0 \leq \gamma_n \leq 1, \phi_1, \phi_2 \geq 0$.

Hence, from (3) applied to u_1, ϕ_1 and u_2, ϕ_2, we get (with $j = 1, 2$)

$$\int_\Omega \sum_{i=1}^N A_i(x, \nabla u_j) \frac{\partial \phi_j}{\partial x_i} + B(x, u_j, \nabla u_j)\phi_j \leq 0,$$

i.e.,

$$\begin{aligned} &\int_\Omega \sum_{i=1}^N A_i(x, \nabla u_1)\left\{-\gamma_n'(w_n)\frac{\partial w_n}{\partial x_i}\,\phi + [1 - \gamma_n(w_n)]\frac{\partial \phi}{\partial x_i}\right\} \\ &\qquad + \int_\Omega B(x, u_1, \nabla u_1)[1 - \gamma_n(w_n)]\phi \leq 0, \end{aligned} \tag{13}$$

and

$$\begin{aligned} &\int_\Omega \sum_{i=1}^N A_i(x, \nabla u_2)\left\{\gamma_n'(w_n)\frac{\partial w_n}{\partial x_i}\,\phi + \gamma_n(w_n)\frac{\partial \phi}{\partial x_i}\right\} \\ &\qquad + \int_\Omega B(x, u_2, \nabla u_2)\gamma_n(w_n)\phi \leq 0. \end{aligned} \tag{14}$$

Now, for almost all $x \in \Omega_2$, we have $w(x) > 0$. It follows from (11) that there exists $n_0 = n_0(x)$ such that $w_n(x) > 1/n$, $\forall n \geq n_0$. (Indeed, choose $n_1 = n_1(x)$ such that $w(x) > 2/n_1$. For $n \geq n_1$ sufficiently large, $|w_n(x) - w(x)| < 1/n_1$, and thus, $w_n(x) - w(x) > -1/n_1$. Hence, $w_n(x) > w(x) - 1/n_1 > 1/n_1 \geq 1/n$.) Therefore, by (8), $\gamma_n(w_n(x)) = 1$ for all n large. It follows that

$$\gamma_n(w_n(x)) \to 1 \quad \text{for a.e.} \quad x \in \Omega_2. \tag{15}$$

Similarly, if $x \in \Omega_1$, then $w(x) < 0$ and $w_n(x) < 0$ for all n large. Again, by (8),

$$\gamma_n(w_n(x)) \to 0 \quad \text{for a.e.} \quad x \in \Omega_1. \tag{16}$$

Now, adding (13) and (14), we get

$$\begin{aligned} 0 \geq & \int_\Omega \sum_{i=1}^N [A_i(x, \nabla u_2) - A_i(x, \nabla u_1)] \gamma_n'(w_n) \frac{\partial w_n}{\partial x_i} \phi \\ & + \int_\Omega \sum_{i=1}^N [A_i(x, \nabla u_2) - A_i(x, \nabla u_1)] \gamma_n(w_n) \frac{\partial \phi}{\partial x_i} \\ & + \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \nabla u_1) \frac{\partial \phi}{\partial x_i} + B(x, u_1, \nabla u_1) \phi \right\} \\ & + \int_\Omega [B(x, u_2, \nabla u_2) - B(x, u_1, \nabla u_1)] \gamma_n(w_n) \phi. \end{aligned} \tag{17}$$

Now, since $B(x, u_2, \nabla u_2) = B(x, u_1, \nabla u_1)$ on Ω_0, by (5),

$$\begin{aligned} & \int_\Omega [B(x, u_2, \nabla u_2) - B(x, u_1, \nabla u_1)] \gamma_n(w_n) \phi \\ & = \left(\int_{\Omega_1} + \int_{\Omega_2} \right) [B(x, u_2, \nabla u_2) - B(x, u_1, \nabla u_1)] \gamma_n(w_n) \phi \\ & = \left(\int_{\Omega_1 \cap \operatorname{supp} \phi} + \int_{\Omega_2 \cap \operatorname{supp} \phi} \right) [B(x, u_2, \nabla u_2) - B(x, u_1, \nabla u_1)] \gamma_n(w_n) \phi. \end{aligned}$$

Since $B(x, u_j, \nabla u_j) \in L^1(\Omega_1 \cap \operatorname{supp} \phi) \cap L^1(\Omega_2 \cap \operatorname{supp} \phi), 0 \leq \gamma_n(u_n) \leq 1$, and ϕ is bounded, (15), (16), and the dominated convergence theorem imply that

$$\begin{aligned} & \int_{\Omega_2 \cap \operatorname{supp} \phi} [B(x, u_2, \nabla u_2) - B(x, u_1, \nabla u_1)] \gamma_n(w_n) \phi \\ & \quad \to \int_{\Omega_2 \cap \operatorname{supp} \phi} [B(x, u_2, \nabla u_2) - B(x, u_1, \nabla u_1)] \phi \\ & \quad = \int_{\Omega_2} [B(x, u_2, \nabla u_2) - B(x, u_1, \nabla u_1)] \phi, \end{aligned}$$

and

$$\int_{\Omega_1 \cap \operatorname{supp} \phi} [B(x, u_2, \nabla u_2) - B(x, u_1, \nabla u_1)] \gamma_n(w_n) \phi \to 0.$$

