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Wald-type tests for two hypotheses concerning parallel mean profiles of several groups

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ABSTRACT. This paper is concerned with profile analysis in two extended growth curve models. The first is a growth curve model with parallel mean profiles, which has a random-effects covariance structure based on a single response variable; the second is a multivariate growth curve model with parallel mean profiles, which has a multivariate random-effects covariance structure based on several response variables. For testing "no condition variation" and "level" hypotheses concerning parallel mean profiles of several groups, we obtain the criteria proposed by Wald [8] along with their asymptotic null distributions. We give a numerical example of these asymptotic results.

1. Introduction

Let X be an $N \times p$ observation matrix of repeated measurements on p occasions for each of N individuals. As an extension of the mean structure in the growth curve model for X proposed by Potthoff and Roy [1], we consider

$$E(X) = A_1 \Xi_1 B + A_2 \Xi_2, \tag{1.1}$$

where A_1 and A_2 are $N \times k_1$ and $N \times k_2$ design matrices, respectively, Ξ_1 and Ξ_2 are unknown $k_1 \times q$ and $k_2 \times p$ parameter matrices, respectively, B is a $q \times p$ design matrix. It may be noted (Verbyla and Venables [7]) that an important application of the mean structure (1.1) arises in analysis of growth curves with parallel profiles.

In this paper we are interested in analyzing growth curves with parallel profiles under random-effects covariance structures. In Section 2 we consider a growth curve model with parallel mean profiles, which has a random-effects covariance structure based on a single response variable. As an alternative of the likelihood ratio ($=$ LR) criteria, Wald's criteria (Wald [8]) for two hypotheses concerning parallel mean profiles are obtained under the random-effects covariance structure. In Section 3 we consider a multivariate growth curve model with parallel mean profiles, which is useful in analyzing multiple-

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response parallel growth curves. The mean structure is a special case of (1.1), but it has a multivariate random-effects covariance structure based on several response variables. The multivariate case of Wald-type tests in Section 2 is discussed under the multivariate random-effects covariance structure. In Section 4 we apply the asymptotic results of Section 2 to a data set of repeated measurements.

2. Analysis of growth curves with parallel profiles

2.1. Growth curve model with parallel mean profiles

Suppose that a response variable x has been measured at p different occasions on each of N individuals, and each individual belongs to one of k groups. Let $\boldsymbol{x}_j{}^{(g)} = (x_{1j}{}^{(g)}, \dots, x_{pj}{}^{(g)})'$ be a p-vector of measurements on the j-th individual in the g-th group, and assume that the $\boldsymbol{x}_j{}^{(g)}$ are independently distributed as $N_p(\boldsymbol{\mu}^{(g)}, \Sigma)$ and the $\boldsymbol{\mu}^{(g)}$ have parallel profiles, i.e., $\boldsymbol{\mu}^{(g)} = \delta^{(g)}\mathbf{1}_p + \boldsymbol{\mu}$, where $\mathbf{1}_p = (1, \dots, 1)'$, $\boldsymbol{\delta} = (\delta^{(1)}, \dots, \delta^{(k-1)})'$ and $\boldsymbol{\mu} = (\mu_1, \dots, \mu_p)'$ are vectors of unknown parameters, Σ is an unknown $p \times p$ positive definite matrix, $j = 1, \dots, N_g$, $g = 1, \dots, k$. Without loss of generality we may assume that $\delta^{(k)} = 0$. Then the model of $X = [\boldsymbol{x}_1^{(1)}, \dots, \boldsymbol{x}_{N_1}{}^{(1)}, \dots, \boldsymbol{x}_1^{(k)}, \dots, \boldsymbol{x}_{N_k}{}^{(k)}]'$ can be written as

$$(2.1) \qquad X \sim N_{N\times p}(A_1\boldsymbol{\delta}\mathbf{1}_p' + \mathbf{1}_N\boldsymbol{\mu}', \Sigma \otimes I_N),$$

where A_1 is an $N \times (k-1)$ between-individual design matrix of rank $k-1$ $(\leq N-p-1)$, $N = N_1 + \cdots + N_k$. The model (2.1) may be simply called a parallel profile model. Further, we assume that Σ in (2.1) has a random-effects covariance structure (see, e.g., Rao [2])

$$(2.2) \qquad \Sigma = \lambda^2\mathbf{1}_p\mathbf{1}_p' + \sigma^2 I_p,$$

where $\lambda^2 \geq 0$ and $\sigma^2 > 0$. Srivastava [5] obtained the LR tests for "no condition variation" hypothesis

$$(2.3) \qquad H_{01} : \boldsymbol{\mu} = \nu\mathbf{1}_p \quad \text{vs.} \quad H_{11} : \boldsymbol{\mu} \neq \nu\mathbf{1}_p$$

and "level" hypothesis

$$(2.4) \qquad H_{02} : \boldsymbol{\delta} = \mathbf{0} \quad \text{vs.} \quad H_{12} : \boldsymbol{\delta} \neq \mathbf{0}$$

when Σ is unknown positive definite, where $-\infty < \nu < \infty$. Yokoyama [11] has obtained the LR criteria for the hypotheses (2.3) and (2.4) under the random-effects covariance structure (2.2). In Section 2.2 we obtain Wald's criteria for these two hypotheses and their asymptotic null distributions under the random-effects covariance structure (2.2).

Let $Q = [p^{-1/2}\mathbf{1}_p\, Q_2]$ and $H = [N^{-1/2}\mathbf{1}_N\, H_2]$ be orthogonal matrices of orders p and N, respectively. Then the model (2.1) can be reduced to a canonical form

$$(2.5)\qquad H'XQ = \begin{bmatrix} z_1 & z_2' \\ y_1 & Y_2 \end{bmatrix} \sim N_{N\times p}\left(\begin{bmatrix} \theta_1 & \theta_2' \\ \tilde{A}_1\gamma & O \end{bmatrix}, \Psi \otimes I_N\right),$$

where $\theta_1 = N^{-1/2}\mathbf{1}_N' A_1\gamma + N^{1/2}p^{-1/2}\boldsymbol{\mu}'\mathbf{1}_p$, $\boldsymbol{\theta}_2' = N^{1/2}\boldsymbol{\mu}' Q_2$, $\tilde{A}_1 = H_2' A_1$, $\gamma = p^{1/2}\boldsymbol{\delta}$,

$$\Psi = \begin{pmatrix} \tau^2 & \mathbf{0} \\ \mathbf{0} & \sigma^2 I_{p-1} \end{pmatrix} \quad \text{and} \quad \tau^2 = p\lambda^2 + \sigma^2.$$

It is known (Yokoyama [11]) that the maximum likelihood estimators (= MLE's) of $\boldsymbol{\theta}_2, \gamma, \lambda^2, \sigma^2$ and τ^2 are given by

$$\hat{\boldsymbol{\theta}}_2 = z_2, \quad \hat{\gamma} = (\tilde{A}_1'\tilde{A}_1)^{-1}\tilde{A}_1' y_1, \quad \hat{\lambda}^2 = \max\left\{\frac{1}{p}\left[\frac{1}{N}s_{11} - \frac{1}{N(p-1)}\operatorname{tr} Y_2'Y_2\right], 0\right\},$$

$$(2.6)\quad \hat{\sigma}^2 = \begin{cases} \dfrac{1}{N(p-1)}\operatorname{tr} Y_2'Y_2, & \text{if } \dfrac{1}{N}s_{11} \geq \dfrac{1}{N(p-1)}\operatorname{tr} Y_2'Y_2, \\ \dfrac{1}{Np}(s_{11} + \operatorname{tr} Y_2'Y_2), & \text{if } \dfrac{1}{N}s_{11} < \dfrac{1}{N(p-1)}\operatorname{tr} Y_2'Y_2, \end{cases}$$