Hence,

$$\int_{\Omega}[B(x,u_2,\nabla u_2)-B(x,u_1,\nabla u_1)]\gamma_n(w_n)\phi \to \int_{\Omega_2}[B(x,u_2,\nabla u_2)-B(x,u_1,\nabla u_1)]\phi. \tag{18}$$

Similarly, since $A_i(\cdot,\nabla u_j)\in L^q(\Omega)$ and $\frac{\partial\phi}{\partial x_i}\in L^\infty(\Omega)$, and $0\le\gamma_n(w_n)\le 1$, one has

$$\begin{aligned}\lim &\int_{\Omega}\sum_{i=1}^{N}[A_i(x,\nabla u_2)-A_i(x,\nabla u_1)]\gamma_n(w_n)\frac{\partial\phi}{\partial x_i}\\ &=\lim\left(\int_{\Omega_1}+\int_{\Omega_2}\right)\sum_{i=1}^{N}[A_i(x,\nabla u_2)-A_i(x,\nabla u_1)]\gamma_n(w_n)\frac{\partial\phi}{\partial x_i}\\ &=\int_{\Omega_2}\sum_{i=1}^{N}[A_i(x,\nabla u_2)-A_i(x,\nabla u_1)]\frac{\partial\phi}{\partial x_i}.\end{aligned} \tag{19}$$

Now, since

$$\sum_{i=1}^{N}[A_i(x,\nabla u_2)-A_i(x,\nabla u_1)]\left(\frac{\partial u_2}{\partial x_i}-\frac{\partial u_1}{\partial x_i}\right)\ge 0,$$

a.e. on Ω by (H2), $\gamma_n'(w_n)\ge 0$, and $\phi\ge 0$, we get

$$\begin{aligned}&\int_{\Omega}\sum_{i=1}^{N}[A_i(x,\nabla u_2)-A_i(x,\nabla u_1)]\gamma_n'(w_n)\frac{\partial w_n}{\partial x_i}\,\phi\\ &=\int_{\Omega}\sum_{i=1}^{N}[A_i(x,\nabla u_2)-A_i(x,\nabla u_1)]\frac{\partial(u_2-u_1)}{\partial x_i}\,\gamma_n'(w_n)\phi\\ &\quad+\int_{\Omega}\sum_{i=1}^{N}[A_i(x,\nabla u_2)-A_i(x,\nabla u_1)]\left(\frac{\partial w_n}{\partial x_i}-\frac{\partial w}{\partial x_i}\right)\gamma_n'(w_n)\phi\\ &\ge-\left|\int_{\Omega}\sum_{i=1}^{N}[A_i(x,\nabla u_2)-A_i(x,\nabla u_1)]\left(\frac{\partial w_n}{\partial x_i}-\frac{\partial w}{\partial x_i}\right)\gamma_n'(w_n)\phi\right|.\end{aligned} \tag{20}$$

Moreover, from (H1) and Hölder's inequality,

$$\begin{aligned}&\left|\int_{\Omega}\sum_{i=1}^{N}[A_i(x,\nabla u_2)-A_i(x,\nabla u_1)]\frac{\partial(w_n-w)}{\partial x_i}\,\gamma_n'(w_n)\phi\right|\\ &\le\sum_{i=1}^{N}\|A_i(x,\nabla u_2)-A_i(x,\nabla u_1)\|_{L^{p'}(\Omega)}\left\|\frac{\partial(w_n-w)}{\partial x_i}\right\|_{L^p(\Omega)}\|\gamma_n'(w_n)\|_{L^\infty(\Omega)}\|\phi\|_{L^\infty(\Omega)}\end{aligned}$$

$$\leq \sum_{i=1}^{N}(\|A_i(x,\nabla u_2)\|_{L^{p'}(\Omega)} + \|A_i(x,\nabla u_1)\|_{L^{p'}(\Omega)})\|w_n$$
$$- w\|_{W^{1,p}(\Omega)}\|\gamma_n'(w_n)\|_{L^\infty(\Omega)}\|\phi\|_{L^\infty(\Omega)}$$
$$\leq \sum_{i=1}^{N}(\| |f_0| + |(c_0(x)| |\nabla u_1|^{p-1}\|_{L^{p'}(\Omega)} + \| |f_0| + |c_0(x)| |\nabla u_2|^{p-1}\|_{L^{p'}(\Omega)})$$
$$\cdot \|w_n - w\|_{W^{1,p}(\Omega)}\|\gamma_n'(w_n)\|_{L^\infty(\Omega)}\|\phi\|_{L^\infty(\Omega)}$$
$$\leq C_1\left\{[\|f_0\|_{L^{p'}(\Omega)} + \|c_0\|_{L^\infty(\Omega)}(\|u_1\|^{p-1}_{W^{1,p}(\Omega)} + \|u_2\|^{p-1}_{W^{1,p}(\Omega)})] \cdot \frac{1}{n^2} \cdot Mn \cdot \|\phi\|_{L^\infty(\Omega)}\right\}$$

(by using (9) and (12))

$$\leq \frac{C_2}{n}, \tag{21}$$

where C_1, C_2 are constants independent of n.