$$\hat{\tau}^2 = \begin{cases} \dfrac{1}{N}s_{11}, & \text{if } \dfrac{1}{N}s_{11} \geq \dfrac{1}{N(p-1)}\operatorname{tr} Y_2'Y_2, \\ \dfrac{1}{Np}(s_{11} + \operatorname{tr} Y_2'Y_2), & \text{if } \dfrac{1}{N}s_{11} < \dfrac{1}{N(p-1)}\operatorname{tr} Y_2'Y_2, \end{cases}$$

where $s_{11} = y_1'(I_{N-1} - P_{\tilde{A}_1})y_1$, $P_{\tilde{A}_1} = \tilde{A}_1(\tilde{A}_1'\tilde{A}_1)^{-1}\tilde{A}_1'$. In the model (2.5), the hypotheses (2.3) and (2.4) are equivalent to

$$(2.7)\qquad H_{01}: \boldsymbol{\theta}_2 = \mathbf{0} \quad \text{vs.} \quad H_{11}: \boldsymbol{\theta}_2 \neq \mathbf{0}$$

and

$$(2.8)\qquad H_{02}: \gamma = \mathbf{0} \quad \text{vs.} \quad H_{12}: \gamma \neq \mathbf{0},$$

respectively.

2.2. Tests for two hypotheses

We consider to test the hypotheses (2.3) and (2.4) in the parallel profile model (2.1). This is equivalent to considering to test the hypotheses (2.7) and (2.8) in the model (2.5). Noting that

$$\hat{\boldsymbol{\theta}}_2 \sim N_{p-1}(\boldsymbol{\theta}_2, \sigma^2 I_{p-1}) \quad \text{and} \quad \hat{\gamma} \sim N_{k-1}(\gamma, \tau^2(\tilde{A}_1'\tilde{A}_1)^{-1}),$$

from (2.6) we can suggest test statistics

$$(2.9)\qquad W_1 = \frac{\hat{\theta}_2'\hat{\theta}_2}{\hat{\sigma}^2} = \begin{cases} R_1, & \text{if } \frac{1}{N} s_{11} \geq \frac{1}{N(p-1)} \operatorname{tr} Y_2'Y_2, \\ R_2, & \text{if } \frac{1}{N} s_{11} < \frac{1}{N(p-1)} \operatorname{tr} Y_2'Y_2 \end{cases}$$

and

$$(2.10)\qquad W_2 = \frac{\hat{\gamma}'\tilde{A}_1'\tilde{A}_1\hat{\gamma}}{\hat{\tau}^2} = \begin{cases} R_3, & \text{if } \frac{1}{N} s_{11} \geq \frac{1}{N(p-1)} \operatorname{tr} Y_2'Y_2, \\ R_4, & \text{if } \frac{1}{N} s_{11} < \frac{1}{N(p-1)} \operatorname{tr} Y_2'Y_2 \end{cases}$$

for testing H_{01} vs. H_{11} and H_{02} vs. H_{12}, respectively, where

$$R_1 = \frac{z_2'z_2}{\operatorname{tr} Y_2'Y_2/\{N(p-1)\}}, \quad R_2 = \frac{z_2'z_2}{(s_{11} + \operatorname{tr} Y_2'Y_2)/(Np)},$$
$$R_3 = \frac{y_1'P_{\tilde{A}_1}y_1}{s_{11}/N}, \quad R_4 = \frac{y_1'P_{\tilde{A}_1}y_1}{(s_{11} + \operatorname{tr} Y_2'Y_2)/(Np)}.$$

The statistics (2.9) and (2.10) can be expressed in terms of the original observations, using

$$z_2'z_2 = N\left\{\bar{x}'\bar{x} - \frac{1}{p}(\bar{x}'\mathbf{1}_p)^2\right\}, \quad y_1'P_{\tilde{A}_1}y_1 = \frac{1}{p}\mathbf{1}_p'(S_t - S_w)\mathbf{1}_p,$$
$$s_{11} = \frac{1}{p}\mathbf{1}_p'S_w\mathbf{1}_p, \quad \operatorname{tr} Y_2'Y_2 = \operatorname{tr} S_t - \frac{1}{p}\mathbf{1}_p'S_t\mathbf{1}_p,$$

where S_t and S_w are the matrices of the sums of squares and products due to the total variation and within variation, i.e.,

$$S_t = \sum_{g=1}^{k}\sum_{j=1}^{N_g}(\boldsymbol{x}_j^{(g)} - \bar{\boldsymbol{x}})(\boldsymbol{x}_j^{(g)} - \bar{\boldsymbol{x}})', \quad S_w = \sum_{g=1}^{k}\sum_{j=1}^{N_g}(\boldsymbol{x}_j^{(g)} - \bar{\boldsymbol{x}}^{(g)})(\boldsymbol{x}_j^{(g)} - \bar{\boldsymbol{x}}^{(g)})',$$

$\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{x}}^{(g)}$ are the sample mean vectors of observations of all the groups and the g-th group, respectively.

THEOREM 2.1. *Let W_1 and W_2 be the test statistics defined by* (2.9) *and* (2.10) *for testing $H_{01} : \boldsymbol{\mu} = v\mathbf{1}_p$ vs. $H_{11} : \boldsymbol{\mu} \neq v\mathbf{1}_p$ and $H_{02} : \boldsymbol{\delta} = \mathbf{0}$ vs. $H_{12} : \boldsymbol{\delta} \neq \mathbf{0}$, respectively. Then it holds that*

(i) *under* H_{01}, $\lim_{N\to\infty} P(W_1 \le c) = P(\chi^2_{p-1} \le c)$,

(ii) *under* H_{02}, $\lim_{N\to\infty} P(W_2 \le c) = P(\chi^2_{k-1} \le c)$,

where χ^2_f *denotes a* χ^2 *variate with* f *degrees of freedom.*

PROOF. From the definition of W_1 we have

$$P(W_1 \le c) = P(R_1 \le c, s_{11}/N \ge \mathrm{tr}\, Y_2'Y_2/\{N(p-1)\}) + P(R_2 \le c, s_{11}/N < \mathrm{tr}\, Y_2'Y_2/\{N(p-1)\}).$$

Let

$$\frac{1}{\sqrt{2N}}\left(\frac{1}{\tau^2}s_{11} - N\right) = U_1, \quad \frac{1}{\sqrt{2N(p-1)}}\left\{\frac{1}{\sigma^2}\mathrm{tr}\, Y_2'Y_2 - N(p-1)\right\} = U_2.$$

Then U_1 and U_2 are independent, and the limiting distribution of U_i is $N(0,1)$, $i = 1, 2$. Note that under H_{01}, $z_2'z_2/\sigma^2$ is distributed as χ^2_{p-1}. Since $\mathrm{tr}\, Y_2'Y_2/\{\sigma^2 N(p-1)\}$ converges in probability to 1, R_1 converges in distribution to χ^2_{p-1}. When $\lambda^2 > 0$, we have

$$\lim_{N\to\infty} P(s_{11}/N \ge \mathrm{tr}\, Y_2'Y_2/\{N(p-1)\}) = 1$$

and hence

$$\lim_{N\to\infty} P(W_1 \le c) = \lim_{N\to\infty} P(R_1 \le c) = P(\chi^2_{p-1} \le c).$$

When $\lambda^2 = 0$, since $(s_{11} + \mathrm{tr}\, Y_2'Y_2)/(\sigma^2 Np)$ converges in probability to 1, R_2 converges in distribution to χ^2_{p-1}. Let

$$Z = \sqrt{\frac{p-1}{p}}U_1 - \sqrt{\frac{1}{p}}U_2.$$

Then the limiting distribution of Z is $N(0,1)$, and $s_{11}/N \ge \mathrm{tr}\, Y_2'Y_2/\{N(p-1)\}$ is equivalent to $Z \ge 0$. Therefore, it holds that