From (20) and (21),

$$\liminf_{n\to\infty} \int_\Omega \sum_{i=1}^{N}[A_i(x,\nabla u_2) - A_i(x,\nabla u_1)]\gamma_n'(w_n)\frac{\partial w_n}{\partial x_i}\phi \geq 0. \tag{22}$$

Letting $n \to \infty$ in (17), and using (18), (19) and (22), we get

$$0 \geq \liminf \int_\Omega \sum_{i=1}^{N}[A_i(x,\nabla u_2) - A_i(x,\nabla u_1)]\gamma_n'(w_n)\frac{\partial w_n}{\partial x_i}\phi$$
$$+ \lim \int_\Omega \sum_{i=1}^{N}[A_i(x,\nabla u_2) - A_i(x,\nabla u_1)]\gamma_n(w_n)\frac{\partial \phi}{\partial x_i}$$
$$+ \lim \int_\Omega [B(x,u_2,\nabla u_2) - B(x,u_1,\nabla u_1)]\gamma_n(w_n)\phi$$
$$+ \int_\Omega \left\{\sum_{i=1}^{N} A_i(x,\nabla u_1)\frac{\partial \phi}{\partial x_i} + B(x,u_1,\nabla u_1)\phi\right\}$$
$$\geq \int_{\Omega_2} \sum_{i=1}^{N}[A_i(x,\nabla u_2) - A_i(x,\nabla u_1)]\frac{\partial \phi}{\partial x_i} + \int_{\Omega_2}[B(x,u_2,\nabla u_2) - B(x,u_1,\nabla u_1)]\phi$$
$$+ \int_\Omega \sum_{i=1}^{N} A_i(x,\nabla u_1)\frac{\partial \phi}{\partial x_i} + \int_\Omega B(x,u_1,\nabla u_1)\phi$$
$$= \int_{\Omega_2} \sum_{i=1}^{N}[A_i(x,\nabla u_2) - A_i(x,\nabla u_1)]\frac{\partial \phi}{\partial x_i} + \left(\int_{\Omega_2} + \int_{\Omega\setminus\Omega_2}\right)\sum_{i=1}^{N} A_i(x,\nabla u_1)\frac{\partial \phi}{\partial x_i}$$
$$+ \int_{\Omega_2}[B(x,u_2,\nabla u_2) - B(x,u_1,\nabla u_1)]\phi + \left(\int_{\Omega_2} + \int_{\Omega\setminus\Omega_2}\right)B(x,u_1,\nabla u_1)\phi$$

$$
\begin{aligned}
&= \int_{\Omega_2} \sum_{i=1}^{N} A_i(x, \nabla u_2) \frac{\partial \phi}{\partial x_i} + \int_{\Omega \setminus \Omega_2} \sum_{i=1}^{N} A_i(x, \nabla u_1) \frac{\partial \phi}{\partial x_i} \\
&\quad + \int_{\Omega_2} B(x, u_2, \nabla u_2)\phi + \int_{\Omega \setminus \Omega_2} B(x, u_1, \nabla u_1)\phi \\
&= \int_{\Omega} \left\{ \sum_{i=1}^{N} A_i(x, \nabla u) \frac{\partial \phi}{\partial x_i} + B(x, u, \nabla u)\phi \right\}.
\end{aligned}
$$

This holds for all $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$. We have checked that (3) is satisfied for $u = \max\{u_1, u_2\}$. Theorem 1 is proved. □

From Theorem 1, we also obtain the following equivalence criterion for L-sub-(super)solutions and C-sub-(super)solutions. For the detailed definitions of L- and C-sub-(super)solutions, we refer the readers to Kura's paper [4].

COROLLARY 1 *$u \in W^{1,p}(\Omega)$ is an L-sub-(super)solution (respectively, C-sub-(super)-solution) of (1) if and only if u is a sub-(super)solution of (1) and moreover,* $u \in L^\infty(\Omega)$ (respectively, $u \in C^{0,1}(\bar{\Omega})$).

To conclude, we note that the above results immediately hold for the case of unbounded domains. They can also be extended to the case where $A_i = A_i(x, u, \nabla u)$ also depend on u, provided certain appropriate assumptions and modifications are considered.

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