$$\lim_{N\to\infty} P(W_1 \le c) = \lim_{N\to\infty}\{P(R_1 \le c, Z \ge 0) + P(R_2 \le c, Z < 0)\} = P(\chi^2_{p-1} \le c),$$

which proves the result (i). Note that under H_{02}, $y_1'P_{\tilde{A}_1}y_1/\tau^2$ is distributed as χ^2_{k-1}, and is independent of s_{11}. Since $s_{11}/(\tau^2 N)$ converges in probability to 1, R_3 converges in distribution to χ^2_{k-1}. When $\lambda^2 = 0$, since $(s_{11} + \mathrm{tr}\, Y_2'Y_2)/(\tau^2 Np)$ converges in probability to 1, R_4 converges in distribution to χ^2_{k-1}. Therefore, the derivation for the result (ii) follows similarly.

We note that the limiting distributions of the test statistics W_1 and W_2 in Theorem 2.1 agree with ones of the LR criteria in Yokoyama [11]. From the limiting distributions of W_1 and W_2, we can use approximate critical values c_1^* and c_2^* of size α tests such that $P(\chi_{p-1}^2 > c_1^*) = \alpha$ and $P(\chi_{k-1}^2 > c_2^*) = \alpha$, respectively.

3. Analysis of multivariate growth curves with parallel profiles

3.1. Multivariate growth curve model with parallel mean profiles

In this section we consider an extension of the parallel profile model (2.1) to the multiple-response case when m response variables have been measured. Let $\boldsymbol{x}_j{}^{(g)} = (x_{11j}{}^{(g)}, \dots, x_{1mj}{}^{(g)}, \dots, x_{p1j}{}^{(g)}, \dots, x_{pmj}{}^{(g)})'$ be an mp-vector of measurements, and assume that the $\boldsymbol{\mu}^{(g)}$ satisfy $\boldsymbol{\mu}^{(g)} = (\mathbf{1}_p \otimes I_m)\boldsymbol{\delta}^{(g)} + \boldsymbol{\mu}$, $g = 1, \dots, k$. Then the model of X can be written as

$$X \sim N_{N\times mp}(A_1 \Delta(\mathbf{1}_p' \otimes I_m) + \mathbf{1}_N\boldsymbol{\mu}', \Omega \otimes I_N), \tag{3.1}$$

where A_1 is the same as described in (2.1), $\Delta = [\boldsymbol{\delta}^{(1)}, \dots, \boldsymbol{\delta}^{(k-1)}]'$ is an unknown $(k-1) \times m$ parameter matrix, $\boldsymbol{\mu}$ is an mp-vector of unknown parameters, Ω is an unknown $mp \times mp$ positive definite matrix. The model (3.1) may be simply called a multivariate parallel profile model. Further, we assume that Ω in (3.1) has a multivariate random-effects covariance structure (see, e.g., Reinsel [3])

$$\Omega = (\mathbf{1}_p \otimes I_m)\Sigma_\lambda(\mathbf{1}_p' \otimes I_m) + I_p \otimes \Sigma_e, \tag{3.2}$$

where Σ_λ and Σ_e are arbitrary $m \times m$ positive semi-definite and positive definite matrices, respectively. In Section 3.2 we consider Wald-type tests for the hypotheses

$$H_{01} : \boldsymbol{\mu} = \mathbf{1}_p \otimes \boldsymbol{v} \quad \text{vs.} \quad H_{11} : \boldsymbol{\mu} \neq \mathbf{1}_p \otimes \boldsymbol{v} \tag{3.3}$$

and

$$H_{02} : \Delta = O \quad \text{vs.} \quad H_{12} : \Delta \neq O \tag{3.4}$$

under the multivariate random-effects covariance structure (3.2), where $\boldsymbol{v}$ is an m-vector of free parameters. The hypotheses (3.3) and (3.4) are extensions of "no condition variation" and "level" hypotheses in the single-response case due to Srivastava [5] to ones in the multiple-response case. Modified LR statistics for the hypotheses (3.3) and (3.4) have been obtained by Yokoyama [10].

Let $G = [p^{-1/2}\mathbf{1}_p, \boldsymbol{g}_2^{(1)}, \dots, \boldsymbol{g}_2^{(p-1)}] = [p^{-1/2}\mathbf{1}_p, G_2]$ be an orthogonal matrix of order p. Then $Q = G \otimes I_m = [Q_1, Q_2^{(1)}, \dots, Q_2^{(p-1)}] = [Q_1, Q_2]$ is an orthogonal matrix of order mp. Further, let $H = [N^{-1/2}\mathbf{1}_N, H_2]$ be an orthogonal matrix of order N. Then, letting $Y = H_2'XQ = [Y_1, Y_2^{(1)}, \dots, Y_2^{(p-1)}] = [Y_1, Y_2], \boldsymbol{z}' = N^{-1/2}\mathbf{1}_N'XQ = [\boldsymbol{z}_1', \boldsymbol{z}_2^{(1)\prime}, \dots, \boldsymbol{z}_2^{(p-1)\prime}] = [\boldsymbol{z}_1', \boldsymbol{z}_2']$, a canonical form of the model (3.1) can be written as

$$(3.5) \qquad H'XQ = \begin{bmatrix} z_1' & z_2' \\ Y_1 & Y_2 \end{bmatrix} \sim N_{N\times mp}\left(\begin{bmatrix} \boldsymbol{\theta}_1' & \boldsymbol{\theta}_2' \\ \tilde{A}_1\Gamma & O \end{bmatrix}, \Psi \otimes I_N\right),$$

where $\boldsymbol{\theta}_1' = N^{-1/2}\mathbf{1}_N' A_1 \Gamma + N^{1/2}\boldsymbol{\mu}' Q_1$, $\boldsymbol{\theta}_2' = N^{1/2}\boldsymbol{\mu}' Q_2$, $\tilde{A}_1 = H_2' A_1$, $\Gamma = p^{1/2}\Delta$,

$$\Psi = \begin{pmatrix} \Psi_{11} & O \\ O & I_{p-1}\otimes \Sigma_e \end{pmatrix} \quad \text{and} \quad \Psi_{11} = p\Sigma_\lambda + \Sigma_e.$$

It is easily seen that the MLE's of $\boldsymbol{\theta}_2$ and Γ are given by

$$(3.6) \qquad \hat{\boldsymbol{\theta}}_2 = z_2, \quad \hat{\Gamma} = (\tilde{A}_1{}'\tilde{A}_1)^{-1}\tilde{A}_1{}' Y_1.$$

However, since the MLE's of Ψ_{11} and Σ_e are complicated, we use

$$(3.7) \qquad \hat{\Psi}_{11} = \frac{1}{N} S_{11}, \quad \hat{\Sigma}_e = \frac{1}{N(p-1)} \sum_{i=1}^{p-1} Y_2^{(i)\prime} Y_2^{(i)},$$

which are the MLE's under a weaker condition that Ψ_{11} is arbitrary positive definite instead of the restriction that $\Psi_{11} - \Sigma_e$ is positive semi-definite, where $S_{11} = Y_1{}'(I_{N-1} - P_{\tilde{A}_1})Y_1$. In the model (3.5), the hypotheses (3.3) and (3.4) are equivalent to

$$(3.8) \qquad H_{01} : \boldsymbol{\theta}_2 = \mathbf{0} \quad \text{vs.} \quad H_{11} : \boldsymbol{\theta}_2 \neq \mathbf{0}$$

and

$$(3.9) \qquad H_{02} : \Gamma = O \quad \text{vs.} \quad H_{12} : \Gamma \neq O,$$

respectively.

3.2. Tests for two hypotheses

We may consider the problems of testing the hypotheses (3.8) and (3.9) in the model (3.5) instead of testing the hypotheses (3.3) and (3.4) in the multivariate parallel profile model (3.1). Noting that

$$\hat{\boldsymbol{\theta}}_2 \sim N_{m(p-1)}(\boldsymbol{\theta}_2, I_{p-1}\otimes \Sigma_e) \quad \text{and} \quad \hat{\Gamma} \sim N_{(k-1)\times m}(\Gamma, \Psi_{11}\otimes (\tilde{A}_1{}'\tilde{A}_1)^{-1}),$$

from (3.6) and (3.7) we can suggest test statistics

$$(3.10) \qquad W_1 = \hat{\boldsymbol{\theta}}_2'(I_{p-1}\otimes \hat{\Sigma}_e)^{-1}\hat{\boldsymbol{\theta}}_2 = \sum_{i=1}^{p-1} z_2^{(i)\prime} \hat{\Sigma}_e^{-1} z_2^{(i)}$$

and

$$W_2 = \mathrm{vec}(\hat{\Gamma}')'((\tilde{A}_1{}'\tilde{A}_1)^{-1} \otimes \hat{\Psi}_{11})^{-1}\mathrm{vec}(\hat{\Gamma}') = \mathrm{tr}\,\tilde{A}_1{}'\tilde{A}_1\hat{\Gamma}\hat{\Psi}_{11}^{-1}\hat{\Gamma}' = \sum_{i=1}^{k-1} \boldsymbol{y}_1^{(i)\prime}\hat{\Psi}_{11}^{-1}\boldsymbol{y}_1^{(i)} \tag{3.11}$$

for testing H_{01} vs. H_{11} and H_{02} vs. H_{12}, respectively, where $(T' \otimes I_m)\mathrm{vec}(Y_1') = [\boldsymbol{y}_1^{(1)\prime}, \dots, \boldsymbol{y}_1^{(N-1)\prime}]'$, and T is an orthogonal matrix of order $N-1$ such that $T'P_{\tilde{A}_1}T = \mathrm{diag}(1, \dots, \underset{k\hat{-}1}{1}, 0, \dots, 0)$. In terms of the original observations, we can write

$$\hat{\boldsymbol{\theta}}_2 = \sqrt{N}(G_2{}' \otimes I_m)\bar{\boldsymbol{x}}, \quad \hat{\Sigma}_e = \frac{1}{N(p-1)}\sum_{i=1}^{p-1}(\boldsymbol{g}_2^{(i)\prime} \otimes I_m)S_t(\boldsymbol{g}_2^{(i)} \otimes I_m),$$

$$\hat{\Gamma} = \frac{1}{\sqrt{p}}[\bar{\boldsymbol{x}}^{(1)} - \bar{\boldsymbol{x}}^{(k)}, \dots, \bar{\boldsymbol{x}}^{(k-1)} - \bar{\boldsymbol{x}}^{(k)}]'(\mathbf{1}_p \otimes I_m),$$

$$\hat{\Psi}_{11} = \frac{1}{Np}(\mathbf{1}_p' \otimes I_m)S_w(\mathbf{1}_p \otimes I_m), (\tilde{A}_1{}'\tilde{A}_1)^{-1} = \mathrm{diag}\left(\frac{1}{N_1}, \dots, \frac{1}{N_{k-1}}\right) + \frac{1}{N_k}\mathbf{1}_{k-1}\mathbf{1}_{k-1}'.$$

THEOREM 3.1. *Let W_1 and W_2 be the test statistics defined by* (3.10) *and* (3.11) *for testing $H_{01} : \boldsymbol{\mu} = \mathbf{1}_p \otimes \boldsymbol{v}$ vs. $H_{11} : \boldsymbol{\mu} \neq \mathbf{1}_p \otimes \boldsymbol{v}$ and $H_{02} : \Delta = O$ vs. $H_{12} : \Delta \neq O$, respectively. Then it holds that*

(i) *under H_{01},* $\lim_{N\to\infty} P(W_1 \le c) = P(\chi^2_{m(p-1)} \le c)$,

(ii) *under H_{02},* $\lim_{N\to\infty} P(W_2 \le c) = P(\chi^2_{m(k-1)} \le c)$.

PROOF. The statistic (3.10) can be written as

$$W_1 = \sum_{i=1}^{p-1} \frac{K_2^{(i)}}{K_1^{(i)}/\{N(p-1)\}},$$

where

$$K_1^{(i)} = N(p-1)\frac{\boldsymbol{z}_2^{(i)\prime}\Sigma_e^{-1}\boldsymbol{z}_2^{(i)}}{\boldsymbol{z}_2^{(i)\prime}\hat{\Sigma}_e^{-1}\boldsymbol{z}_2^{(i)}}, \quad K_2^{(i)} = \boldsymbol{z}_2^{(i)\prime}\Sigma_e^{-1}\boldsymbol{z}_2^{(i)}.$$

Note that under H_{01}, $\boldsymbol{z}_2^{(i)}$'s are independent,

$$\boldsymbol{z}_2^{(i)} \sim N_m(\mathbf{0}, \Sigma_e) \quad \text{and} \quad \hat{\Sigma}_e \sim W_m\left((N-1)(p-1), \frac{1}{N(p-1)}\Sigma_e\right).$$

It is easy (see, e.g., Siotani, Hayakawa and Fujikoshi [4, p. 74]) to verify that

$K_1^{(i)}$ is distributed as $\chi^2_{(N-1)(p-1)-m+1}$, and $K_1^{(i)}/\{N(p-1)\}$ converges in probability to 1. Since $K_2^{(i)}$'s are independently distributed as χ^2_m, W_1 converges in distribution to $\chi^2_{m(p-1)}$, which proves the result (i). Note that under H_{02}, $y_1^{(i)}$'s are independent,

$$y_1^{(i)} \sim N_m(\mathbf{0}, \Psi_{11}) \quad \text{and} \quad \hat{\Psi}_{11} \sim W_m\left(N-k, \frac{1}{N}\Psi_{11}\right).$$

Therefore, the proof of the result (ii) follows similarly.

We note that the limiting distributions of the test statistics W_1 and W_2 in Theorem 3.1 agree with ones of modified LR statistics in Yokoyama [10].

4. Numerical example

In this section we apply the results of Section 2 to the data (see, e.g., Srivastava and Carter [6, p. 227]) of the price indices of hand soaps packaged in 4 ways, estimated by 12 consumers. Each consumer belongs to one of 2 groups. The adequacy of the parallel profile model (2.1) with the random-effects covariance structure (2.2) (in the case $p = 4$, $k = 2$ and $N = 12$) to the data has been examined by Yokoyama [9]. Therefore, we may consider to test the hypotheses (2.3) and (2.4) in this model. Since

$$z_2'z_2 = N\left\{\bar{x}'\bar{x} - \frac{1}{p}(\bar{x}'\mathbf{1}_p)^2\right\} = .78204, \quad y_1'P_{\tilde{A}_1}y_1 = \frac{1}{p}\mathbf{1}_p'(S_t - S_w)\mathbf{1}_p = 1.0468,$$

$$s_{11} = \frac{1}{p}\mathbf{1}_p'S_w\mathbf{1}_p = .76635, \quad \operatorname{tr} Y_2'Y_2 = \operatorname{tr} S_t - \frac{1}{p}\mathbf{1}_p'S_t\mathbf{1}_p = .35130$$

and $s_{11}/N \geq \operatorname{tr} Y_2'Y_2/\{N(p-1)\}$, it follows from Theorem 2.1 that

$$W_1 = \frac{z_2'z_2}{\operatorname{tr} Y_2'Y_2/\{N(p-1)\}} = 80.141 > \chi^2_{p-1}(.01) = 11.345,$$

$$W_2 = \frac{y_1'P_{\tilde{A}_1}y_1}{s_{11}/N} = 16.391 > \chi^2_{k-1}(.01) = 6.635.$$

Hence, both hypotheses H_{01} and H_{02} are rejected at $\alpha = .01$. On the other side, it is known (Yokoyama [11]) that the LR criteria also reject both hypotheses in this example.

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